Lecture 8

Analytic compactification and modular forms

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1 The classical modular curves

As a motivation for the analytic construction of the Drinfeld modular curves we review the analytic construction of the classical modular curve corresponding to full level \( n \) with \( n \in \mathbb{Z}, \ n \geq 3 \). This curve, considered as an affine curve over the field of complex numbers \( \mathbb{C} \), will be denoted by \( Y(n) \).

The complex valued points of \( Y(n) \) are in one to one correspondence with the isomorphy classes of the pairs \( (E, \lambda) \), where \( E \) denotes an elliptic curve over \( \mathbb{C} \); \( E[n] := \{ e \in E \mid ne = 0 \} \) is the group of the \( n \)-torsion points and \( \lambda: n^{-1}\mathbb{Z}^2/\mathbb{Z}^2 \to E[n] \) is an isomorphism of groups.

Every elliptic curve \( E \) over \( \mathbb{C} \) has the form \( \mathbb{C}/\Lambda \) where \( \Lambda \subset \mathbb{C} \) is a lattice of rank two. Two lattices \( \Lambda_1 \) and \( \Lambda_2 \) define isomorphic curves if and only if there is a constant \( c \in \mathbb{C}^\ast \) with \( c\Lambda_1 = \Lambda_2 \). Moreover the group of the \( n \)-torsion points of \( E \) can be identified with \( (\frac{1}{n}\Lambda)/\Lambda \subset \mathbb{C}/\Lambda \). Consider now the set

\[
Z := (\mathbb{C} \setminus \mathbb{R}) \times \text{GL}(n^{-1}\mathbb{Z}^2/\mathbb{Z}^2).
\]

This can be interpreted as the disjoint union of \( 2 \cdot \#\text{GL}(n^{-1}\mathbb{Z}^2/\mathbb{Z}^2) \) copies of the upper half plane \( \mathcal{H} := \{ z \in \mathbb{C} \mid \text{Im}(z) > 0 \} \). The group \( \text{GL}(2, \mathbb{Z}) \) acts on this space by

\[
\sigma := \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in \text{GL}(2, \mathbb{Z}) \text{ acts on } (\tau, \alpha) \in Z \text{ as } \sigma(\tau, \alpha) = \left( \frac{a\tau + b}{c\tau + d}, \frac{a\tau}{c\tau + d} \right),
\]

where \( \sigma \) denotes the image of \( (\sigma^t)^{-1} \) (i.e. the inverse of the transposed matrix) in \( \text{GL}(n^{-1}\mathbb{Z}^2/\mathbb{Z}^2) \).

To a point \( (\tau, \alpha) \in Z \) we associate a pair \( (E, \lambda) \). Let \( f_\tau: \mathbb{Z}^2 \to \mathbb{C} \) denote the map given by \( f_\tau(a, b) = \alpha \tau + b \). This map extends to a \( \mathbb{Q} \)-linear map \( \mathbb{Q}^2 \to \mathbb{C} \). Then \( E = \mathbb{C}/f_\tau(\mathbb{Z}^2) \) and

\[
\lambda := n^{-1}\mathbb{Z}^2/\mathbb{Z}^2 \to n^{-1}\mathbb{Z}^2/\mathbb{Z}^2 \cong E[n],
\]
where \(\alpha\) denotes the canonical map induced by \(f_r\). It can be seen that the corresponding map \(Z \to Y(n)\) is surjective and that two elements \((\tau_1, \lambda_1)\) and \((\tau_2, \lambda_2)\) have the same image in \(Y(n)\) if and only if there exists a \(\sigma \in \text{GL}(2, \mathbb{Z})\) with \(\sigma(\tau_1, \alpha_1) = (\tau_2, \alpha_2)\). In other words we have found a bijection

\[ \text{GL}(2, \mathbb{Z}) \backslash Z \to Y(n). \]

This construction induces on \(Y(n)\) the structure of a Riemann surface. Let \(\Gamma(n)\) denote the subgroup of \(\text{GL}(2, \mathbb{Z})\) consisting of the matrices \(\sigma\) such that \(\sigma\) is congruent to the identity matrix modulo \(n\). Clearly \(\Gamma(n)\) is a normal subgroup of \(\text{SL}(2, \mathbb{Z})\). The action of \(\Gamma(n)\) on \(Z\) leaves every component of \(Z\) (setwise) invariant. Let us write \(\mathbb{C} \backslash \mathbb{R}\) as \(\mathcal{H} \times \{\pm 1\}\). Then \(\Gamma(n) \backslash Z\) can be identified with \((\Gamma(n) \backslash \mathcal{H}) \times \{\pm 1\} \times \text{GL}(n^{-1} \mathbb{Z}^2 / \mathbb{Z}^2)\). This space has to be divided out by the action of the subgroup \(\Gamma(n) \backslash \text{GL}(2, \mathbb{Z})\) of \(\text{GL}(2, \mathbb{Z} / n \mathbb{Z})\) \(\cong \text{GL}(n^{-1} \mathbb{Z}^2 / \mathbb{Z}^2)\), consisting of the matrices with determinant \(\pm 1\). The action of this finite group on the collection of the components, i.e. on the set \(\{\pm 1\} \times \text{GL}(n^{-1} \mathbb{Z}^2 / \mathbb{Z}^2)\), is faithful. The action of an element \(\sigma\) of this finite group on the factor \(\{\pm 1\}\) is multiplication by the determinant of \(\sigma\) and the action on the second factor is left multiplication by \(\sigma\). Let \(\Xi\) denote the finite set \(\text{SL}(2, \mathbb{Z}) \backslash \text{GL}(2, \mathbb{Z} / n \mathbb{Z})\). One easily sees that \(\Xi\) is isomorphic to the group \((\mathbb{Z} / n \mathbb{Z})^*\), which has cardinality \(\phi(n)\). From the above one derives an isomorphism

\[ (\Gamma(n) \backslash \mathcal{H}) \times \Xi \to Y(n). \]

The Riemann surface \(Y(n)\) has therefore \(\phi(n)\) connected components.

In the classical literature the (affine) modular curve with full level \(n\) is often defined as the quotient \(\Gamma(n) \backslash \mathcal{H}\). This quotient is known to be the analytification of an affine curve over \(\mathbb{C}\). The (complex valued) points on this (more restricted) curve correspond to the isomorphism classes of the pairs \((E, \lambda)\) with \(\lambda\) a symplectic isomorphism. Our definition of \(Y(n)\) is a finite disjoint union of \(\phi(n)\) copies of an affine irreducible curve.

## 2 An analytic moduli space

We recall the standard notations \(p, q, A, K, K_\infty, \mathbb{C}\) which are used here:

\(p\) is a prime number and \(q\) is a power of it; \(K\) is a function field in one variable over \(\mathbb{F}_q\); a place \(\infty\) of \(K\), called the place at infinity, is chosen; \(A\) is the subring of \(K\) consisting of the elements which have no poles outside \(\infty\); the completion of \(K\) with respect to the place \(\infty\) is written as \(K_\infty\); the completion of the algebraic closure of \(K_\infty\) is denoted by \(\mathbb{C}\). The degree \(\text{deg}(a)\) of a nonzero element \(a \in A\) is chosen such that \(a \mapsto -\text{deg}(a)\) is the normalized additive valuation
associated with the place $\infty$.

We remark that $A, K, K_\infty, C$ are analogues of $\mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$. The analogue of the complex upper half plane (more precisely, of $\mathbb{C}\setminus\mathbb{R}$) is the rigid analytic space $\Omega$, which is defined over $K_\infty$. This is Drinfeld’s upper half plane introduced in lecture 6. By abuse of notation $\Omega$ will also denote the set $\Omega(C) = C \setminus K_\infty$.

As we have seen, any Drinfeld module of rank two over $C$ can be obtained from an $A$-lattice $\Lambda \subset C$ of rank two. We recall the definitions and the construction (see also lecture 3):

1. An $A$-lattice $\Lambda$ of rank two is a finitely generated $A$-submodule of $C$ such that $A_0$ is discrete as subset of $C$ and such that $K \otimes_A A$ has dimension two over $K$. For two $A$-lattices $\Lambda_1, \Lambda_2$ of rank two we define $\text{Hom}(\Lambda_1, \Lambda_2)$ to be the additive group of the $c \in C$ such that $c\Lambda_1 \subset \Lambda_2$. This makes the set of $A$-lattices of rank two in $C$ into a category.

2. Let $\Lambda$ be an $A$-lattice of rank two. One associates to this lattice the function $e_{\Lambda} : C \to C$, given by the formula

$$e_{\Lambda}(z) = z \prod_{\lambda \in \Lambda, \lambda \neq 0} (1 - \frac{z}{\lambda}).$$

$e_{\Lambda}$ is an entire function on $C$ and has an expansion $\sum_{n \geq 0} c(\Lambda)n^z$. This function is a surjective $A_0$-linear map : $C \to C$ and its kernel is $\Lambda$. The induced isomorphism of groups $C/\Lambda \to C$ is actually an isomorphism of rigid analytic spaces.

3. The homomorphism

$$\phi : A \to C \to C/\Lambda \cong C$$

induces a new $A$-module structure on $C$ for which $C$ is a Drinfeld module of rank two over $C$. The definition of $\phi$ implies that for any $a \in A$; $a \neq 0$ the map $\phi_a(z)$ has the form $\sum_{n=0}^{\deg(a)} c(\Lambda, a)n^z$. By construction $\phi_a(e_{\Lambda}(z)) = e_{\Lambda}(az)$. Let $D(\Lambda)$ denote this Drinfeld module. For a homomorphism $\text{Hom}(\Lambda_1, \Lambda_2)$ one has an induced map $c : C/\Lambda_1 \to C/\Lambda_2$ and as a consequence a homomorphism $D(c) : D(\Lambda_1) \to D(\Lambda_2)$ of Drinfeld modules.

4. The functor $D$ is an equivalence of the category of the $A$-lattices of rank two in $C$ with the category of the Drinfeld $A$-modules over $C$ of rank two.

5. Since $A$ is a Dedekind ring and $\Lambda$ has no torsion it follows that $\Lambda$ is a projective module of rank two. Such a module is isomorphic to $Y = A \oplus N$.
where $N$ is a projective module of rank one. The class group of $A$ (or Picard group $Pic(A)$) is finite and can be represented by a finite set of nonzero ideals \( \{N_1, \ldots, N_s\} \) of $A$. In the sequel we will suppose that $Y$ has the form $A \oplus N$ for some $N \in \{N_1, \ldots, N_s\}$.

6. An $A$-lattice $\Lambda$ will be called of type $Y$ if $\Lambda$ is isomorphic to $Y$ as an $A$-module. The corresponding Drinfeld module $D(\Lambda)$ is also called of type $Y$. Let $n \neq 0$ be an ideal of $A$. A (full) level $n$-structure on an $A$-lattice $\Lambda$ of rank two and of type $Y$ is given by an $A$-linear isomorphism $n^{-1}Y/Y \to n^{-1}\Lambda/\Lambda$. A (full) level $n$-structure on a Drinfeld module $\phi : A \to C$ over $C$ and of type $Y$ is defined to be an $A$-linear isomorphism $n^{-1}Y/Y \to \{c \in C | \phi_a(c) = 0 \text{ for all } a \in n\}$.

The last set is easily seen to be the image of $n^{-1}\Lambda \subset C$ under the map $\phi$. One obtains in this way a 1-1 correspondence between the level $n$-structures on $\Lambda$ and on $D(\Lambda)$. Hence the operation $\Lambda \mapsto \phi \Lambda$ induces an equivalence between the category of the $A$-lattices of rank two in $C$ with level structure $n$ and the category of the Drinfeld $A$-modules of rank two over $C$ with level $n$-structure.

**Some comments**

We note some differences with the classical situation. First of all $C$ contains $A$-lattices of any rank because $C$ is an infinite extension of $K_{\infty}$. Moreover rank two lattices are in general not free since $Pic(A)$ can be nontrivial. As a consequence we will have to consider a Drinfeld modular curve for each element in $Pic(A)$. Finally, the rigid analytic space $\Omega$ is connected and $\mathbb{C} \setminus \mathbb{R}$ has two components.

Our aim is to describe the set of equivalence classes of $A$-lattices in $C$ with level structure $n$. Fix a choice for $Y = A \oplus N$ and a level $n$. For $\omega \in \Omega$ one considers the map $f_\omega : Y \to C$, given by $f_\omega(a + n) = a\omega + n$. This map extends to a $K$-linear map $f_\omega : K \otimes Y \to C$. Then $\Lambda = f_\omega(Y)$ and the map $n^{-1}Y/Y \to n^{-1}\Lambda/\Lambda$, induced by $f_\omega$, define an $A$-lattice of type $Y$ with level structure $n$.

In this way, every equivalence class of $A$-lattices of type $Y$ in $C$ is obtained. The obvious homomorphism $GL(Y) \to GL(n^{-1}Y/Y)$ is in general not surjective. This has the consequence that the map from $\Omega$ to the equivalence classes of $A$-lattices of type $Y$ and with level $n$ is not surjective. We consider the map $\Omega \times GL(n^{-1}Y/Y) \to A$-lattices of type $Y$ and with level $n$ given by $(\omega, \alpha) \mapsto \Lambda = A\omega + N$ and with the level structure given by $n^{-1}Y/Y \xrightarrow{\phi} n^{-1}Y/Y \xrightarrow{\phi} n^{-1}\Lambda/\Lambda$. 4
The last isomorphism is the one induced by $f_*$ defined above.

The action of the group $\text{Gl}(Y)$ on $\Omega \times \text{Gl}(\mathfrak{n}^{-1}Y/Y)$ is defined by the formula $g(\omega, \alpha) = (g(\omega), g\alpha)$, where $g$ denotes the image of $(g')^{-1}$ (i.e., the inverse of the transposed map) in $\text{Gl}(\mathfrak{n}^{-1}Y/Y)$. It is easy to verify that two elements $(\omega_1, \alpha_1)$ and $(\omega_2, \alpha_2)$ define equivalent $A$-lattices of type $Y$ and with level $\mathfrak{n}$, if and only if there is a $g \in \text{Gl}(Y)$ with $g(\omega_1, \alpha_1) = (\omega_2, \alpha_2)$.

Let $\text{Gl}(Y; \mathfrak{n})$ denote the kernel of the map $\text{Gl}(Y) \to \text{Gl}(\mathfrak{n}^{-1}Y/Y)$. One can verify that $\text{Gl}(Y; \mathfrak{n})$ consists of the $\sigma$ such that $\sigma - 1$ maps $Y$ into $\mathfrak{n}Y$. Put $\Xi := \text{Gl}(Y) \setminus \text{Gl}(\mathfrak{n}^{-1}Y/Y)$. Then the quotient $\text{Gl}(Y) \setminus \text{Gl}(\mathfrak{n}^{-1}Y/Y)$ can also be written as a finite disjoint union over the $\xi \in \Xi$ of the quotients $(\text{Gl}(Y; \mathfrak{n}) \setminus \Omega) \times \xi$. Thus we have shown:

**Proposition 2.1** The set of Drinfeld $A$-modules of rank two with type $Y$ and level $\mathfrak{n}$ over the field $\mathbb{C}$ is parametrized by

$$\text{Gl}(Y) \setminus \text{Gl}(\mathfrak{n}^{-1}Y/Y).$$

This rigid analytic space is isomorphic to $(\text{Gl}(Y; \mathfrak{n}) \setminus \Omega) \times \Xi$ where $\Xi$ denotes the finite set $\text{Gl}(Y) \setminus \text{Gl}(\mathfrak{n}^{-1}Y/Y)$.

This result is the first step in showing that the rigid space

$$\mathcal{M}_Y(\mathfrak{n}) := (\text{Gl}(Y; \mathfrak{n}) \setminus \Omega) \times \Xi$$

is a moduli space (in the rigid analytic category) for the Drinfeld modules of type $Y$ and level $\mathfrak{n}$.

We note further that the homomorphism $\text{Sl}(2, A) \to \text{Sl}(2, A/\mathfrak{n})$ is surjective. It follows from this that the group $\text{Gl}(2, A) \setminus \text{Gl}(2, A/\mathfrak{n})$ is isomorphic to $(A/\mathfrak{n})^*/\mathbb{F}_q^*$. Thus the number of connected components of the space $\text{Gl}(2, A) \setminus (\Omega \times \text{Gl}(2, A/\mathfrak{n})) = (\text{Gl}(2, A) \setminus \Omega) \times \Xi$ is equal to $\# \Xi = \#(A/\mathfrak{n})^*/\mathbb{F}_q^*$.

To show that $\mathcal{M}_Y(\mathfrak{n})$ is the desired moduli space, one has to construct a universal Drinfeld module (of type $Y$ and level $\mathfrak{n}$) above it. This will be obtained from the construction of a Drinfeld module above $\Omega \times \text{Gl}(\mathfrak{n}^{-1}Y/Y)$ (of type $Y$ and with level $\mathfrak{n}$) and then dividing out the action of the group $\text{Gl}(Y)$.

Write again $Y = A \oplus N$. To a point $\omega \in \Omega$ we have associated the lattice $A\omega + N$ and the “Carlitz exponential” of the lattice

$$e(\omega, z) = z \prod_{\alpha \in A\omega + N, \alpha \neq 0} \left(1 - \frac{z}{\alpha}\right).$$

This function is holomorphic in the two variables $\omega$ and $z$. It has a convergent expansion $\sum_{n \geq 0} e(\omega)_n z^n$, where the $e(\omega)_n$ are holomorphic functions on
\( \Omega \). For \( a \in A, a \neq 0 \), the function \( \phi_a(\omega, z) \), which is defined by the equality 
\( \phi_a(\omega, e(\omega, z)) = e(\omega, a\omega) \), is also holomorphic in the two variables and has the expansion 
\( \phi_a(\omega, z) = \sum_{n=0}^{\infty} \frac{a^n}{n!} e(\omega, \alpha)^n \) with respect to the variable \( z \).

Line bundles (rigid analytic or algebraic) above a space \( X \) can be either defined as a sheaf on \( X \) which is locally isomorphic to the structure sheaf or as a space \( L \to X \) which is locally isomorphic to a product \( X \times \mathbb{C} \). The latter description will be used now since it is more convenient. The trivial analytic line bundle \( L \) above \( \Omega \times GL(\mathbb{C}^{-1}Y/Y) \) is \( \Omega \times GL(\mathbb{C}^{-1}Y/Y) \times \mathbb{C} \). (In fact any analytic line bundle above \( \Omega \times GL(\mathbb{C}^{-1}Y/Y) \) is trivial.) For any \( a \in A, a \neq 0 \) the map 
\( \phi_a : (\omega, \alpha, z) \to (\omega, \alpha, \phi_a(\omega, z)) \) is an endomorphism of this line bundle. This defines the Drinfeld module above \( \Omega \times GL(\mathbb{C}^{-1}Y/Y) \).

As usual the group \( GL(Y) \) operates on \( \Omega \) by
\[
\begin{pmatrix} a & b \\ c & d \end{pmatrix} \omega = \frac{a\omega + b}{c\omega + d}.
\]

We extend this to an action on the trivial line bundle \( L \) by
\[
\begin{pmatrix} a & b \\ c & d \end{pmatrix} (\omega, \alpha, z) = \begin{pmatrix} a \omega + b \\ c \omega + d \end{pmatrix}^{-1} \alpha, (c\omega + d)^{-1} z).
\]

We note that \( GL(Y) \) is the subgroup of the matrices of \( GL(K) \) satisfying 
\( a, d \in A, b \in N^{-1}, c \in N \) and \( ad - bc \in A^* \).

For \( \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(Y) \) one has the following formulas:
\[
(c\omega + d) e(\gamma(\omega), (c\omega + d)^{-1} z) = e(\omega, z) \text{ and }
\]
\[
(c\omega + d) \phi_a(\gamma(\omega), (c\omega + d)^{-1} z) = \phi_a(\omega, z).
\]

The first formula is correct since the left hand side and the right hand side of the formula have expansions which start with \( z \) and the kernels of the two \( \mathbb{F}_q \)-linear maps are \( A\omega + N \). The formula concerning \( \phi_a(\omega, z) \) is a consequence of the first formula. One can now easily verify that the Drinfeld module \( \phi : A \to End(L) \) above \( \Omega \times GL(\mathbb{C}^{-1}Y/Y) \) commutes with the action of \( GL(Y) \) on \( L \).

The next step is the division of the Drinfeld module \( \Omega \times GL(\mathbb{C}^{-1}Y/Y), L, \phi \) by the group \( GL(Y) \). Here we have to make a general remark about moduli. For the existence of a moduli space for certain objects it is necessary that the objects have no automorphisms (different from the identity). In our situation this means that no \( A \)-lattice of type \( Y \) and level structure \( \mathfrak{a} \) has an automorphism different from the identity. The automorphisms (and endomorphisms) of
an $A$-lattice of type $Y$ (and of Drinfeld modules) are discussed in the lectures 1, 3 and 5. One can show that for a level structure $\mathfrak{n}$ such that the dimension of the $\mathbb{F}_q$-vector space $A/\mathfrak{n}$ is large enough the $A$-lattices with level $\mathfrak{n}$-structure have no automorphisms. This condition on the $A$-lattices of type $Y$ with level $\mathfrak{n}$ can be translated into:

The group $\text{Gl}(Y)$ has no fixed points on $\Omega \times \text{Gl}(\mathfrak{n}^{-1}Y/Y)$

or into the equivalent statement:

The group $\text{Gl}(Y, \mathfrak{n})$ has no fixed points for its action on $\Omega$.

The standard example.
For $A = \mathbb{F}_q[T]$ and $\mathfrak{n}$ generated by a (monic) polynomial of degree $\geq 1$, the condition above is satisfied.

In the following we will assume that the group $\text{Gl}(Y, \mathfrak{n})$ has no fixed points on $\Omega$. Then $\Omega \times \text{Gl}(\mathfrak{n}^{-1}Y/Y)$ and $L$ can be divided by $\text{Gl}(Y)$ and one obtains a (rigid analytic) line bundle

$$\text{Gl}(Y) \backslash L \rightarrow \text{Gl}(Y) \backslash \Omega \times \text{Gl}(\mathfrak{n}^{-1}Y/Y).$$

Since $\phi$ is equivariant for the action of $\text{Gl}(Y)$ one obtains a Drinfeld module with level $\mathfrak{n}$ structure above the space $\text{Gl}(Y) \backslash (\Omega \times \text{Gl}(\mathfrak{n}^{-1}Y/Y))$.

Theorem 2.2 Suppose that the group $\text{Gl}(Y, \mathfrak{n})$ has no fixed points for its action on $\Omega$. Then the Drinfeld module above the rigid space $\text{Gl}(Y) \backslash (\Omega \times \text{Gl}(\mathfrak{n}^{-1}Y/Y))$, constructed above, is the universal one for type $Y$ and level $\mathfrak{n}$ in the category of the rigid spaces over $K_\infty$.

Before indicating a proof we return to the “standard example”.

The standard example. Put $A = \mathbb{F}_q[T]$ and $\Gamma = \text{Gl}(A^2)$. The function

$$e(\omega, z) = z \prod (1 - \frac{z}{\alpha \omega + b}) = \sum_{n \geq 0} e(\omega) z^n$$

satisfies

$$e(\omega, z) = z(1 - \sum_{m \geq 1} s_m z^{m(1-1)})^{-1}, \quad \text{with} \quad s_m = \sum_{\langle 0,0 \rangle \neq (a,b) \in A^2} (\alpha \omega + b)^{m(1-1)}.$$

One finds this formula by calculating $\frac{d e(\omega, z)}{d (\omega, z)} = \frac{1}{z (\omega, z)}$ and by observing that $\sum_{\langle 0,0 \rangle \neq (a,b) \in A^2} (\alpha \omega + b)^{-k}$ is 0 for $k > 0$ and $k$ not divisible by $q - 1$. 

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The formula implies that $e(\omega)_1 = s_1$ and $e(\omega)_2$ is some polynomial in $s_1, \ldots, s_{q+1}$.

Write $\phi_T(\omega, z) = Tz + c_1(\omega)z^q + c_2(\omega)z^{q+1}$. From $\phi_T(\omega, e(\omega, z)) = e(\omega, Tz)$ it follows that

$$c_1(\omega) = e(\omega)_1(T^q - T) \quad \text{and} \quad c_2(\omega) = e(\omega)_2(T^{q+1} - T) - e(\omega)_1^{q+1}(T^q - T).$$

The $c_2(\omega)$ is invertible on $\Omega$ and we can form the expression $j(\omega) = \frac{c_1(\omega)^{q+1}}{c_2(\omega)}$. This expression is a holomorphic function on $\Omega$ and is invariant under the action of $GL(A^2)$. Hence $j$ induces a holomorphic function $j : \Gamma \backslash \Omega \to C$. This map is a bijection.

Indeed, the set $\Gamma \backslash \Omega$ parametrizes all Drinfeld $A$-modules of rank two over $C$. Such a Drinfeld module is characterized by the $\phi_T = T + c_1\tau + c_2\tau^2$. The polynomials $T + c_1\tau + c_2\tau^2$ and $T + d_1\tau + d_2\tau^2$ define isomorphic Drinfeld modules if and only if there exists a $c \in C^*$ with $c^{-1}(T + c_1\tau + c_2\tau^2)c = T + d_1\tau + d_2\tau^2$. This last condition is equivalent with $\frac{c_1^{q+1}}{c_2} = \frac{d_1^{q+1}}{d_2}$. This proves the statement.

One can prove a stronger result, namely :

**Proposition 2.3** Let $A_{K_w}^{1,an}$ denote the affine line over $K_\infty$, considered as a rigid space over $K_\infty$. Then $j : PGL(2, A) \backslash \Omega \to A_{K_w}^{1,an}$ is an isomorphism.

**Proof.** The space $PGL(2, A) \backslash \Omega$ has been identified with the affine line over $K_\infty$ (see lecture 7). Any point $b \in F_\infty^* \backslash F_\omega$ is a zero of the function $c_1(\omega)$. It can be calculated that it is a zero of order one. The stabilizer in $PGL(2, A)$ of $b$ is a cyclic group of order $q + 1$. Let $\xi \in PGL(2, A) \backslash \Omega$ denote the image of $b$. Using the expression $j = \frac{c_1^{q+1}}{c_2}$ one finds that $j$ has a zero of order one at $\xi$. Thus $j$ is a bijective analytic map on the rigid analytic affine line over $K_\infty$ with a simple zero at $\xi$. Let $z$ denote a coordinate of this affine line with $z(\xi) = 0$. Then $j$ is equal to a convergent power series $\sum_{n>0} a_n z^n$ with $a_1 \neq 0$. Since $j$ is bijective, this expression has only one zero and so $\sum_{n>0} a_n z^{n-1}$ has no zeroes on the affine line. It is known that the only invertible holomorphic functions on $A_{K_w}^{1,an}$ are the constant functions (see lecture 3). Thus $\sum_{n>0} a_n z^{n-1}$ is constant and equals $a_1$. This shows that $j$ is an isomorphism. \qed

**Proof.** **Sketch of the proof of theorem 2.2**
We first observe that the analytic construction is given over the field $K_\infty$. Let a Drinfeld module $(\psi, L, S)$ of type $Y$ and level $\underline{2}$ be given in the category of the rigid analytic spaces over $K_\infty$. This means that $S$ is a rigid analytic space over $K_\infty$, that $L$ is an analytic line bundle over $S$ and that there is given a
Drinfeld module of rank two on $L$, of type $Y$ with a level $\mathfrak{m}$ structure. It has to be shown that there exists a unique analytic map

$$f : S \to GL(Y) \backslash (\Omega \times GL(\mathfrak{m}^{-1}Y/Y))$$

such that $f$ transports the Drinfeld module above $GL(Y) \backslash (\Omega \times GL(\mathfrak{m}^{-1}Y/Y)$ to one above $S$ which is isomorphic to the given one (by a uniquely determined isomorphism).

We will restrict ourselves to the case $A = F_q[T]$ and some level $\mathfrak{m}$ generated by a polynomial $f$ of degree $\geq 1$. Write $\mathcal{M}(\mathfrak{m})$ for the space

$$GL(A^2) \backslash (\Omega \times GL(\mathfrak{m}^{-1}A^2/A^2)) = \cup_{\xi \in \mathfrak{m}} (GL(A^2, \mathfrak{m}) \backslash \Omega) \times \xi,$$

and $\mathcal{M}(1)$ for $PGl(A^2) \backslash \Omega$. On $\mathcal{M}(\mathfrak{m})$ the group $GL(2, A/\mathfrak{m})/F_q^*$ acts in the obvious way and the quotient by this action is $\mathcal{M}(1)$.

Let a Drinfeld module $\psi$ with level $\mathfrak{m}$-structure above a rigid space $S$ (over $K^\infty$) be given. We have to construct an analytic map $S \to \mathcal{M}(\mathfrak{m})$ with the required properties. Using the $j$-invariant of this Drinfeld module and proposition 2.4 one finds an analytic map $f : S \to \mathcal{M}(1)$. We note further that the line bundle $L$ is trivial since the level $\mathfrak{m}$-structure provides a nowhere vanishing section of $L$.

Let $S'$ be the fibered product of $S$ and $\mathcal{M}(\mathfrak{m})$ over $\mathcal{M}(1)$. The projection $pr_1 : S' \to S$ is finite and $S$ is in fact the quotient of $S'$ under the action of $GL(2, A/\mathfrak{m})/F_q^*$ on $S'$. Further $pr_1$ induces a Drinfeld structure $pr_1^* \psi$ on $S'$ (with respect to the trivial line bundle). The second projection $pr_2 : S' \to \mathcal{M}(\mathfrak{m})$ induces a Drinfeld structure $pr_2^* \phi$ on $S'$ (again with respect to the trivial line bundle). For each point of $S'$ the two Drinfeld structures are isomorphic. This implies that $\ker(pr_1^* \psi_j, O_{S'}) = \ker(pr_2^* \phi_j, O_{S'})$. Let us denote this constant subsheaf of $O_{S'}$ by $L$.

From the level structure on $S$ we obtain an isomorphism of constant sheaves $\iota_1 : \mathfrak{m}^{-1}A^2/A^2 \to L$ and from the level structure on $\mathcal{M}(\mathfrak{m})$ one obtains another isomorphism of constant sheaves $\iota_2 : \mathfrak{m}^{-1}A^2/A^2 \to L$. This induces an automorphism of the constant sheaf $\mathfrak{m}^{-1}A^2/A^2$ on $S'$. This automorphism is a section $\sigma$ of the constant sheaf $GL(2, A/\mathfrak{m})$ on $S'$. Then $S'$ can be written as a disjoint union $\cup S'_g$ over the elements $g$ of the group $GL(2, A/\mathfrak{m})$. The subspace $S'_g$ is the subset of $S'$ where $\sigma$ is equal to $g$. The group $GL(2, A/\mathfrak{m})$, acting on $S'$, permutes the $S'_g$'s as follows: $h(S'_g) = S'_{hg}$. It follows from this that each $pr_1 : S'_g \to S$ is an isomorphism. In particular, the isomorphism $pr_1 : S'_1 \to S$ induces a morphism $S \to \mathcal{M}(\mathfrak{m})$ which has the required properties as one easily verifies. \hfill $\square$

**Corollary 2.4** Assume that the group $GL(Y, \mathfrak{m})$ has no fixed points for its action on $\Omega$. Let $\mathcal{M}^2(\mathfrak{m})$ denote the moduli space for Drinfeld $A$-modules of rank two with level structure $\mathfrak{m}$ and type $Y$. 

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Then the analytic space over $K_\infty$ associated to the affine algebraic curve $M^0_Y(\underline{n}) \otimes_A K_\infty$ together with its Drinfeld module and level $\underline{n}$-structure is isomorphic to $\tilde{M}_Y(\underline{n})$ with its Drinfeld module and level $\underline{n}$-structure.

Proof. Generalities about representable functors imply that the analytification of $M^0_Y(\underline{n}) \otimes_A K_\infty$ with its Drinfeld module and level structure is also universal in the rigid category. This yields the isomorphism of the corollary. \qed

3 Compactifying $Gl(\mathcal{Y}, \underline{n}) \setminus \Omega$

As a didactical excursion we recall first the compactification of the classical modular curve. The notation of section 1 will be used.

3.1 Compactifying $\Gamma(n) \setminus \mathcal{H}$

The integer $n$ is supposed to be $\geq 3$. Since all the components of the curve $Y(n)$ are isomorphic we may restrict our attention to the component $\Gamma(n) \setminus \mathcal{H}$. This object is an affine nonsingular irreducible curve over $\mathbb{C}$. Its compactification $\overline{\Gamma(n) \setminus \mathcal{H}}$ is simply the corresponding projective nonsingular curve. The new points are called the cusps.

Each rational point on the “boundary” of $\mathcal{H}$, i.e. an element of $\mathbb{P}^1(\mathbb{Q}) = \mathbb{Q} \cup \{\infty\}$, produces a cusp of $\Gamma(n) \setminus \mathcal{H}$. The simplest cusp to consider is $\infty$. A “horicylce neighbourhood” of $\infty$ is the set $U := \{z \in \mathbb{C} \mid \text{Im}(z) > r\}$. For $r \geq 1$ one has:

Two points $u_1, u_2 \in U$ are equivalent under the action of $\Gamma(n)$ if and only if they are equivalent under the action of the subgroup

$$H_n = \left\{ \begin{pmatrix} 1 & na \\ 0 & 1 \end{pmatrix} \right\} \mid a \in \mathbb{Z} \right\} \text{ of } \Gamma(n).$$

The image of $U$ in $\Gamma(n) \setminus \mathcal{H}$ is therefore isomorphic to $H_n \setminus U$. The last space is isomorphic to $\{q \in \mathbb{C} \mid 0 < |q| < r'\}$ (for some $r'$) by means of the exponential map $\tilde{z} \in U \mapsto q = e^{2\pi i z/n}$. The complex analytic way to compactify $\Gamma(n) \setminus \mathcal{H}$ at the cusp $\infty$ consists of gluing $\{q \in \mathbb{C} \mid |q| < r'\}$ to $\Gamma(n) \setminus \mathcal{H}$ over the open subset $H_n \setminus U \cong \{q \in \mathbb{C} \mid 0 < |q| < r'\}$.

The group $\Gamma(1) = PGU(2, \mathbb{Z})$ has only one cusp. Therefore the cusps of $\Gamma(n)$ are conjugated under the action of $\Gamma(1)$. This has the consequence that the set of cusps of $\Gamma(n)$ can be identified with $\Gamma(n) \setminus \mathbb{P}^1(\mathbb{Q})$.

Above the curve $Y(n)$ (or above the more restricted curve $\Gamma(n) \setminus \mathcal{H}$) there is a universal family of elliptic curves with full level $\underline{n}$-structure. One can extend
this family by placing a “generalized elliptic” curve above every cusp. In the case
of the cusp ∞, considered above, this would mean that one has to find out what
the natural extension of the elliptic family above \( H_n\setminus U \equiv \{ q \in \mathbb{C} \mid 0 < |q| < r' \} \)
to \( q = 0 \) is.

This family is essentially the “Tate elliptic curve” \( \mathbb{C}(q)^*/\{ q^n \mid a \in \mathbb{Z} \} \),
which has the affine equation \( y^2 + xy = x^3 + Bx + C \) where \( B, C \in q^n \mathbb{C}[q^n] \).
Using this equation one can see the Tate-curve as a closed subset of \( \mathbb{P}^2 \mathbb{C}[q] \).
Above the open subset \( q \neq 0 \) it is a honest elliptic curve. The fibre for \( q = 0 \) is
equal to a rational curve with an ordinary double point (i.e. given by the affine
equation \( y^2 + xy = x^3 \)). After deleting the double point (i.e. the point \( (0,0) \))
one finds a group structure and this group is the multiplicative group \( \mathbb{G}_m \) over
\( \mathbb{C} \). This is not quite the correct object to put above the cusp, since the group
of the \( n \)-torsion points \( \mathbb{G}_m[n] \) of \( \mathbb{G}_m \) is too small.

The remedy consists of replacing the Tate curve by its Néron model (or equivalently blowing up the double point a number of times). This Néron model
is obtained by dividing out a canonical model for the multiplicative group over
the field \( \mathbb{C}(q) \), with respect to its valuation ring \( \mathbb{C}[q] \), by the action of \( q^n \).
The fibre at \( q = 0 \) is the “generalized elliptic curve” \( \tilde{E} \) that fits into this family
of elliptic curves with level \( n \).

We will now describe \( \tilde{E} \). As an algebraic variety, \( \tilde{E} \) is an \( n \)-gon of projective
lines over \( \mathbb{C} \). After deleting the \( n \) ordinary double points of \( E \) one obtains a
group \( E^* \) which is isomorphic to the product \( \mathbb{G}_m \times \mathbb{Z}/n\mathbb{Z} \). The group of the
\( n \)-torsion points \( E^*[n] \) of \( E^* \) is isomorphic to \( n^{-1}\mathbb{Z}^2/\mathbb{Z}^2 \). A level \( n \)-structure
on the generalized elliptic curve is an isomorphism \( n^{-1}\mathbb{Z}^2/\mathbb{Z}^2 \to E^*[n] \). The
generalized elliptic curve has automorphisms. An automorphism will be an isomorphism
\( f : \tilde{E} \to \tilde{E} \) which respects the group law on \( E^* \). It is not difficult to see
that the group of automorphisms is generated by the two elements:
(1) \( i : a \mapsto a^{-1} \) on \( E^* \), extended in the obvious way to \( \tilde{E} \).
(2) \( \sigma : E^* \to E^* \), extended in the obvious way to \( \tilde{E} \), where \( \sigma(a, i) = (\zeta a^i, i) \)
with \( \zeta \) a primitive \( n \)th root of unity, \( a \in \mathbb{G}_m, i \in \mathbb{Z}/n\mathbb{Z} \).
Thus the group of automorphisms of the generalized elliptic curve has order \( 2n \).

Now one can count the number of cusps in two ways. The first method
counts the number of level \( n \) structures on the generalized elliptic curve. This
number is clearly \( 2^{2n} \mathbb{G}(n^{-1}\mathbb{Z}^2/\mathbb{Z}^2) \).
The other method is to count the number of cusps of \( Y(n) \) as the number of the
components of \( Y(n) \) (i.e. \( \phi(n) \)) multiplied by the number of cusps of \( \Gamma(n) \setminus \mathbb{H} \)
(i.e. the cardinal of \( \mathbb{G}(n) \setminus \mathbb{P}^1(\mathbb{Q}) \)). Those two numbers are equal as they should
be. We refer to [D-R] for more details. In the exposition above we have de-
scribed the situation over the field of complex numbers. In lecture 9 we will continue the description of the cusps of modular curves and Néron models of the Tate curve over a general base ring.

3.2 Describing the cusps of \( GL(Y, \mathbb{Z}) \backslash \Omega \)

A consequence of 2.5 is that \( GL(Y, \mathbb{Z}) \backslash \Omega \) is the analytification of a (smooth, connected) affine curve over \( K_{\infty} \). This algebraic curve has a natural compactification to a projective curve. The analytification of the latter is of course equal to the analytic compactification of \( GL(Y, \mathbb{Z}) \backslash \Omega \) as rigid analytic space. The new points are called the cusps of \( \overline{GL(Y, \mathbb{Z}) \backslash \Omega} \) (or of \( GL(Y, \mathbb{Z}) \)). Thus the existence of this analytic compactification follows from the algebraic situation. However we will describe the analytic compactification of \( GL(Y, \mathbb{Z}) \backslash \Omega \) in more detail. The analytic compactification is also useful for the construction of the local parameters at the cusps.

An element \( g \in PGl(2, K_{\infty}) \), \( g \neq 1 \), is called parabolic if \( g \) has only one fixed point in \( P^1(\mathbb{C}) \). Such a \( g \) can be represented by a matrix having the single eigenvalue 1. As a consequence, the order of a parabolic element is \( p \). Conversely, an element of order \( p \) is a parabolic element. Let \( \Gamma \) denote a discrete subgroup of \( PGl(2, K_{\infty}) \). A point of \( P^1(\mathbb{C}) \) will be called a parabolic point of \( \Gamma \) if it is the fixed point of infinitely many parabolic elements of \( \Gamma \). Such a point belongs to \( P^1(K_{\infty}) \). We will now investigate the parabolic points of \( PGl(Y) \).

**Lemma 3.1**

(1) Let \( \Gamma \) be a subgroup of finite index in \( PGl(Y) \). Then the set of parabolic points of \( \Gamma \) is equal to \( P^1(K) \).

(2) There is a natural bijection \( PGl(Y) \backslash P^1(K) \to Pic(A) \).

**Proof.** (1) Two groups \( \Gamma \) and \( \Gamma' \) have the same set of parabolic points if \( \Gamma \cap \Gamma' \) is a subgroup of finite index in both \( \Gamma \) and \( \Gamma' \). It suffices therefore to consider \( Y = A^2 \) and the group \( \Gamma := PGl(A^2) \). Any parabolic point lies in \( P^1(K) \) since it is the fixed point of a parabolic element in \( PGl(2, K) \). It is further clear that 0 and \( \infty \) are parabolic points. Let \( \lambda \in K, \lambda \neq 0 \). Choose any \( c \notin A \), \( c \neq 0 \) such that \( c\lambda \in A \) and \( c\lambda^2 \in A \). Then \( \lambda \) is the fixed point of the parabolic element

\[
\begin{pmatrix}
c\lambda - 1 & -c\lambda^2 \\
c & c\lambda - 1
\end{pmatrix}
\]

This shows that \( \lambda \) is a parabolic point.

(2) \( Y \) is considered as a projective rank two \( A \)-submodule of \( K^2 \). Every point of \( P^1(K) \) is represented by a line \( L \subset K^2 \). We associate to \( L \) the class in \( Pic(A) \) of the rank one projective module \( L \cap Y \). For every \( g \in PGl(Y) \), one has \( g(L \cap Y) = (L \cap Y) \). Hence \( L \) and \( gL \) have the same image in \( Pic(A) \).
Let the two lines $L_1, L_2$ have the same image in $Pic(A)$. The projective submodules $L_1 \cap Y$ are direct factors of $Y$. Thus we can write $Y = (L_1 \cap Y) \oplus M_1 = (L_2 \cap Y) \oplus M_2$. Since $L_1 \cap Y \cong L_2 \cap Y$, also $M_1 \cong M_2$. This leads to an automorphism $g$ of $Y$ with $g(L_1 \cap Y) = (L_2 \cap Y)$ and $gM_1 = M_2$. Thus we found an injective map $PGl(Y) \backslash \mathbb{P}^1(K) \to Pic(A)$.

Finally, let $M$ be a rank one projective module representing an element of $Pic(A)$. Then $M \oplus (M^{-1} \otimes N) \cong Y = A \oplus N$, since the determinants of the two modules are isomorphic. Let $B \subset Y$ be the image of $M$ under an isomorphism and put $L := K \otimes B \subset K \otimes Y = K^2$. Then $L \in \mathbb{P}^1(K)$ is mapped to the class of $M$ in $Pic(A)$. \hfill \Box

**Proposition 3.2** Let $\Gamma$ be a subgroup of finite index in $PGl(Y)$. There is a bijective map from $\Gamma \backslash \mathbb{P}^1(K)$ to the set of cusps of $\Gamma$.

**Proof.** We will sketch the proof. A full proof can be obtained by using [S]. Every point of $\mathbb{P}^1(K)$ is represented by a line $L = Ke \subset K^2$. This line induces an end, called $end(L)$ in the tree $\mathcal{T}$, which can be represented by the sequence of modules $Oe + \pi Y \supset Oe + \pi^2 Y \supset Oe + \pi^3 Y \supset \ldots$ (see the lectures 6 and 7).

Let the subgroup $H \subset \Gamma$ consist of the matrices $h$ in $\Gamma \subset PGl(Y)$ with $h(e) = e$ and $det(h) = 1$. As we have seen this is an infinite group consisting of parabolic elements (indeed, $L$ is a parabolic point). It is not difficult to show that $end(L)$ has the property:

For large $n, m$ the modules $Oe + \pi^n Y$ and $Oe + \pi^m Y$ are $\Gamma$-equivalent only if $n = m$.

Thus the image of $end(L)$ in $\Gamma \backslash \mathcal{T}$ is also an end of $\Gamma \backslash \mathcal{T}$. This provides an injective map from $\Gamma \backslash \mathbb{P}^1(K)$ to the ends of $\Gamma \backslash \mathcal{T}$.

It is more difficult to see that this map is surjective. The structure of $PGl(Y) \backslash \mathcal{T}$ is completely given in [S]. In particular, he shows that the ends of $PGl(Y) \backslash \mathcal{T}$ are in one-to-one correspondence with the elements of $Pic(A)$. This implies the statement for the group $PGl(Y)$.

Let $\Gamma \subset PGl(Y)$ be a normal subgroup of finite index. The factor group $PGl(Y) / \Gamma$ acts on the graph $\Gamma \backslash \mathcal{T}$. The obvious map $pr : \Gamma \backslash \mathcal{T} \to Gl(Y) \backslash \mathcal{T}$, is just the quotient map under the action of $PGl(Y) / \Gamma$. The map from $\mathbb{P}^1(K)$ to the ends of $\Gamma \backslash \mathcal{T}$ is surjective, since the map

$$\mathbb{P}^1(K) \to \text{the ends of } \Gamma \backslash \mathcal{T} \to \text{the ends of } PGl(Y) \backslash \mathcal{T}$$

is surjective.

For any subgroup $\Gamma \subset PGl(Y)$ of finite index, there is a normal subgroup $\Gamma' \subset PGl(Y)$ of finite index which is contained in $\Gamma$. Then

$$\mathbb{P}^1(K) \to \text{the ends of } \Gamma' \backslash \mathcal{T} \to \text{the ends of } \Gamma \backslash \mathcal{T}$$

is surjective.
Remark
In the case of the standard example $A = F_q[T]$, we have already shown that $PGl(2,A) \setminus \Omega$ has one cusp. The more complicated results of [S] are not needed in this case.

3.3 The analytic compactification of the curve $\Gamma \setminus \Omega$

$\Gamma$ is supposed to be a subgroup of finite index in $PGl(Y)$. The aim is to compactify the rigid analytic space $\Gamma \setminus \Omega$ by “adding” a point for each end of the graph $\Gamma \setminus \mathcal{T}$. Thus we have to “add a point” for every element of $\Gamma \setminus P^1(K)$. The idea is the following.

An element $c$ of $\Gamma \setminus P^1(K)$ is represented by a parabolic point $Ke \in P^1(K)$. One considers the subgroup $H$ of $\Gamma$ consisting of the matrices $h$ in $\Gamma \subset PGl(Y)$ with $h(Ke) = Ke$. One constructs a suitable subset $U$ of $\Omega$ (called a “horicycle neighbourhood of the cusp”) with the properties:

$U$ is (setwise) invariant under $H$. Let $u_1, u_2 \in U$ and let $g \in \Gamma$ satisfy $g(u_1) = u_2$, then $g \in H$.

In other words, the image of $U$ in $\Gamma \setminus \Omega$ is isomorphic to $H \setminus U$. It is shown that there is an isomorphism $f_c : U_c := H \setminus U \to \{ z \in \mathbb{C} | 0 < |z| \leq 1 \}$. The disk \{ $z \in \mathbb{C} | |z| \leq 1$ \} is glued to $\Gamma \setminus \Omega$ using $f_c$. This is done for every $c \in \Gamma \setminus P^1(K)$.

The result is an analytic space $\overline{\Gamma \setminus \Omega}$ which is proper and smooth over $K_{\infty}$. The rigid analytic version of GAGA states that $\overline{\Gamma \setminus \Omega}$ is the analytification of a smooth projective connected curve over $K_{\infty}$. The local parameter of this curve at the cusp $c$ is the function $f_c$.

We will now give some more information about this horicycle neighbourhood. For notational convenience, we suppose that $Ke$ is the point at $\infty$. The group $H$ consists of certain (classes of) matrices \( \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \). The subgroup $H_0$ of $H$ of the matrices with $a = d = 1$ is a normal subgroup and $H/H_0$ is a finite cyclic group. The group $H_0$ can be identified with an infinite discrete subgroup of $K \subset K_{\infty}$. It has the property that $H_0 \setminus K_{\infty}$ is a compact set. For the construction of the admissible subset $U \subset \Omega$ we will use again the real valued function $d(\cdot)$ on $\Omega$ given by $d(z) = \min \{ |z + \lambda| | \lambda \in K_{\infty} \}$. (See lecture 7)

Lemma 3.3 There exists an element $r \in |K_{\infty}^*|$ such that the “horicycle neighbourhood of $\infty$”, i.e. the admissible subset $U := \{ z \in \Omega | d(z) \geq r \}$ of $\Omega$, has the
properties: 
$U$ is (setwise) invariant under $H$. 
Let $u_{1}, u_{2} \in U$ and $g \in \Gamma$ be such that $g(u_{1}) = u_{2}$, then $g \in H$.

Proof. This is a combinatorial exercise. See [G-vdP].

The quotient space $H_{0}\backslash U$ can be identified with a subset of the quotient space $H_{0}\backslash \mathbb{C}$. The latter is isomorphic with $\mathbb{C}$. Indeed, the function

$$e_{H_{0}}(z) = z \prod_{h \in H_{0}, \ h \neq 0} \left(1 - \frac{z}{h}\right)$$

is known to be a surjective homomorphism of $\mathbb{C} \to \mathbb{C}$ with kernel $H_{0}$ (See lecture 3). Thus $e_{H_{0}}$ provides an analytic isomorphism of $H_{0}\backslash \mathbb{C}$ with $\mathbb{C}$. Using that $H_{0}\backslash K_{\infty}$ is compact one can show that the image of $U$ under $e_{H_{0}}$ is equal to \{\(z \in \mathbb{C} | r' \leq |z|\} for some $r' \in |K_{\infty}|$. This proves that $H_{0}\backslash U$ is isomorphic to \{\(z \in \mathbb{C} | 0 < |z| \leq 1\}. The action of the cyclic group $H/H_{0}$ on this space is multiplication of $z$ by roots of unity. We conclude that also $H\backslash U$ is isomorphic to \{\(z \in \mathbb{C} | 0 < |z| \leq 1\).

The function $e_{H_{0}}^{-m}$, where $m$ is the order of $H/H_{0}$, is defined and analytic on $\overline{\Gamma\backslash \Omega}$ in some neighbourhood of the cusp $\infty$. It has a zero of order one at the cusp $\infty$ and can be taken as local parameter at the cusp $\infty$.

This finishes the construction of the analytic compactification of $\Gamma\backslash \Omega$.

4 The standard example

We consider here $A = \mathbb{F} [T]$ and the level is given by $f \in A$, a monic polynomial of degree $n$.

4.1 The number of cusps

The calculation of the number of cusps for the group $\Gamma(f) := Gl(A^{2}, (f)) \subset Gl(2, A)$ is somewhat delicate. First of all, in the case $f = 1$ there is only one cusp. Indeed, $PGl(2, A)$ and also $SL(2, A)$ acts transitively on $\mathbb{P}^{1}(K)$.

Let $f$ of degree $\geq 1$ have the decomposition $f = f_{1}^{d_{1}} \ldots f_{s}^{d_{s}}$, with $f_{1}, \ldots, f_{s}$ distinct monic irreducible polynomials of degrees $n_{i}$. The degree of $f$ is $n = \sum d_{i} m_{i}$. 

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Note that $\mathbf{P}^1(K)$ is the orbit of $\infty$ under $\text{SL}(2,A)$. Hence we can identify $\mathbf{P}^1(K)$ with $\text{SL}(2,A)/H$, where $H$ is the group of the matrices of the form
\[
\begin{pmatrix}
a & b \\
0 & a^{-1}
\end{pmatrix},
\]
with $b \in A$ and $a \in A^\times$. The set of cusps of the congruence subgroup $\Gamma(f)$ is then equal to the family of double cosets \( \{ g \in \text{SL}(2,A) \mid g \equiv 1 \mod f \} \backslash \text{SL}(2,A) \backslash H \).

The map $\text{SL}(2,A) \to \text{SL}(2,A/\mathcal{A})$ is surjective. Hence our set of cusps is equal to $\text{SL}(2,A/\mathcal{A}) / H_1$, where $H_1$ is the image of $H$ in $\text{SL}(2,A/\mathcal{A})$. The group $H_1$ is equal to the group of matrices
\[
\left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \mid a \in \mathbf{F}_q^*, b \in A/\mathcal{A} \right\}.
\]

It is known that $\text{SL}(2,A/\mathcal{A})$ is isomorphic to the product $\prod_{i=1}^s \text{SL}(2,A/\mathcal{A}_i)$. The number of elements of each factor is equal to $q^{2n_i}d_i(1 - q^{-2n_i})$. Counting leads to the following formula for the number of cusps of $\Gamma(f)$:
\[
\frac{q^{2n} - 1}{q - 1} \prod_{i=1}^s (1 - q^{-2n_i}).
\]

The complete Drinfeld modular curve $G\text{M}((\Omega) \times G\text{M}(f^{-1}A^2/A^2))$ has $\#(A/fA)^*/\mathbf{F}_q^*$ connected components, each one isomorphic to $\Gamma(f) \backslash \Omega$. And thus the number of cusps of this curve is
\[
\frac{\prod_{i=1}^s (q^{n_i} - 1)q^{n_i(d_i - 1)}}{q - 1} \cdot \frac{q^{2n}}{q - 1} \prod_{i=1}^s (1 - q^{-2n_i}).
\]

### 4.2 Elliptic points and a formula for the genus

The covering
\[
X(f) := G\text{M}(\mathcal{A}^2,f) \backslash \Omega \to X(1) := \text{PGF}(2,A) \backslash \Omega \cong \mathbf{P}^1_{\mathbf{Q}}
\]
can be used to find the formula for the genus of the curve $G\text{M}(\mathcal{A}^2,f) \backslash \Omega$. The degree $d$ of the morphism is equal to the order of the group $G\text{M}(\mathcal{A}^2,f) / \text{PGF}(\mathcal{A}^2)$, which is also equal to the order of the group $\text{SL}(2,A/\mathcal{A}(f))$. This gives a formula for the degree
\[
d = q^{3n} \prod_{i=1}^s (1 - q^{-2n_i}).
\]

Above the point $\infty$ of $X(1)$ there are the $\frac{d}{e}$ cusps, where $e$ denotes the ramification of any cusp of $X(f)$ above $\infty \in X(1)$. The stabilizer of the cusp $\infty$ of $X(f)$ has the form
\[
\left( \begin{pmatrix} 1 & b \\ 0 & d \end{pmatrix} \right) \mid d \in \mathbf{F}_q^*, b \in A/\mathcal{A}(f) \}
\]
This implies that \( e = (q - 1)q^n \) and that the number of cusps of \( X(f) \) is equal to \( \frac{d}{q^n} \). We have already found this number.

The contribution \( B \) of the cusp \( \infty \) of \( X(f) \) for the Riemann-Hurwitz-Zeuthen formula can be found by considering the morphism \( P_{K_{\infty}}^1 \to P_{K_{\infty}}^1 \) obtained by dividing the first projective line by a group of the form

\[
\begin{pmatrix}
1 & b \\
0 & d
\end{pmatrix} \in F_q^* \quad b \in W,
\]

where \( W \subset K_{\infty} \) is a linear subspace over \( F_q \) of dimension \( n \). In this situation, \( \infty \) is the only point above \( \infty \) and gives a contribution \( B \) for the Riemann-Hurwitz-Zeuthen formula. The points of \( W \) are ramified points of order \( q - 1 \). There is no further ramification. This leads to \( -2 = -2(q - 1)q^n + q^n(q - 2) + B \) and thus \( B = q^{n+1} - 2 \).

The fixed points in \( \Omega \) for the group \( PGL(2, A) \) are called the “elliptic points” of \( \Omega \) (we deviate here somewhat from the classical use of the expression “elliptic point”). Take a point \( b \in F_q \setminus F_q^* \). This is an elliptic point. Its stabilizer is a cyclic group of order \( q + 1 \). Any other elliptic point of \( \Omega \) is \( PGL(2, A) \)-equivalent to \( b \). Let \( \xi \) denote the image of \( b \) in the curve \( X(1) \). Then the points of \( X(f) \) which lie above \( \xi \in X(1) \) are ramified with order \( q + 1 \). Their number is clearly \( \frac{q}{q+1} \) and their contribution in the Riemann-Hurwitz-Zeuthen formula is \( \frac{q}{q+1} \).

Let \( g \) denote the genus of \( X(f) \). According to the Riemann-Hurwitz-Zeuthen formula one has \( 2g - 2 = -2d + \frac{dq}{q+1} + \frac{d(q^{n+1} - 2)}{q^n(q - 1)} \). This gives the formula

\[
g = 1 + \frac{q^{2n}(q^n - q - 1)}{q^2 - 1} \prod_{i=1}^{n} (1 - q^{-2m}).
\]

### 4.3 The Drinfeld module with level \( f \) at a cusp

The universal Drinfeld module with level \( f \)-structure above the space \( GL(A)^2 \setminus (\Omega \times GL(f^{-1}A^2/A^2)) \), constructed in theorem 2.2, extends in a certain way to the cusps. In this section we study this extension. The cusps are equivalent under the automorphisms of this curve. Hence it suffices to consider the cusp at \( \infty \) of the curve \( X(f) \), which is the compactification of \( GL(A^2, (f)) \setminus \Omega \).

Our first concern is to find out in what sense the Drinfeld structure \( \phi \) extends at \( \infty \). Let \textbf{cusps} denote the set of the cusps of the curve \( X(f) \). The construction of theorem 2.2 produces a line bundle, which we will denote by \( L \), on \( X(f) \setminus \textbf{cusps} \) and a homomorphism \( \phi : A \to End(L) \). We want to extend
this line bundle $L$ and the homomorphism $\phi$ to all of $X(f)$. It suffices to describe this extension at the cusp $\infty$.

Let us first consider the case $f=1$.

As we know, the cusp has the following description. Consider the horocycle neighbourhood at $\infty$, i.e. $U = \{\omega | d(\omega) \geq R \} \subset \Omega$ (for some $R > 1$), and put

$$H := \{ \omega \mapsto \alpha \omega + \beta | \alpha \in F^*, \beta \in A \}.$$

The space $H\backslash U \subset X(1)$ can be identified with $\{ \omega \in C | 0 < |c| \leq r \}$ for some $r > 0$ be means of the local parameter $t$ at $\infty \in X(1)$. The local parameter $t$ has the form

$$t(\omega) = \prod_{a \in A, a \neq 0} (1 - \frac{\omega}{a})^{-\rho}.$$

The action of $H$ on the trivial line bundle $U \times C$ is given as $h(\omega, z) = (h(\omega), z)$. Hence the restriction of $L$ to $H\backslash U$ coincides with the structure sheaf. The extension of $\mathcal{L}$ to $\{ \omega \in C | |c| \leq r \}$ is chosen to be again the structure sheaf.

In order to find the extension of $\phi$ above $\{ \omega \in C | |c| \leq r \}$ we consider again the function

$$e(\omega, z) = z \prod_{(a, b) \neq (0, 0)} (1 - \frac{z}{\omega a + b}) = z (1 - \sum_{m \geq 1} s_m z^{m(\rho-1)})^{-1},$$

where $s_m = \sum_{(a, b) \neq (0, 0)} 1 \prod_{(\omega a + b)^{m+1}}$. This function and the $s_m$ are invariant under the group $H$. Hence $s_m$ is a convergent Laurent series in $t$. In fact $s_m$ is holomorphic at $t = 0$ and $s_m(t = 0) \neq 0$. Indeed, for $(a, b) \in \mathbb{A}^2, (a, b) \neq (0, 0)$, one has $|\omega a + b| \geq |c|$ if $\alpha \neq 0$ and $|\omega a + b| \geq |c|$ if $\alpha = 0$. Hence, each term $\frac{1}{(\omega a + b)^{m+1}}$ of $s_m$ has absolute value $\leq 1$. As a consequence $s_m$ is bounded on $U$ and has as function of $t$ no pole at $t = 0$. The limit of $s_m(\omega)$ for $\omega \to \infty, \omega \in U$ is the expression $\sum_{0 \in A, 0 \neq 0} \frac{1}{(\omega a + b)^{m+1}}$. This is the value of $s_m$ at $t = 0$. One easily sees that $|s_m(t = 0) + 1| < 1$.

Also the function $e(\omega, z)$ can therefore be seen as a holomorphic function of the two variables $t, z$. The value of $e$ for $t = 0$ is equal to $z \prod_{0 \in A, 0 \neq 0} (1 - \frac{z}{\omega a + b})$. This is the “Carlitz exponential function” of the rank one lattice $A \subset C$.

The Drinfeld module of rank two above $H\backslash U$ is determined by $\phi_T = T + c_1 T + c_2 T^2$ with $c_1 = (T^2 - T)s_1$ and some $c_2$. We know that the $j$-function $j = \frac{c_1}{c_2}$ has a pole of order 1 at $t = 0$. Thus $c_2$ has a zero of order 1 at $t = 0$. Further $c_1$
is not zero at \( t = 0 \). We conclude that \( \phi : \mathcal{A} \to End(L) \) has a natural extension at the cusp \( \infty \) and that \( \phi(t = 0) \) is the rank one Drinfeld module corresponding to the lattice \( \mathcal{A} \subseteq \mathbf{C} \).

The case where the level is given by a monic polynomial \( f \in \mathcal{A} \) of degree \( n \geq 1 \).

The group \( H_f \) is defined as \( \{ \omega \mapsto \omega + b \mid b \in \mathcal{A} f \} \). The space \( H_f \backslash \mathcal{U} \) is also isomorphic to \( \{ c \in \mathbf{C} \mid 0 < |c| \leq r \} \) (for some \( r > 0 \)). This isomorphism is given by the local parameter \( t_f \) of \( X(f) \) at the cusp \( \infty \). This local parameter is given by the formula

\[
t_f = (\omega \prod_{b \in (f), b \neq 0} (1 - \frac{\omega}{b}))^{-1}.
\]

Again \( c_1, c_2 \) are holomorphic functions of \( t_f \). \( c_1 \) is not zero for \( t_f = 0 \). Further \( c_2 \) has a zero of order \((q - 1)q^n\) at \( t_f = 0 \) since the ramification index of \( X(f) \to X(1) \) at the point \( \infty \in X(f) \) is equal to \((q - 1)q^n\). We note that this is also the index of \( H_f \) in \( H \). We conclude that the Drinfeld module of rank two above \( X(f) \backslash \{ \text{cusps} \} \) has an extension to all of \( X(f) \) as a “stable Drinfeld module”. The last expression means that at any cusp the rank of the Drinfeld module is equal to 1.

We now investigate the level \( f \)-structure. Above \( \mathcal{U} \) the kernel of \( \phi_f \) is isomorphic to the constant sheaf \( f^{-1} \mathcal{A}^2/\mathcal{A}^2 \). Its generators as \( \mathcal{A}/(f) \)-module are the functions \( e(\omega, \frac{1}{f}) \) and \( e(\omega, \frac{\omega}{f}) \). Both functions are invariant under \( H_f \). For the first function this is clear. For the second one, this follows from:

\[
e(\omega + bf, \omega + \frac{bf}{f}) = e(\omega, \frac{\omega + bf}{f}) = e(\omega, \frac{\omega}{f}) + e(\omega, b) = e(\omega, \frac{\omega}{f}).
\]

Hence both functions are defined on \( H_f \backslash \mathcal{U} \) and are Laurent series in \( t_f \). The first function \( e(\omega, \frac{1}{f}) \) can be written in the form

\[
\frac{1}{f}(1 - \sum_{m \geq 1} s_m f^{m(1-\theta)})^{-1}.
\]

This is clearly holomorphic and nonzero at \( t_f = 0 \). This function generates a constant subheaf (isomorphic to \( f^{-1} \mathcal{A}/\mathcal{A} \) and moreover a direct summand) of the constant sheaf \( \ker \phi_f \) above \( H_f \backslash \mathcal{U} \). This subheaf extends as a constant subheaf (isomorphic to \( f^{-1} \mathcal{A}/\mathcal{A} \)) of the structure sheaf on \( \{ c \in \mathbf{C} \mid |c| \leq r \} \), (i.e. it extends at \( t_f = 0 \)).
The second generator of the level structure $e(\omega, \frac{\omega}{f})$ can be written as

$$\frac{\omega}{f} \prod (1 - \frac{\omega}{a\omega + b}).$$

On the subset of $U$ given by $d(\omega) = |\omega| \geq R$, the absolute value of the expression above can be given as

$$\frac{|\omega|}{|f|} \prod_{b \in A_1} \frac{|\omega|}{|b_f|}.$$

For $\omega$ as above, the function $\omega f(\omega)^{-1} = \frac{\omega}{f} \prod b \in A_1, b \neq 0 (1 - \frac{\omega}{b_f})$ has the same absolute value. It follows that $e(\omega, \frac{\omega}{f})$ has a pole of order 1 at $t_f = 0$.

We investigate now the behaviour of the more general expression $e(\omega, \frac{\omega + \varepsilon}{f})$, with $\varepsilon, d$ polynomials of degree $< n$. (Again $n$ denotes the degree of $f$.)

This expression is a $F_q$-linear combination of the terms $v_i := e(\omega, \frac{T^i}{f})$ and $w_j := e(\omega, \frac{T^j}{f})$ for $0 \leq i, j < n$. We have shown that the order of $v_0$ at the cusp is $-1$. Further we know that $\phi_T = T + c_1 \tau + c_2 \tau^2$ with $c_1$ invertible at the cusp and the order of $c_2$ at the cusp is $q^n(q - 1)$. It follows from this that the order of $v_i$ at the cusp is $-q^{i-1}$ and that the $w_j$ have no poles at the cusp. We conclude that the order of $e(\omega, \frac{\omega + \varepsilon}{f})$ at the cusp is $\geq 0$ if and only if $c = 0$ and that for $c \neq 0$ of degree $i$ this order is $-q^{i-1}$.

What we have found is a rank two Drinfeld module over $K_\infty[[t_f]]$ given by $\phi_T = T + c_1 \tau + c_2 \tau^2$ with the following properties:

1. $c_1, c_2 \in K_\infty[[t_f]]$ have orders 0 and $q^n(q - 1)$
2. $\ker(\phi_f, K_\infty((t_f)))$ is isomorphic to the $A$-module $(f^{-1}A/A)^2$.
3. $\ker(\phi_f, K_\infty[[t_f]])$ is isomorphic to the $A$-module $f^{-1}A/A$.

We will call this structure a Tate-Drinfeld module (rank two and level $f$) over the field $K_\infty$. In lecture 9, a “universal” Tate-Drinfeld module (rank two and level $f$) will be constructed.

### 4.4 Geometric interpretation at a cusp

In the following the valuation of the field $K_\infty$ will play no role. The field $L := K_\infty((t_f))$ is considered as a discrete valued field. Its valuation ring is $R := K_\infty[[t_f]]$.
The Tate-Drinfeld module above will be given a geometric interpretation. We will describe this by using the rigid analytic terminology. The algebraic counterpart will be explained in lecture 9.

The affine line over $L$ is replaced by the projective line $\mathbb{P}^1_L$. The ring $A$ acts, via $\phi$, on the affine part of this projective line. We are interested in the set $F := \ker(\phi_f, L) \cup \{\infty\} \subset \mathbb{P}^1_L$. The projective line $\mathbb{P}^1_L$ is considered as a rigid analytic space over the discrete valued field $L$. The obvious reduction (see lecture 6) of this space is the projective line over the residue field, i.e. $K_\infty$, of $L$. On this reduction $\phi$ has rank one and many elements of $F$ are mapped to the point $\infty$ of $\mathbb{P}^1_{K_\infty}$. We would like to find another reduction of the analytic space $\mathbb{P}^1_L$ such that the images of the points of $F$ are distinct. We will briefly explain the concept of reductions.

4.4.1 Reductions

The canonical reduction of an affinoid space $X = \text{max}(B)$, where $B$ is an affinoid algebra over a complete valued field $L$ is the affinoid space $\bar{X} = \text{max}(\bar{B})$ over the residue field $l$ of $L$. We recall that $\bar{B} = B^n/B^n$ with $B^n = \{b \in B| ||b||_p < 1\}$ and $B^{\infty} = \{b \in B| ||b||_p = 1\}$. Let $c_X : X \to \bar{X}$ denote the (canonical) reduction map.

For a general rigid space $X$ we call a covering $\mathcal{U} = \{U_i\}_{i \in I}$ pure (or formal) if the following holds:

1. Each $U_i$ is affinoid and the covering $\mathcal{U}$ is admissible.

2. For each pair $U_i, U_j$ such that $U_i \cap U_j \neq \emptyset$ there is a Zariski open subset $V_{i,j} \subset \bar{U}_i$ such that $U_i \cap U_j$ is the preimage of $V_{i,j}$ under the canonical reduction map $c_{U_i} : U_i \to \bar{U}_i$.

The reduction of $X$ with respect to this pure covering $\mathcal{U}$ is denoted by $(X, \mathcal{U})$. It is defined as the gluing of the affinoid space $\bar{U}_i$ over the open subsets $V_{i,j}$. There is a reduction map $r_{\mathcal{U}} : X \to \text{(X, U)}$, which is obtained by gluing the reduction maps $c_{U_i}$.

The "obvious" reduction of $\mathbb{P}^1_L$ is given by

$$\mathcal{U} = \{\{z \in \mathbb{P}^1_L| |z| \leq 1\}, \{z \in \mathbb{P}^1_L| |z| \geq 1\}\}.$$ 

The space $(\mathbb{P}^1_L, \mathcal{U})$ is equal to the projective line over the residue field $l$. The reduction map $r_{\mathcal{U}} : \mathbb{P}^1_L \to \mathbb{P}^1_l$ is given by $r_{\mathcal{U}} : (x_0 : x_1) \mapsto (x_0 : x_1)$, where the $x_0, x_1$ are chosen such that $\max(|x_0|, |x_1|) = 1$. 

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For certain pure finite affinoid coverings \( \mathcal{U} \) of \( \mathbf{P}^1_L \) one has a corresponding analytic reduction \( (\mathbf{P}^1_L, \mathcal{U}) \) which is a tree of projective lines over the residue field of \( L \). By induction on the cardinality of \( F \) one can show the following statement:

**Lemma 4.1** For any finite set \( F \) (with cardinality \( \geq 3 \)) of points in \( \mathbf{P}^1(L) \) there is a unique reduction map \( r_{\mathcal{U}} \) (the covering \( \mathcal{U} \) is not quite unique) such that:

1. The space \( (\mathbf{P}^1_L, \mathcal{U}) \) is a tree of lines over the residue field of \( L \).
2. \( r_{\mathcal{U}} \) is injective on \( F \) and \( r_{\mathcal{U}}(F) \) contains no double point.
3. On every irreducible component of \( (\mathbf{P}^1_L, \mathcal{U}) \) there are at least three special points. Here special point means either a double point or an element in \( r_{\mathcal{U}}(F) \).

In the terminology of \([\text{G-H-vdP}]\), \( (\mathbf{P}^1_L, \mathcal{U}) \) with the set of points \( r_{\mathcal{U}}(F) \) is a stable \( \#F \)-pointed tree.

The algebraic version of this lemma reads as follows:

There is a model \( M \) over the valuation ring \( R \) of \( L \) such that

1. \( M \) is projective and flat over \( R \).
2. \( M \otimes L \cong \mathbf{P}^1_L \).
3. The special fibre \( M_s = M \otimes l \) is a tree of lines over the residue field \( l \) of \( L \).
4. The map \( r : \mathbf{P}^1_L(L) = M(R) \to M_s(l) \) is the reduction map.
5. \( r \) is injective on \( F \) and \( r(F) \) contains no double point of \( M_s \).
6. On every irreducible component of \( M_s \) there are at least three special points (i.e., double points or points of \( r(F) \)).

The lemma is applied to the field \( F_q(x) \) with as valuation ring the local ring at \( x = 0 \), i.e., the local ring \( F_q[x]_{(0)} \). For the set \( F \) we choose

\[
\{ F_q + F_qx^{-1} + F_qx^{-q} + \ldots + F_qx^{-q^{-1}} \} \cup \{ \infty \}.
\]

Let \( RT = RT(q, n) \) denote the reduction given by the lemma. We will describe \( RT \) in more detail.

Each projective line in \( RT \) is identified with \( \mathbf{P}^1_{F_q} \) in a suitable way. The intersection graph \( \mathcal{G} \) of \( RT \) is by definition the graph with vertices the irreducible components of \( RT \) and with edges the double points of \( RT \).

There is one vertex of \( \mathcal{G} \), the “root” \( R \), which has \( q \) edges. This root is the irreducible component on which the image of \( \infty \) lies. The image of \( \infty \in S \)
is the infinite point of $R$. The $q$ edges correspond to the intersection of the line $R$ with the other irreducible components of $RT$. The points of intersection on $R$ are the points $F_q \subseteq R$. A neighbour of $R$ can be labelled as $V_{a_1}$, with $a_1 \in F_q$, according to the point of intersection $R$. The point of intersection of $V_{a_1}$ with $R$ is, as point on $V_{a_1}$, the infinite point. Each $V_{a_1}$ has $q$ new neighbours corresponding to the new double points which form the set $F_q \subseteq V_{a_1}$. The new neighbours of $V_{a_1}$ can be labelled by $V_{a_1,a_2}$ with $a_2 \in F_q$, according to their point of intersection on $V_{a_1}$. This point, seen as a point of $V_{a_1,a_2}$, is the infinite point. And so on. We stop at the $n$th level. The following picture illustrates the situation for $q = 2$ and $n = 2$.

Now we introduce the following definition. Let $k$ be any field containing $F_q$. The $q$, $n$-rooted tree over $k$ is the $k$-scheme $RT(q,n) \otimes_{F_q} k$.

4.4.2 Structure on the reduction $B$

We apply lemma 4.1 to the subset $F = \ker(\phi_f, L) \cup \{\infty\}$ of $P^1_k(L)$. Let us write $r : P^1_k \to B$ for the corresponding reduction. The set $F$ is rather special, apart from $\infty$ it consists of

$$\ker(\phi_f, R) + F_qv + F_q\phi_T(v) + \ldots + F_q(\phi_T)^{n-1}v,$$

with $v = e(\omega, \frac{e}{f})$. The order of $(\phi_T^i)^{\ast}v$ is equal to $-q^i$ for $i = 0, \ldots, n - 1$. It is a combinatorial exercise to show that the data of $F$ imply that $B$ is isomorphic to $RT(q,n) \otimes K_{\infty}$, the $q$, $n$-rooted tree over $K_{\infty}$.

The point $\infty$ is mapped (under $r$) to (the projective line corresponding with) the root $R$. The other points of $F$ are mapped to the end vertices $V_{a_1}, \ldots, a_n$. One can choose the labelling of the vertices of $\mathcal{G}$ such that the points

$$\ker(\phi_f, R) + a_nv + a_{n-1}\phi_T(v) + \ldots + a_1(\phi_T)^{n-1}v$$

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are mapped to the (projective line corresponding to the) vertex \( V_{a_1, \ldots, a_n} \).

The reduction \( B \) has much more structure. Let \( B^* \) denote the open subset of \( B \) obtained by deleting all the lines on which no points of \( \ker(\phi_f, L) \) lie. \( B^* \) is the disjoint union of affine lines over \( K_\infty \) (corresponding to the vertices \( V_{a_1, \ldots, a_n} \)). The line which contains the images of \( \ker(\phi_f, R) \) is denoted by \( B^{**} \) (corresponding to the vertex \( V_0, \ldots, 0 \)).

The preimage \( r^{-1}(B^*) \) is the affinoid subset
\[
\{ z \in \mathbb{P}_L^1 \mid |z| \leq 1 \} + \mathcal{F}_g v + \mathcal{F}_g \phi_T(v) + \ldots + \mathcal{F}_g (\phi_T)^{n-1} v
\]
of \( \mathbb{P}_L^1 \). Further \( r^{-1}(B^{**}) \) is equal to \( \{ z \in \mathbb{P}_L \mid |z| \leq 1 \} \). In fact \( r^{-1}(B^*) \) and \( r^{-1}(B^{**}) \) are affinoid subgroups of \( \mathbb{A}_L^1 \). Hence \( B^* \) and \( B^{**} \) are algebraic groups over the residue field \( K_\infty \). The second group is the additive group over \( K_\infty \) and has a Drinfeld structure of rank one induced by \( \phi \). The algebraic group \( B^* \) is isomorphic to \( B^{**} \times f^{-1}A/A \) and has therefore an \( A \)-action induced by the Drinfeld structure on \( B^{**} \) and the obvious action on \( f^{-1}A/A \).

We note that the \( A \)-module of the \( f \)-torsion points of \( B^* \) is equal to the image of \( \ker(\phi_f, L) \). We note further that the group law \( B^* \times B^* \to B^* \) extends in a unique way to an algebraic group action \( B^* \times B \to B \). Indeed, the group law \( r^{-1}(B^*) \times r^{-1}(B^*) \to r^{-1}(B^*) \) extends to a group action \( r^{-1}(B^*) \times \mathbb{P}_L^1 \to \mathbb{P}_L^1 \), which respects the pure covering and induces the required algebraic group action on the reduction.

### 4.4.3 Generalized Drinfeld modules over a field

The calculations above concerning the reduction \( B \) lead to a definition. Let \( k \) be any field which has an \( A \)-algebra structure. Fix a level \( (f) \) of degree \( n \). A generalized Drinfeld module (of rank two and with level \( (f) \)) is a special \( q \), \( n \)-rooted tree \( N \) over \( k \) with the additional structure:

1. Let \( N^* \) be the open subset obtained by deleting all the lines of \( N \) which are not corresponding to end vertices. Then \( N^* \) is given a structure of an algebraic group over \( k \) for which it is isomorphic to the commutative algebraic group \( \mathbb{G}_{a,k} \times f^{-1}A/A \).

2. The action \( N^* \times N^* \to N^* \) extends to an algebraic action \( N^* \times N \to N \).

3. The component \( N^{**} \) of the identity of \( N^* \) has the structure of a rank one Drinfeld module \( \psi : A \to \text{End}_{\mathcal{O}_q}(N^{**}) \). The kernel of \( \psi_f \) acting upon \( N^{**} \) is isomorphic to \( f^{-1}A/A \).
Let $N_1, N_2$ be two generalized Drinfeld modules (of rank two and with level $(f)$). An isomorphism $h : N_1 \to N_2$ is an isomorphism of schemes over $k$ such that $h : N_1^* \to N_2^*$ is an isomorphism of algebraic groups and commutes with the $A$-action.

Suppose that the field $k$ is separably algebraically closed. Then any two generalized Drinfeld modules (rank two and level $(f)$) are isomorphic. Let $N$ denote the generalized Drinfeld module. Then $\ker(f, N^*)$ is isomorphic to the $A$-module $(f^{-1}A/A)^2$.

It can be seen that the automorphisms $h$ of $N$ are parametrized by the matrices $\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$ with $a, d \in F_0^*$ and $b \in A/fA$. For such a matrix the corresponding $h$ on $N^* \cong k \times f^{-1}A/A$ is given by the formula

$$h(z, \frac{g}{f} \mod A) = (az + \lambda \frac{bg}{f}, \frac{dg}{f} \mod A),$$

where $\lambda : f^{-1}A/A \to \ker(f, N^*)$ is a chosen isomorphism of $A$-modules. This action $h$ is extended to an isomorphism of the $k$-scheme $N$.

### 4.4.4 The universal generalized Drinfeld module

We will briefly describe this universal object which lives above the compactification $Z := M^2(f) \otimes K_\infty$ of the affine modular curve $M^2(f) \otimes K_\infty$ over $K_\infty$. We have already constructed the universal Drinfeld module of rank two and with level $f$. (See 2.2 and 2.5) The line bundle $L$ above $M^2(f) \otimes K_\infty$ is extended to a line bundle, also called $L$, on the compactification $M^2(f) \otimes K_\infty$. Using “GAGA” this line bundle is algebraic. We will continue the description in the category of algebraic varieties over $K_\infty$.

The line bundle $L \to Z$ is made into a bundle of projective lines, say $PL \to Z$, by adding an infinite section. The fibre of $PL$ above an ordinary point of $Z$ is a projective line with a Drinfeld structure of rank two and level $f$ on the corresponding affine line. The fibre of $PL$ above a cusp does not give the correct structure. After “blowing up” suitable points of $PL$ above the cusps one obtains a new bundle $PL \to Z$. The fibre above any cusp is isomorphic with the reduction $B$ with all its structure, in other words it is a generalized Drinfeld module (rank two, level $f$) over the field $K_\infty$. The ring $A$ acts on a big open part $PL^*$ of $PL$ (i.e. the complement of the infinite section and for every cusp the irreducible components which are no “end lines”). The isomorphism of sheaves of $A$-modules $(f^{-1}A/A)^2 \to \ker(\phi_f, L)$ on the
open subset $M^2(f) \otimes K_\infty$ of $Z$ is extended to an isomorphism $(f^{-1}A/A)^2 \rightarrow \ker(f, PL^*)$. The universal Drinfeld module of rank two and with level $f$ is the object $PL \rightarrow Z$ with the additional data:

1. A structure of algebraic group on $PL$.
2. The group law $PL^* \times PL^* \rightarrow PL^*$ extends to an algebraic action $PL^* \times PL \rightarrow PL$.
3. An action of $A$ on $PL^*$. Above an ordinary point of $Z$ this comes from a Drinfeld module of rank two. Above a cusp the fibre is isomorphic to $G_{r,K_\infty} \times f^{-1}A/A$, where the first factor has a Drinfeld module structure of rank one.
4. An isomorphism of the constant sheaf $(f^{-1}A/A)^2$ with $\ker(f, PL^*)$.

For a suitable definition of a generalized Drinfeld module (rank two and level $f$) above any $K_\infty$-scheme, one can show that the universal Drinfeld module described above is indeed universal.

4.4.5 Remarks

One can describe the line bundle $L$ on $X(f)$ (which is a component of $M^2(f) \otimes K_\infty$) in terms of the cusps and the “elliptic points” of $X(f)$. The set of elliptic points, in our terminology this is the preimage of the point $\xi \in X(1)$ under the canonical map $X(f) \rightarrow X(1)$, will be denoted by $\text{elliptic}$. The term $\phi_1 \in \text{End}(L)$ has the form $T + c_1 \tau + c_2 \tau^2$ with $c_1$ a section of $L^{1-q}$ and $c_2$ a section of $L^{1-q^2}$. The section $c_1$ has zeroes of order one at the set $\text{elliptic}$. Hence $L^{1-q} \cong O(\text{elliptic})$.

The section $c_2$ has zeroes of order $q^2(q-1)$ at the set $\text{cusps}$. Hence $L^{1-q^2} \cong O(\text{cusps})$.

(There is indeed the possibility of replacing $L$ by $L \otimes M$ where $M$ is a line bundle of order $q-1$. This seems to correspond with an “automorphy factor” of the form $\det^m$). Hence $L^2 \cong O(\text{elliptic})O(\text{cusps})^{-q}$.

5 Modular forms

A modular function of weight $k$ and type $m$ for a group $\Gamma$ of finite index in $Gl(Y)$ is a meromorphic function $f$ on $\Omega$ satisfying

$$f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k(ad-bc)^{-m}f(z),$$

for every \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \) $\in \Gamma$. For every point $c \in \mathbf{P}^1(K)$ one considers the subgroup $\Gamma_c$ of $\Gamma$ consisting of the matrices with a double eigenvalue 1 and having $c$ as
fixed point.
The subgroup $\Gamma_\infty$ can be represented as $\{z \mapsto z + h| h \in H\}$, where $H \subset K$ is a discrete subgroup such that $H \backslash K_\infty$ is compact. Let, as before, $e_H : \mathbb{C} \to \mathbb{C}$ be given by the formula

$$e_H(z) = z \prod_{h \in H, \delta > 0} \left(1 - \frac{z}{h}\right).$$

This function identifies the analytic space $H \backslash \mathbb{C}$ with $\mathbb{C}$. Moreover for suitable $R$ one has that $e_H^{-1}(R < |z| < \infty) \subset \Omega$. The restriction of the modular function $f$ to $e_H^{-1}(R < |z| < \infty)$ is $H$-invariant and is therefore equal to a Laurent series in the function $t = e_H^{-1}$, i.e. equal to a convergent infinite sum $\sum_{n=-\infty}^{\infty} a_n t^n$.

The modular function is called holomorphic (meromorphic) at the cusp $\infty$ if $a_n = 0$ for $n < 0$ ($a_n = 0$ for $n < 0$ respectively).

For any other cusp of $\Gamma$ represented by a $c \in K$ one chooses a $g \in GL(2, K)$ with $g(\infty) = c$. The modular function $f$ is called holomorphic (meromorphic) at the cusp $c$ if the function $z \mapsto f(g(z))$ is holomorphic (meromorphic) at the cusp $\infty$. For the group $g \Gamma g^{-1}$.

The modular function $f$ is called a modular form if moreover, $f$ is holomorphic on $\Omega$ and is holomorphic at the cusps of $\Gamma$.

Examples
A meromorphic differential form on $\Omega$ can be written in the form $f(z)dz$. For

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2, K_\infty)$$

one has $g(f(z)dz) = f(g(z)) \frac{ad - bc}{(cz + d)^2}dz$.

Let $\Gamma$ be a subgroup of finite index in $GL(Y)$. The differential $f(z)dz$ is $\Gamma$-invariant if and only if $f$ is a modular function for $\Gamma$ of weight 2 and type 1. In this case $f(z)dz$ represents a meromorphic differential on $\Gamma \backslash \Omega$. If $f$ is moreover meromorphic at the cusps then $f(z)dz$ extends to an analytic meromorphic differential on the projective curve $\overline{\Gamma \backslash \Omega}$. By GAGA such a differential is also an algebraic meromorphic differential.

Suppose now that $f(z)$ is a modular form of weight 2 and type 1. Then $f(z)dz$ is a differential form on $\overline{\Gamma \backslash \Omega}$ which has at most poles at the cusps of a prescribed order. The prescription of this order depends on the choice of $\Gamma \subset GL(Y)$.

A differential form of degree $k$ on $\Omega$ is an expression $f(z)(dz)^k$. This expression is $\Gamma$-invariant if and only if $f$ is a modular function of weight $2k$ and type $k$. An invariant $f(z)(dz)^k$, with $f$ meromorphic on $\Omega$ and at the cusps, is a meromorphic differential of degree $k$ on $\overline{\Gamma \backslash \Omega}$.

The sheaf of the holomorphic differential forms on $\overline{\Gamma \backslash \Omega}$ is a line bundle $\mathcal{L}$. The
sheaf of the holomorphic forms of degree $k$ is equal to the line bundle $\mathcal{L}^k$. In a similar way one can construct, for any $k, m$, a line bundle, which we denote for the moment by $\mathcal{L}_{k,m}$, on $\mathbf{H}/\Gamma$ such that the space of the modular forms of weight $k$ and type $m$ are sections of this line bundle with prescribed poles at the cusps.

We cite from [G] more precise information:

Let $\Gamma = \operatorname{Gl}(Y)$. The line bundle of the differentials on $\mathbf{H}/\Gamma$ is denoted by $\mathcal{L}$. The space of the modular forms of weight $2k$ and type $k$ is equal to the sections of $\mathcal{L}^k$ with poles at the cusps of order $\leq \frac{k}{k+1}$ and at the elliptic points of order $\leq \frac{k}{k+1}$.

Let $\Gamma = \Gamma_Y(n)$ with $n$ a proper ideal of $A$. The line bundle of the differentials on $\mathbf{H}/\Gamma$ is again denoted by $\mathcal{L}$. The space of the modular forms of weight $2k$ and type $k$ is equal to the sections of $\mathcal{L}^k$ with poles at the cusps of order $\leq 2k$.

References


