Lecture 9

Algebraic compactification and modular interpretation

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1 Introduction

In section 9 of Drinfeld's seminal 1974 paper [D], results concerning the algebraic compactification of the moduli space of rank 2 Drinfeld modules may be found. We will state part of this here and give some comments. Next, some proofs and supplements to his results will be given. In particular we will give a precise description how the universal Drinfeld module with a level structure degenerates over a cusp. This appears to be new; it is quite analogous to the generalized elliptic curves (n-gons) which Deligne and Rapoport use in [DR] although it seems technically more tricky. The results are strongly motivated by the descriptions found in the (rigid) analytic theory (Lecture 8). In our present setup, compared with Drinfeld's work matters will be simplified in two ways. Firstly, only Drinfeld $\mathbf{F}_q[T]$-modules are considered. Secondly, when dealing with level (f)-structures we restrict to $\mathbf{F}_q[T][\frac{1}{f}]$-schemes.

The first simplification allows one to avoid considerations concerning class groups (in our setup projective modules are free). An additional benefit is that Drinfeld $\mathbf{F}_q[T]$-modules are in a sense more explicit. Although the necessary modifications needed to cover the case of more general rings could be rather involved or messy, this first simplification does not seem to be a very essential one.

With the second restriction one avoids an explanation of "level structures in a characteristic dividing the level". Of course Drinfeld's idea of introducing what is now called a Drinfeld level structure has been very important; it had a great impact on reduction theory of classical modular curves (compare [KM], etc.). Nevertheless, even without this technical aspect it seems worth while to try to describe a compactification.

1.1 Drinfeld's results

Unlike in the other sections, here A will be general; so $C/\mathbf{F}_q$ is a smooth, complete absolutely irreducible curve, ∞ is a place of the function field $\mathbf{F}_q(C)$ and $A \subset \mathbf{F}_q(C)$ is the ring of functions which are regular outside ∞.

Suppose $S \to \mathrm{Spec}(A)$ is an A-scheme, $\mathcal{L}$ is a line bundle on S, and ϕ : $A \to \mathrm{End}_S(\mathcal{L})$ is a rank 2 Drinfeld A-module over S.

For $I \subset A$ a nonzero ideal, Drinfeld defines a level I-structure on ϕ to be a homomorphism of A-modules

$$\varphi\ :\ (A/I)^2 \to \mathcal{L}(S),$$

such that $\sum_{f\in(A/I)^2}\varphi(f)$, seen as effective Cartier divisor over S, is the same as $\mathrm{Ker}(\phi(I))$.

We will not explain this; for more information about such a "Drinfeld level structure" one may consult [KM]. If the points in $\mathrm{Spec}(A)$ on which I vanishes are *not* in the image of the structure morphism $S \to \mathrm{Spec}(A)$, then a level I-structure is simply an isomorphism from the constant group scheme $\underline{(A/I)^2}_S$ over S to $\mathrm{Ker}(\phi(I))$.

Drinfeld's result is as follows.

Theorem 1.1 (Drinfeld) *Suppose that at least two prime ideals of A contain I.*

Then the functor assigning to an A-scheme S the set of isomorphism classes of rank 2 Drinfeld A-modules over S with level I-structure is representable by an A-scheme M_I^2.

This M_I^2 is of finite type over A, with fibres of dimension 1*, and $M_I^2 \to$ Spec(A) is smooth outside the primes containing I.*

Moreover, there exists a (unique upto isomorphism) scheme $\overline{M}_I^2$ of finite type over A, containing M_I^2 as an open and everywhere dense subscheme, such that (again) $\overline{M}_I^2 \to$ Spec(A) is smooth outside primes containing I, and $\overline{M}_I^2 \to$ Spec(A) is proper.

Remarks.

1. The assumption that I is contained in at least two different prime ideals is used by Drinfeld because he wants to construct $M_I^2, \overline{M}_I^2$ over *all* of $\mathrm{Spec}(A)$. The idea is that if $I \subset \underline{m} \cap \underline{n}$, then the level I-structure yields a level $\underline{m}$-structure as well, and this structure survives over $\underline{n} \in \mathrm{Spec}(A)$.

   A similar idea is well known in the theory of elliptic curves. Here one uses the 'Legendre family' given by $y^2 = x(x-1)(x-\lambda)$ to describe curves over rings in which 2 is invertible and the 'Hessian family' given by $x^3 + y^3 + z^3 - 3\mu xyz = 0$ in cases where 3 is invertible. After gluing over rings in which both are invertible (and if necessary taking quotients in order to get rid of level 2– and level 3–structures in the families above), one finally obtains moduli spaces over all of $\mathbb{Z}$; compare [KM].

2. If one inverts the primes containing I, a ring B is obtained. The same proof sketched by Drinfeld for the theorem above shows, that whenever I is such that Drinfeld modules with level I-structure have no automorphisms, the restriction of our functor to B-schemes is representable. The corresponding moduli space and compactification are again denoted $M_I^2, \overline{M}_I^2$, respectively. These are smooth schemes over $\mathrm{Spec}(B)$. In fact, they are simply the restrictions (base changes) to $\mathrm{Spec}(B) \subset \mathrm{Spec}(A)$ of the A-schemes which exist by Drinfeld's theorem.

1.2 Some examples

From now on we restrict ourselves to the case $A = \mathbf{F}_q[T]$. Motivated by the analytic compactification (Lecture 8), we try to classify rank 2 Drinfeld modules with a level (f)-structure and their degenerations, for $f = T$ and $f = T^2$.

1.2.1 Example: level T

The general way to describe a rank 2 Drinfeld A-module is by giving the endomorphism $\phi_T = T + A_1\tau + A_2\tau^2$. This makes sense over the ring $A[\frac{1}{T}, A_1, A_2, \frac{1}{A_2}]$ and as a line bundle on its spectrum one takes the trivial bundle (which corresponds to the structure sheaf). A level (T)-structure means that two solutions X_1, X_2 of $\phi_T X = 0$ are given, which are independent over $\mathbf{F}_q$. This independence says that no $a, b \in \mathbf{F}_q$ exist, not both 0, such that $aX_1 + bX_2 = 0$. In other words, it precisely states that the projective point $(X_1 : X_2) \in \mathbf{P}^1$ is not $\mathbf{F}_q$-rational, hence that $X_1^q X_2 - X_1 X_2^q \neq 0$. We can therefore construct rank 2 modules with level (T)-structure precisely over rings which are algebras over

$$\mathbf{F}_q[T, T^{-1}][A_1, A_2, X_1, X_2]_{X_1 X_2^q - X_1^q X_2}/I,$$

where I is the ideal generated by the two elements $\phi_T(X_i) = TX_i + A_1 X_i^q + A_2 X_i^{q^2}$. This ring is in fact equal to

$$\mathbf{F}_q[T, T^{-1}][X_1, X_2]_{X_1 X_2^q - X_1^q X_2},$$

and $A_1 = T\frac{X_1^{q^2} X_2 - X_1 X_2^{q^2}}{(X_1 X_2^q - X_1^q X_2)^q}$ and $A_2 = T\frac{X_1 X_2^q - X_1^q X_2}{(X_1 X_2^q - X_1^q X_2)^q}$. This ring is not quite what we want, since it classifies A-Drinfeld modules of rank 2 and level T rather than isomorphism classes of them. The multiplicative group $\mathbf{G}_m =$

$\mathrm{Spec}(\mathbf{F}_q[u,u^{-1}])$ acts on the ring above by $X_i \mapsto u \otimes X_i$. This precisely describes all possible isomorphisms. On the sections A_1, A_2 the action is given by $A_1 \mapsto u^{1-q} \otimes A_1$ and $A_2 \mapsto u^{1-q^2} \otimes A_2$. Dividing the algebra above by this action gives the affine ring of $\mathrm{Proj}(\mathbf{F}_q[T,T^{-1}][X_1,X_2]) \setminus V(X_1^qX_2 - X_1X_2^q)$. In other words, $M^2_{(T)}$ is the projective line over $\mathbf{F}_q[T,T^{-1}]$ minus the $q+1$ points $\mathbf{P}^1(\mathbf{F}_q)$. The scheme $\overline{M}^2_{(T)}$ must be the projective line over $\mathbf{F}_q[T,T^{-1}]$. The cusps are the $q+1$ points above.

The formulas for ϕ_T and A_1, A_2 do not make sense at the cusps. We want to change the formulas in such a way that locally at a cusp ϕ looks like a "Tate-Drinfeld module" as described in Lecture 8. The obvious way to do this is to replace ϕ_T by $(X_1X_2^q - X_1^qX_2)^{-1}\phi_T(X_1X_2^q - X_1^qX_2)$. This yields

$$\phi_T = T + T\frac{X_1^{q^2}X_2 - X_1X_2^{q^2}}{(X_1X_2^q - X_1^qX_2)}\tau + T(X_1X_2^q - X_1^qX_2)^{q^2-q}\tau^2.$$

Outside the cusps the level structure is given by $\frac{X_i}{X_1X_2^q - X_1^qX_2}$ with $i = 1,2$. The line bundle L on the projective line over $\mathbf{F}_q[T,T^{-1}]$ must be such that A_1 is a section of L^{1-q} and A_2 is a section of L^{1-q^2} and the $\frac{X_i}{X_1X_2^q - X_1^qX_2}$ are meromorphic sections of L. The degree of L is $-q$. We take $L = O(-q)$. Clearly A_1 is a section of $O(-q)^{1-q} = O(q^2-q)$ and its set of zeroes is $\mathbf{F}_{q^2} \setminus \mathbf{F}_q$. Further A_2 is a section of $O(-q)^{1-q^2} = O(q^3-q)$ with zeroes of order q^2-q at the points $\mathbf{P}^1(\mathbf{F}_q)$.

The $\frac{X_i}{X_1X_2^q - X_1^qX_2}$ with $i = 1,2$ are meromorphic sections of $O(-q)$ with poles of order at most one at the points of $\mathbf{P}^1(\mathbf{F}_q)$. On the open part $X_1X_2^q - X_1^qX_2 \neq 0$ we have the level structure $\lambda : (A/TA)^2 \to O(-q)(\mathbf{P}^1_{\mathbf{F}_q[T,T^{-1}]})$ which maps (a,b) to the section $\frac{aX_1 + bX_2}{X_1X_2^q - X_1^qX_2}$. For every cusp there is a unique line in $(A/TA)^2$ for which λ extends at the cusp. For the cusp 0, i.e. the point $(1:0)$ this line is $0 \times A/TA$. Thus we have found a structure like a rank 2 Drinfeld module on the line bundle $O(-q)$ on $\mathbf{P}^1_{\mathbf{F}_q[T,T^{-1}]}$, which locally at the cusps $\mathbf{P}^1(\mathbf{F}_q)$ behaves exactly like what was found in the analytic theory (Lecture 8).

We now suppose that we have chosen coordinates in such a way, that $X = 0$ defines a cusp. Moreover, we will assume that the sections in $\mathrm{Ker}(\phi_T)$ which extend over this cusp, in fact reduce to $\mathbf{F}_q$. Since the reduced Drinfeld module is given by $\bar{\phi}_T = T + c\tau$ for some c, this implies $0 = \bar{\phi}_T(1) = T + c$, hence $c = -T$. Finally, we demand that the "original" module has the form $\phi_T = T - T\tau + XA_2\tau^2$. By Hensel's lemma, the section 1 lifts uniquely to a T-torsion section of ϕ over the completion of the local ring at the cusp. In the analytic theory and also above, we found an other generator of the torsion with a pole of order 1 at the cusp. If we want this other generator to be $1/X$, we obtain the Drinfeld module with

$$\phi_T = T - T\tau - TX^{q^2-q}(X^{q-1}-1)\tau^2.$$

This is the "Tate-Drinfeld module of rank 2 and level T", as will be defined in Definition 2.4 below.

1.2.2 Example: level T^2

We now consider rank 2 Drinfeld A-modules with level T^2 structure. The analytic theory shows that the moduli spaces $M^2_{(T^2)} \otimes K_\infty$ is not connected (and has in fact q connected components). We will see this via an algebraic approach as well for $M^2_{(T^2)} \otimes K^{sep}$. We note that $M^2_{(T^2)} \otimes K$ *is* connected. This follows from a more general result proven in section 5.

We have $A/T^2A \cong \mathbf{F}_q[\epsilon]$, with $\epsilon^2 = 0$. The diagram

$$\begin{array}{ccc} \mathbf{F}_q[\epsilon] \times \mathbf{F}_q[\epsilon] & \xrightarrow{\sim} & \mathrm{Ker}\,\phi_{T^2} \\ \cup \;\; \downarrow \cdot\epsilon & & \cup \;\; \downarrow \phi_T \\ \epsilon \cdot (\mathbf{F}_q \times \mathbf{F}_q) & \xrightarrow{\sim} & \mathrm{Ker}\,\phi_T \\ \cong & & \\ \mathbf{F}_q \times \mathbf{F}_q & & \end{array}$$

shows that rank 2 Drinfeld modules with T^2-structure also carry a level T-structure. Hence it is natural to describe $M^2_{(T^2)}$ as a covering of $M^2_{(T)}$.

To this end, start with the ring $R = \mathbf{F}_q[T, T^{-1}][X_1, X_2]_{X_1X_2^q - X_1^qX_2}$ from the previous example. Recall that over R the rank 2 Drinfeld module ϕ is given by $\phi_T = T + A_1\tau + A_2\tau^2$, and A_1, A_2 are as follows. Write

$$\delta = X_1X_2^q - X_1^qX_2 = -X_2 \prod_{\alpha \in \mathbf{F}_q} (X_1 - \alpha X_2)$$

and

$$\delta_2 = X_1^{q^2}X_2 - X_1X_2^{q^2} = X_2 \prod_{\alpha \in \mathbf{F}_{q^2}} (X_1 - \alpha X_2).$$

Then $A_1 = T\frac{\delta_2}{\delta^q} = -T\prod_{\alpha\in\mathbf{F}_{q^2}\setminus\mathbf{F}_q}(X_1 - \alpha X_2)\cdot\frac{1}{\delta^{q-1}}$ and $A_2 = T \cdot \frac{1}{\delta^{q-1}}$.

To obtain a level T^2 structure one precisely needs Y_1, Y_2, possibly in some extension ring of R, which satisfy $\phi_T(Y_i) = X_i$. The fact that the X_i are independent over $\mathbf{F}_q$ then implies that the Y_i generate a free module over $\mathbf{F}_q[\epsilon]$ of rank 2. Clearing denominators one finds that Y_i is a zero of

$$\delta^{q-1}X_i - \delta^{q-1}TY_i + T \prod_{\alpha\in\mathbf{F}_{q^2}\setminus\mathbf{F}_q} (X_1 - \alpha X_2)Y_i^q - TY_i^{q^2}$$

for $i = 1, 2$. These two polynomials define a homogeneous ideal J in $R[Y_1, Y_2]$. So as before, rings over which a rank 2 A-Drinfeld module with level T^2-structure are defined, are precisely the $\tilde{R} = R[Y_1, Y_2]/J$-algebras.

An isomorphism of such a Drinfeld module is given by the usual $\mathbf{G}_m$-action on the X_i and Y_i, so we find a natural model for $\overline{M}^2_{(T^2)}$ in $\mathbf{P}^3_{A[\frac{1}{T}]}$, defined by the ideal J. The projection on the X_i coordinates is a well defined morphism, corresponding to the natural map from a level T^2 structure to the underlying level T-structure. From the analytic theory one knows that on each absolutely irreducible component, this projection has degree q^3, and there are q such components. This nicely agrees with the fact that visibly our projection map has total degree q^4.

The group $\mathbf{F}_q \times \mathbf{F}_q \times \mathbf{F}_q \times \mathbf{F}_q = \operatorname{Ker}\phi_T \times \operatorname{Ker}\phi_T$ acts on our model. This is obvious from the equations $\phi_T(Y_i) = X_i$ and $\phi_T(X_i) = 0$; the action is given by $Y_1 \mapsto Y_1 - aX_1 - bX_2$ and $Y_2 \mapsto Y_2 - cX_1 - dX_2$. The projection to the X-coordinates is precisely the quotient map for this action. This means that the map from $\overline{M}^2_{(T^2)}$ to $\overline{M}^2_{(T)}$ is unramified except possibly over the cusps.

Note that over the cusps of $\overline{M}^2_{(T)}$, the closure in $\mathbf{P}^3$ of our model is singular. Hence one cannot immediately read off the total number of cusps for level T^2 from our model. From the analytic theory we know this number to be $q^3(q+1)$, and the model given here contains only $q^2(q+1)$ (singular!) points over the cusps for level T.

Write $M = \mathbf{F}_q[\epsilon]e_1 + \mathbf{F}_q[\epsilon]e_2$ for a fixed free rank 2 $\mathbf{F}_q[\epsilon]$-module. The level structure on our Drinfeld module ϕ is given by $\varphi : e_i \mapsto Y_i$. The action of $\mathbf{F}_q \times \mathbf{F}_q \times \mathbf{F}_q \times \mathbf{F}_q$ changes this to a level structure $\tilde{\varphi}$, with $e_1 \mapsto Y_1 - aX_1 - bX_2$ and $Y_2 \mapsto Y_2 - cX_1 - dX_2$. Hence the action yields an automorphism $\tilde{\varphi}^{-1}\varphi$ of M, given by a matrix $\begin{pmatrix} 1+a\epsilon & c\epsilon \\ b\epsilon & 1+d\epsilon \end{pmatrix}$. From the analytic theory we know that the two level structures define points on the same component of the moduli space precisely when this matrix is in the $Gl(2, A)$-orbit of the identity matrix. This is the case precisely when $a + d = 0$.

The above observation allows one to find the components of our model, at least for $q = 2$. Consider the affine open part $X_2 \neq 0$, and write $x = X_1/X_2, y = Y_1/X_2, \xi = Y_2/X_2$. We will work over the field $L = \overline{\mathbf{F}_2(T)}(x)$. The equation for y reads

$$x^2(x+1) - Tx(x+1)y + T(x^2+x+1)y^2 - Ty^4 = 0.$$

This defines a degree 4 field extension $L(y)/L$ which is Galois with group $\mathbf{F}_2 \times \mathbf{F}_2$ (the equation for y is equivalent to $\phi_T(y) = x$, from which the assertion is obvious).

Proving that the moduli space in this case consists of $q = 2$ components now boils down to showing that the remaining polynomial

$$x(x+1) - Tx(x+1)\xi + T(x^2+x+1)\xi^2 - T\xi^4$$

factors in $L(y)[\xi]$. Suppose η is a zero of an irreducible factor. Then (y, η) corresponds to a level structure. Any zero of the polynomial is of the form $\eta + cx + d$ with $c, d \in \mathbf{F}_2$. Hence $\eta + cx + d$ is a zero of the same factor precisely when the corresponding level structure $(y, \eta + cx + d)$ gives a point on the same component as (y, η). We have seen above that this is the case when $d = 0$. This means that the zeroes of a factor are $\eta, \eta + x$ and hence a factorization should have the form

$$(\xi^2 + x\xi + a)(\xi^2 + x\xi + b)$$

for certain $a, b \in L(y)$. Comparing coefficients shows that a, b are in fact in a quadratic extension of L. These subfields are easily found, and one obtains $a = (x+1)g + (y^2+y)/x$ and $b = x + 1 + a$, in which g satisfies $g^2 + g = T^{-1}$.

This g is not an element of $K = \mathbf{F}_2(T)$, but it is contained in $K_\infty = \mathbf{F}_2((1/T))$. Namely, here $g = \sum_{n \geq 0} T^{-2^n}$. This confirms the fact that in the analytic theory all components of the moduli space are defined over K_∞.

We now try to compute the Tate-Drinfeld module of rank 2 and level T^2, as we did for level T. For this, we start with the rank 1 module given by $\bar{\phi}_T = T + c\tau$. As before, we want that 1 generates the T^2-torsion (as a module over $A/(T^2)$). This means that $(T + c\tau)(T + c\tau)(1) = 0$, but $(T + c\tau)(1) \neq 0$. In other words $c \neq -T$ but $c^{q+1} + c(T^q + T) + T^2 = 0$. Factoring $c^{q+1} + c(T^q + T) + T^2 = (c + T)(c(c+T)^{q-1} + T)$, this implies that c is any root of $c(c+T)^{q-1} + T = 0$. Note that for $q = 2$, this defines the same extension of $\mathbf{F}_2(T)$ which we need to separate the components of $M^2_{T^2}$. This is not a coincidence.

Next, we want $\phi_T = T + c\tau + eX^{q^3 - q^2}\tau^2$ and $\phi_{T^2}(1/X) = 0$. This becomes a polynomial in X and e after multiplication by X^{q^2}. In fact, $X^{q^2}\phi_{T^2}(1/X) \equiv ec^{q^2} + c^{q+1} \bmod X$. Using Hensel's lemma again, this implies that a unique $e \in A[c][[x]]$ exists such that the T^2-torsion of the corresponding Drinfeld module is indeed generated by $1/X$ and a unit $\equiv 1 \bmod X$.

2 Tate-Drinfeld modules

2.1 Drinfeld modules of rank one and class field theory

In lecture 4 a general survey of class field theory over function fields is given. In the sequel we will provide more details about the case $A = \mathbf{F}_q[T]$ and abelian extensions of the field $K = \mathbf{F}_q(T)$ which are totally split at ∞. Those results are needed for the description of the Tate-Drinfeld modules of rank two with level f.

Let R be an A-algebra with structure homomorphism $\gamma : A \to R$. A rank one Drinfeld module $\psi : A \to R\{\tau\}$ is determined by $\psi_T = \gamma(T) + c\tau$ with $c \in R^*$. Let $f \in A$ be a monic polynomial of degree $n \geq 1$. A (strong) level f-structure is an isomorphism λ of (the constant sheaf) $f^{-1}A/A$ to $ker(\psi_f, R)$ (as a sheaf on $Spec(R)$). Since the polynomial $\psi_f(z)$ has the form

$$\gamma(f)z + *z^q + \ldots + c^{1+q+\ldots+q^{n-1}} z^{q^n}$$

the existence of such a level structure implies that $\gamma(f)$ is invertible in R. In the sequel we suppress γ from the notation and we will consider $A[1/f]$-algebras R. The triple (R, ψ', λ') is called equivalent to (R, ϕ, λ) if there is an $r \in R^*$ such that $\psi' = r^{-1}\psi r$ and $\lambda' = r^{-1}\lambda$. This defines a functor $\mathcal{M}^1(f)$ from the category of the $A[1/f]$-algebras to the category of sets, by $\mathcal{M}^1(f)(R)$ is the set of equivalence classes of Drinfeld modules of rank one and with level f above R. (The functor extends to the category of $A[1/f]$-schemes). As we know, this functor is representable and we will make this explicit. For this purpose we study the Carlitz module given by $\psi : A \to A\{\tau\}$ with $\psi_T = T + \tau \in A\{\tau\}$.

From an analytic point of view the Carlitz module is derived from a lattice $\xi A \subset \mathbf{C}$ for a certain $\xi \in \mathbf{C}$. The Carlitz exponential function

$$e = e_{\xi A} = z \prod_{0 \neq a \in A} (1 - \frac{z}{\xi a}) = z + \sum_{0 \neq a \in A} (\frac{1}{\xi a})^{q-1} z^q + *z^{q^2} + \ldots$$

is supposed to have the property $e(Tz) = Te(z) + e(z)^q$. The sum $\alpha := \sum_{0 \neq a \in A} \frac{1}{a^{q-1}} \in K_\infty$ satisfies $|\alpha + 1| < 1$. A calculation shows then

$$\xi = T \sqrt[q-1]{1 - T^{1-q}} \sqrt[q-1]{-\alpha} \sqrt[q-1]{-T}.$$

In particular $\xi \in K_\infty(\sqrt[q-1]{-T})$. Thus e induces an isomorphism between $K_\infty(\sqrt[q-1]{-T})/\xi A$ and $K_\infty(\sqrt[q-1]{-T})$.

The solutions of the equation $\psi_f(z) = 0$ in the field $K_\infty(\sqrt[q-1]{-T})$ are the $e(\frac{g\xi}{f})$ with $g \in A$ of degree $< n$. Hence the equation splits in $K_\infty(\sqrt[q-1]{-T})$. One defines the polynomial

$$\chi(f) := \prod_{a \in (A/fA)^*} (z - e(\frac{a\xi}{f})),$$

note that $\chi(1) = z$. It is clear that the degree of $\chi(f)$ equals $\#(A/fA)^*$. Also, when f is not constant,

$$\chi(f) = \prod_{\substack{a \in (A/fA)^* \\ a \equiv 1 \bmod f}} \prod_{c \in \mathbf{F}_q^*} (z - ce(\frac{a\xi}{f})) = \prod_{\substack{a \in (A/fA)^* \\ a \equiv 1 \bmod f}} (z^{q-1} - e(\frac{a\xi}{f})^{q-1})$$

hence $\chi(f)$ is a polynomial in z^{q-1}. We now show that $\chi(f) \in A[z^{q-1}]$. Since the zeroes of $\chi(f)$ are also zeroes of the monic polynomial $\psi_f(z)$ which has coefficients in A, they are integral over A. Hence we only need to show that $\chi(f) \in K(z)$. Note that $\psi_f(z) = \prod_{d|f,\, d \text{ monic}} \chi(d)$. Let $\mu : A \setminus \{0\} \to \{0, 1, -1\}$ denote the Möbius function, given by $\mu(c) = 1$ for $c \in \mathbf{F}_q^*$ and $\mu(f_1^{e_1} \dots f_s^{e_s})$ (with irreducible inequivalent $f_1, \dots, f_s$) is equal to 0 if some $e_i > 1$ and is equal to $(-1)^s$ if all $e_i = 1$. Then $\chi(f) = \prod_{d|f,\, d \text{ monic}} \psi_{f/d}^{\mu(d)}$. This implies $\chi(f) \in K(z)$, and hence $\chi(f) \in A[z^{q-1}]$ for $f \in A$ not constant, as we wanted to prove.

We note that $\chi(f)$ is the equivalent for the Carlitz module (A, ψ) of the classical cyclotomic polynomial Φ_n. In fact,

Lemma 2.1 *1. $\chi(f)$ is the minimal monic polynomial of $e(\frac{\xi}{f})$ over K.*

2. *The splitting field L of $\psi_f(z)$ over K is equal to $L = K(e(\frac{\xi}{f})) \cong K[z]/(\chi(f))$.*

3. *The Galois group of L/K acts A-linearly on $ker(\psi_f, L)$. The resulting homomorphism $Gal(L/K) \to (A/fA)^*$ is an isomorphism. We will identify the two groups.*

4. *The integral closure of A in L is $A[z]/(\chi(f))$.*

5. *The finite ring extension $A[1/f] \subset A[1/f, z]/(\chi(f))$ is unramified.*

6. *$e(\frac{\xi}{f})$ is invertible in $A[1/f, z]/(\chi(f))$ for non-constant f.*

7. *The extension L/K is the abelian extension of K which by class field theory corresponds to the subgroup*

$$\left((1 + \frac{1}{T}\mathbf{F}_q[[\frac{1}{T}]]) \times \prod_{\pi | f} (1 + fA_\pi) \prod_{\text{other } \pi} A_\pi^* \right) / \mathbf{F}_q^*$$

of the idele class group $\mathcal{I}_K / K^* \cong \left(\mathbf{F}_q[[\frac{1}{T}]]^* \times \prod_\pi A_\pi^*\right) / \mathbf{F}_q^*$. Here the π are primes of A and A_π denotes the corresponding completion.

Proof. (1) It is obvious that $\chi(f)$ is a monic polynomial with $e(\frac{\xi}{f})$ as a root. For constant f it is z, so we may assume $n = \deg(f) > 0$. Consider any prime π dividing f. Note that modulo π, the Carlitz module ψ reduces to a rank one module $\overline{\psi}$ over $A/\pi A$. The $\overline{\psi}_f$-torsion is a free module over $A/\pi A$ of rank at most 1, hence since $\overline{\psi}_f(z)$ is not separable, it is 0. This implies that $\psi_f(z) \equiv z^{q^n} \bmod \pi$. Hence the divisor $\chi(f)$ of $\psi_f(z) \equiv z^{q^n} \bmod \pi$ also reduces modulo π to a power of z, which shows that all its coefficients except the leading one are divisible by π. Moreover, the constant term of $\psi_f(z)/\chi(1)$ is f, hence the divisor $\chi(f)$ of this has a constant term which is a divisor of f. In case π can be chosen such that it divides f exactly once, this implies that $\chi(f)$ is an Eisenstein polynomial at the prime π.

If $f = \pi^{n+1}$ for $n > 0$ and π irreducible, then observe $\chi(f) = \chi(\pi)(\psi_{\pi^n}(z))$. This is true because both sides have as zeroes precisely the elements of $\mathrm{Ker}(\psi_f)$ which are not in $\mathrm{Ker}(\psi_{\pi^n})$. It follows that $\chi(f)(0) = \chi(\pi)(0)$, hence one concludes that also in the present case $\chi(f)$ is Eisenstein at π.

The general case now easily follows: write $f = \pi_1^{n_1} \cdot \ldots \cdot \pi_t^{n_t}$. The splitting field L of $\chi(f)$ contains the splitting fields L_i of $\chi(\pi_i^{n_i})$. These fields are linearly disjoint over K, because L_i is totally ramified at π_i while L_j is unramified at π_i for $j \neq i$. Hence

$$\#(A/fA)^* \geq [L:K] \geq \prod [L_i:K] = \prod \#(A/\pi_i^{n_i}A)^* = \#(A/fA)^*$$

which shows $[L:K] = \#(A/fA)^*$ and hence $\chi(f)$ is irreducible.
(2 and 3) are immediate using the argument above.
(4) We may assume that f is not constant. It suffices to show that $\chi(f) = 0$ defines a smooth affine variety in $\mathbf{A}^2_{\mathbf{F}_q}$. In other words, we want to verify that $\chi(f)$ and its derivatives to T and z have no common zero. Suppose a common zero would exist. Then we have a maximal ideal $M \subset A$ containing $\chi(f)$ and its derivatives. Since $\psi_f(z) = g \cdot \chi(f)$ for some g, one sees by taking the derivative to z that $f \in M$. Hence $\psi_f(z) \equiv z^{q^n} \bmod M$, and therefore $z \in M$. A similar computation, now using that the derivative to T of $\psi_f(z)/z$ is in M, shows that the derivative f' of f to T is in M. This already leads to a contradiction whenever all roots of f are simple.

In case $f = \pi^m$ and π irreducible, we computed in (1) that $\chi(f) = u\pi + z*$ for a unit $u \in \mathbf{F}_q^*$, hence again taking the derivative to T shows that $\pi' \in M$. Since $\pi \in M$ as well, this again gives a contradiction.

Finally, the remaining case is that $f = \pi^m g$ with π irreducible not dividing g, and $\pi \in M$. Since the zeroes of $\chi(f)$ are also zeroes of $h := \chi(\pi^m)(\psi_g(z))$, the latter polynomial is a multiple of the former hence an element of M. Modulo π, so modulo M as well, we have seen that h is a power of $\psi_g(z)$, hence also $\psi_g(z) \in M$. Again taking the derivative to T it follows that $\pi' \in M$, a contradiction.

(5 and 6) are again immediate: If f is not constant, $\psi_f(z)/z$ is a monic polynomial with constant term f. It is separable over any field in which $f \neq 0$.
(7) The computation in (1) above shows that the conductor of our extension L/K is as asserted, at each prime dividing f. Moreover, by (5) L/K is unramified at the primes of A not in f. At the infinite prime one may argue as follows. Consider the rank one module $\varphi : A \to K\{\tau\}$ given by $\varphi_T = T - T\tau$. A calculation shows that this module corresponds to a lattice λA with $\lambda \in K_\infty$. Hence all torsion of φ is inside K_∞. Take u in an extension of K satisfying $u^{q-1} = -1/T$. Then ψ and φ are isomorphic over $K(u)$. More explicitly, $\psi_f = u^{-1}\varphi_f u$ for any $f \in A$. This means that α is an f-torsion point for ψ precisely when $u\alpha$ is f-torsion for φ. Hence for any non-constant f the f-torsion generates the extension $K_\infty(u)$ of K_∞. This extension is totally ramified of degree $q-1$, which is what we wanted to prove.

Finally, consider a monic irreducible $\pi \in A$ which does not divide f. Then $\pi \bmod f \in (A/fA)^*$ corresponds in (3) to the automorphism σ_π of L defined by $\sigma_\pi(e(\frac{\xi}{f})) = e(\frac{\pi\xi}{f}) = \psi_g(e(\frac{\xi}{f}))$. Write $m = \deg(\pi)$ and let M be a prime of L over π. From (1) we know that $\psi_g(z) \equiv z^{q^m} \bmod M$. This shows that σ_π induces the Frobenius on the residue field modulo M, completing the proof of (7). □

Remark. Consider the rank one module φ given by $\varphi_T = T - T\tau$. The proof of Lemma 2.1 (7) implies, that the torsion of this module generates the maximal abelian extension of K which is totally split at infinity. The Galois group of this extension is isomorphic to $\hat{A}^*/\mathbf{F}_q^*$. The f-torsion of φ generates the abelian extension of K corresponding for $f(0) \neq 0$ to the quotient $(A/fA)^*$. In case $f(0) = 0$, the quotient $(A/fA)^*/\mathbf{F}_q^*$ is obtained. The drawback of using the module φ is that it is in fact a module over $A[1/T]$. Hence, for instance, one cannot expect that torsion points are integral at the place $T = 0$. One way to cope with this, is to use a second module with also period in K_∞ but now defined over $A[1/(T-1)]$.

The ring $A[1/f, z]/(\chi(f))$ with the Drinfeld structure $\psi_T = T + \tau$ is somewhat to big for our purposes. We normalize the Drinfeld structure to $\phi := d^{-1}\psi d$ where $d := e(\frac{\xi}{f})$. Put $c = d^{q-1}$. The minimal monic equation of c over K is $\chi(f)^*$, defined by $\chi(f)^*(z^{q-1}) = \chi(f)(z)$.
Then $\phi_T = T + c\tau$ and the generator of the level f-structure is now 1. The relation $\phi_f(1) = 0$ is a monic polynomial equation for c of degree $\frac{q^n-1}{q-1}$.
Let us introduce a variable C and define formally $\tilde{\phi} : A \to A[C]\{\tau\}$ by $\tilde{\phi}_T = T + C\tau$. The expression $\tilde{\phi}_f(1)$ is a monic polynomial in C of degree $\frac{q^n-1}{q-1}$. For a monic divisor d of f one finds that $\tilde{\phi}_d(1)$ is a divisor of $\tilde{\phi}_f(1)$. Further for any irreducible monic $f \in A$ one has $\tilde{\phi}_f(1) = \chi(f)^*$. It follows that the polynomial

$\tilde{\phi}_f(1)$ in C is the product of the $\chi(d)^*$ over all monic divisors d of f.

Thus we obtain the Drinfeld module $(A[1/f,c],\phi,can)$ which has a canonical level structure given by $can: 1/f \in f^{-1}A/A \mapsto 1 \in A[1/f,c]$. From Lemma 2.1 above one easily derives the following properties:

1. The subgroup $\mathbf{F}_q^*$ of the Galois group of L/K acts on $e(\frac{\xi}{f})$ as multiplication and thus $A[c] \subset K_\infty$.
2. The finite extension $A^f := A[1/f,c] \supset A[1/f]$ is unramified.
3. c is invertible in A^f.
4. The field of quotients K^f of A^f is an extension of K with Galois group $(A/fA)^*/\mathbf{F}_q^*$ and is totally split at ∞.
5. The field K^f is the field associated by class field theory to the group $(A/fA)^*/\mathbf{F}_q^*$.

Proposition 2.2 *The functor $\mathcal{M}^1(f)$ is represented by $A^f = A[1/f,c]$ (or its spectrum). The universal rank one Drinfeld module with level f is (A^f,ϕ,can).*

Proof. Let an element $B \in \mathcal{M}^1(f)(R)$ be given. The equivalence class B contains a unique Drinfeld module (R,ψ,λ) such that λ maps $1/f \in f^{-1}A/A$ to $1 \in R$. Write $\psi_T = T + \tilde{c}\tau$ with $\tilde{c} \in R^*$. Then $\psi_f(1) = 0$ and for every proper divisor d of f one has that $\psi_d(1) \neq 0$. As shown above, this implies that $\chi(f)^*(\tilde{c}) = 0$. Hence there is a A-homomorphism $h: A^f \to R$, given by $h(c) = \tilde{c}$, which transports the Drinfeld structure above A^f to (R,ψ,λ). Clearly h is unique. □

Examples:
We give the polynomial $\chi(f)^*$ which $c \in A[1/f,c]$ satisfies, for some f's.
(1) Let $f = T^2+1 \in \mathbf{F}_3[T]$. Then c must satisfy the equation $c^4 + c(T^3+T) + T^2+1 = 0$. This defines indeed an étale extension of $A[1/f]$ of degree 4 in which c is invertible.
(2) For $f = T \in \mathbf{F}_q[T]$ one has $c = -T$.
(3) Take $f = T^2 \in \mathbf{F}_q[T]$. Let $z \neq 0$ be a solution of $\phi_T(z) = 0$ and let $\phi_T(1) = z$. Then $z^{q-1} = \frac{-T}{c}$ and $T + c = z$. Hence $c(c+T)^{q-1} + T = 0$, as was already shown in section 1. This equation is irreducible and separable. In fact $A[1/T] \subset A[1/T][c]$ is étale of degree q.
(4) Put $f = T(T+1) \in \mathbf{F}_q[T]$. Let $z \neq 0$ satisfy $\phi_T(z) = 0$ and $\phi_{T+1}(1) = z$. This yields the equation $c(c+T+1)^{q-1} + T = 0$. A trivial solution is $c = -T$. The irreducible monic polynomial $\chi(T(T+1))^*$ is $\frac{c(c+T+1)^{q-1}+T}{c+T}$.

2.2 The Tate-Drinfeld module without level

Let an A-algebra R be given with structure homomorphism $\gamma : A \to R$. A Drinfeld module of rank two over R is given by a homomorphism $\phi : A \to R\{\tau\}$ with $\phi_T = \gamma(T) + c_1\tau + c_2\tau^2$ such that $c_1 \in R, c_2 \in R^*$. We will attach to this a *geometric object*, namely the projective line $\mathbf{P}^1_R$ over R with its infinite section ∞, the line bundle $\mathbf{A}^1_R = \mathbf{P}^1_R \setminus \{\infty\}$ and the A-action on this line bundle given by ϕ. Let $(t_0 : t_1)$ denote the homogenous coordinates of the projective line over R. For $f \in A$ of degree $n \geq 0$ one can extend this action by the formula

$$f * (t_0 : t_1) = (t_0^{q^{2n}} : t_0^{q^{2n}} \phi_f(\frac{t_1}{t_0})).$$

This is seen as an analogue of an elliptic curve over R with its natural action of $\mathbb{Z}$.

Tate's elliptic curve E over $\mathbb{Z}[[q]]$ lies in $\mathbf{P}^2_{\mathbb{Z}[[q]]}$ and is given by the affine equation $y^2 + xy = x^3 + Ax + B$ with certain $A, B \in q\mathbb{Z}[[q]]$. Above the open set $q \neq 0$ of $Spec(\mathbb{Z}[[q]])$ this defines a "true" elliptic curve. Above $q = 0$ the affine equation reduces to $y^2 + xy = x^3$, which defines a rational curve with an ordinary double point. After removal of this double point the group structure above $q = 0$ is the multiplicative group over $\mathbb{Z}$.

The obvious analogue for Drinfeld modules is the *Tate-Drinfeld module of rank two* given by $\phi : A \to A[[x]]\{\tau\}$ with $\phi_T = T + \tau + x\tau^2$. The geometric object corresponding to this is the projective line $\mathbf{P}^1_{A[[x]]}$ with its section at infinity ∞ and the line bundle $\mathbf{A}^1_{A[[x]]} = \mathbf{P}^1_{A[[x]]} \setminus \{\infty\}$ with its A-action given by ϕ. The action of $f \in A$ of degree $n \geq 0$ extends to the open set $\mathbf{P}^1_{A[[x]]} \setminus V(x, t_0)$ by the formula

$$f * (t_0 : t_1) = (t_0^{q^{2n}} : t_0^{q^{2n}} \phi_f(\frac{t_1}{t_0})).$$

Above the open subset $x \neq 0$ of $Spec(A[[x]])$ the structure above is an ordinary Drinfeld module of rank two. Above $x = 0$ the structure reduces to a rank one Drinfeld module, namely the *Carlitz module* given by $\psi_T = T + \tau$. This seems to be the canonical choice for the Tate-Drinfeld module of rank two without level structure.

2.3 The Tate-Drinfeld module with level (f)

Tate's elliptic curve E, described above, has no level structure. Fix an integer $n \geq 3$. A level n-structure would be an isomorphism of (the constant sheaf) $(n^{-1}\mathbb{Z}/\mathbb{Z})^2$ with the group of the n-torsion points. We have to extend the base

$\mathbb{Z}[[q]]$ in order to obtain enough torsion points. Above the open subset $q \neq 0$ one needs the extension $\mathbb{Z}[1/n, \zeta_n][[q^{1/n}]]$ of $\mathbb{Z}[[q]]$ for this. Above $q^{1/n} = 0$, the group of the n-torsion points is equal to the group of the n-torsion points of the multiplicate group (over $\mathbb{Z}[1/n, \zeta_n]$). This group is isomorphic to $n^{-1}\mathbb{Z}/\mathbb{Z}$. In order to obtain also above $q^{1/n} = 0$ enough n-torsion points one has to blow up the special fibre of $E \otimes \mathbb{Z}[[1/n, \zeta_n]][[q^{1/n}]]$. This defines the Néron model of *the Tate curve with level* n.
The universal family $\mathcal{E}$ of "generalized elliptic curves of level n" lives above the modular curve $X(n)$ corresponding to level n. For any cusp c of $X(n)$ the completion of the fibre of $\mathcal{E}$ at c is the Néron model of the Tate curve of level n, described above.

Fix a monic $f \in A$ of degree $n \geq 1$. The Tate-Drinfeld module with level f-structure should have approximately the form $\phi : A \to R[[x]]\{\tau\}$, with $\phi_T = T + c_1\tau + c_2\tau^2$ with $c_1 \in R[[x]]^*$ and $0 \neq c_2 \in xR[[x]]$. For $x = 0$ the structure should reduce to the universal rank one Drinfeld module with level f. Thus we have $R = A^f = A[1/f, c]$ and we may normalize the situation such that $c_1 = c$. Above the open set $x \neq 0$ the set $ker(\phi_f, A^f[[x]][x^{-1}])$ should be isomorphic to the A-module $(f^{-1}A/A)^2$. The element $1 \in R[[x]]/(x)$ is a generator of the f-torsion of the rank one Drinfeld module. This generator lifts in a unique way to a generator of $ker(\phi_f, A^f[[x]]) \cong f^{-1}A/A$. We try to normalize the situation by imposing that x^{-1} is a generator of the factor module $ker(\phi_f, A^f[[x]][x^{-1}])/ker(\phi_f, A^f[[x]])$. According to the next lemma this is possible and defines a unique c_2. We first note that c_2 must have the form $ex^{q^n(q-1)}$ with $e \in A^f[[x]]^*$. Indeed, replace A^f by any quotient field $k = A^f/\underline{m}$. A calculation over the field k shows that c_2 mod $\underline{m}$ has order $q^n(q-1)$ with respect to the variable x.

Lemma 2.3 *The equation* $x^{q^n}\phi_f(x^{-1}) = 0$ *can be seen as a polynomial equation for* e *over the ring* $A[c][x]$*. It has a unique solution in* $A^f[[x]]$*. This solution has the form* $e \in -c^{1+q-q^n} + x^{q^n-q^{n-1}}A^f[[x]]$.

Proof. Put $f = \sum_{i=0}^n f_iT^i$ (with $f_n = 1$) and put

$$P(e) = c^{-q^2-q^3-\ldots-q^n} \sum_{i=0}^{n} f_i x^{q^n} (T + c\tau + ex^{q^n(q-1)}\tau^2)^i x^{-1}.$$

One computes that $(T+c\tau+ex^{q^n(q-1)}\tau^2)^i x^{-1}$ has the form $c^{1+q+\ldots+q^{i-1}}x^{-q^i}+\ldots$ for $i = 1, \ldots, n-1$ and $c^{1+q+\ldots+q^{n-1}}x^{-q^n} + ec^{q^2+\ldots+q^n}x^{-q^n} + *x^{-q^{n-1}} + \ldots$ for $i = n$. Hence $P(e)$ is a polynomial in e with coefficients in the ring $A[c][x]$. From the above it follows that $P(e) \in (e + c^{1+q-q^n}) + x^{q^n-q^{n-1}}A^f[[x]][e]$ and that the derivative satisfies $P'(e) \in 1+x^{q^n-q^{n-1}}A^f[[x]][e]$. If e exists then $e \in -c^{1+q-q^n}+$

$x^{q^n-q^{n-1}}A^f[[x]]$. Now we observe that $P(-c^{1+q-q^n}) \in x^{q^n-q^{n-1}}A^f[[x]]$ and that $P'(-c^{1+q-q^n}) \in 1 + x^{q^n-q^{n-1}}A^f[[x]]$. Applying Newton's approximation one finds the unique solution $e \in A^f[[x]]$ of the equation. □

Definition 2.4 *The Tate-Drinfeld module of rank two and level f is defined by $\phi : A \to A^f[[x]]\{\tau\}$ with $\phi_T = T + c\tau + ex^{q^n(q-1)}\tau^2$, where A^f, c and e are defined in* (2.1) *and* (2.2)*.*

By Newton's approximation, the canonical map

$$ker(\phi_f, A^f[[x]]) \to ker(\phi_f, A^f[[x]]/(x)) \cong f^{-1}A/A$$

is an isomorphism. Further the elements $x^{-1}, \phi_T x^{-1}, \ldots, (\phi_T)^{n-1}x^{-1}$ in $ker(\phi_f, A^f[[x]][x^{-1}])$ have the form

$$x^{-1}, cx^{-q} + \ldots, c^{1+q}x^{-q^2} + \ldots, \ldots, c^{\frac{q^{n-1}-1}{q-1}}x^{-q^{n-1}} + \ldots$$

They are linearly independent over $\mathbf{F}_q$ in the factor module $ker(\phi_f, A^f[[x]][x^{-1}])/ker(\phi_f, A^f[[x]])$. Thus the A-module $ker(\phi_f, A^f[[x]][x^{-1}])$ is isomorphic to $(f^{-1}A/A)^2$ as required.

The Tate-Drinfeld module that we have constructed above has a certain universal property. We formulate a weak form of this.

Proposition 2.5 *Let R be an $A[1/f]$-algebra without zero divisors and let a Drinfeld structure $\psi : A \to R[[y]]\{\tau\}$ be given by $\psi_T = T + c_1\tau + c_2\tau^2$. Suppose further that*
(1) $c_1 \in R[[y]]^*$ *and* $c_2 \in yR[[y]]$.
(2) $ker(\psi_f, R[[y]])$ *is isomorphic to* $f^{-1}A/A$.
(3) $ker(\psi_f, R[[y]][y^{-1}])$ *is isomorphic to* $(f^{-1}A/A)^2$.
Then there is a homomorphism of A-algebras $h : A^f[[x]] \to R[[y]]$ such that $h(\phi)$ is equivalent over $R[[y]]$ with ψ.

Proof. The canonical map $ker(\psi_f, R[[y]]) \to ker(\psi_f, R[[y]]/(y))$ is an isomorphism. After changing ψ we may suppose that the last A-module is generated by $1 \in R$. The universality of the rank one module $(A^f, \phi \bmod (x))$ induces a unique A-linear homomorphism $h : A^f \to R$. Let $v \in R[[y]][y^{-1}] \setminus R[[y]]$ have "minimal negative" order among the elements of $ker(\psi_f, R[[y]][y^{-1}])$. Then we extend the homomorphism h to $h : A^f[[x]] \to R[[y]]$ be sending x to v^{-1} (which belongs indeed to $yR[[y]]$). Then $h(\phi)$ and ψ have the same f-torsion groups on $R[[y]][y^{-1}]$. From this it follows that $h(\phi) = \psi$. □

Remarks
The condition in the proposition that R has no zero divisors will be removed in

the sequel by means of the notion of generalized Drinfeld module (of rank two and with level (f)-structure).
The homomorphism h of the proposition induces of course an isomorphism $\tilde{h}$ between the A-modules $ker(\phi_f, A^f[[x]][x^{-1}])$ and $ker(\psi_f, R[[y]][y^{-1}])$. And in fact h is determined by $\tilde{h}$.

3 Models for the projective line over a discrete valuation ring

Let R be a discrete valuation ring with field of fractions L and residue field l. The projective line over L, i.e. $\mathbf{P}^1_L$, has a standard model $\mathbf{P}^1_R$ over R. Attached to this model there is a reduction map for points, namely the map $\mathbf{P}^1_L(L) \tilde{\to} \mathbf{P}^1_R(R) \to \mathbf{P}^1_l(l)$. The models M of $\mathbf{P}^1_L$ over R that we consider here are supposed to have the properties:
(a) M is a proper, flat scheme over R.
(b) The generic fibre $M \otimes_R L$ is isomorphic to $\mathbf{P}^1_L$.
(c) The special fibre $M_s := M \otimes_R l$ has at most ordinary double points as singularities.

Since the genus of the special fibre M_s is 0, this fibre must be a finite tree of projective lines over the residue field l. For any model M one defines a reduction map r_M for points

$$r_M : \mathbf{P}^1_L(L) \tilde{\to} M(R) \to M_s(l)$$

Lemma 3.1 *Let F be a finite subset of $\mathbf{P}^1_L(L)$ with cardinality ≥ 3. Then there is a unique model M which has the following properties:*
(1) The reduction map r_M is injective on F and $r_M(F)$ does not contain any double point of M_s.
(2) The special fibre M_s together with the points $r_M(F)$ form a “stable tree”. This means that every irreducible component of M_s contains at least three special points. The special points of M_s are defined to be the union of the double points of M_s with $r_M(F)$.

Proof. The statement is well known, see [GHP]. We will sketch another proof of the lemma.

Suppose for convenience that $\{0, 1, \infty\} \subset F$. The standard model gives as reduction a projective line over l and F is mapped to some points on this line. If the reduction map (restricted to F) is injective then we are finished. If the reduction map (restricted to F) is not injective then there is a point a on the projective line which is the image of more than one point of F. One blows up the point a, seen as a point of the surface M. The new special fibre is a tree with two lines; on each line there are at least two special points. If F is mapped injectively to the regular points of this tree then we stop. If not then one blows up another point et cetera. One stops when condition (1) is satisfies. Condition (2) does not hold necessarily, since one may have introduced irreducible components which carry only two special points. Those irreducible components are blown down until (2) is also satisfied. It is an exercise to show the uniqueness of the model M for F. □

The lemma is applied to the field $\mathbf{F}_q(x)$ with as valuation ring the local ring at $x = 0$, i.e. the local ring $\mathbf{F}_q[x]_{(x)}$. For the set F we choose

$$\{\mathbf{F}_q + \mathbf{F}_q x^{-1} + \mathbf{F}_q x^{-q} + \ldots + \mathbf{F}_q x^{-q^{n-1}}\} \cup \{\infty\}.$$

The special fibre RT of the model, given by the lemma and the F above, has been described in Lecture 8 in detail.

In the sequel we will also use the concept of *stable n-pointed tree*. Let S be a scheme and let $n \geq 3$. A stable n-pointed tree over S is a morphism $\alpha : C \to S$ which has the following properties:

1. $\alpha : C \to S$ is a flat and projective morphism of relative dimension one.
2. There are given n sections $f_1, \ldots, f_n : S \to C$.
3. For every geometric point $\xi = Spec(k)$ of S, the geometric fibre C_ξ is a stable tree of projective lines over the field k, i.e.
   (a) C_ξ is reduced
   (b) Every irreducible component of C_ξ is a projective line over k.
   (c) The only singular points of C_ξ are ordinary double points.
   (d) The images of the f_i in C_ξ are distinct and do not contain a double point.
   (e) On every irreducible component of C_ξ there are at least three special points. By a special point we mean either a double point or an image of some section f_i.

4 A Néron model of the Tate-Drinfeld module

The Tate-Drinfeld module, rank two and level f, was given by $\phi_T = T + c\tau + ex^{q^n(q-1)}\tau^2$ with $c \in A^f \subset K^f$ and $e \in A^f[[x]]^*$. Put $F = ker(\phi_f, A^f[[x]][x^{-1}]) \cup \{\infty\}$. We can consider this over the discrete valuation ring $K^f[[x]]$. The previous section produces a model M_0 of $\mathbf{P}^1_{K^f[[x]]}$ over $K^f[[x]]$ with respect to F.

Let $\underline{m}$ denote a maximal ideal of A^f with residue field, say k. Write $\bar{F}$ for the image of F in $\mathbf{P}^1_{k[[x]]}$. Section 3 produces again a model M_k of $\mathbf{P}^1_{k[[x]]}$ over $k[[x]]$ with respect to $\bar{S}$. One can combine all those models in a model M over $A^f[[x]]$ of $\mathbf{P}^1_{A^f[[x]]}$ with respect to F which has the properties:
(1) M is projective and flat over $Spec(A^f[[x]])$.
(2) $M \otimes A^f[[x]][x^{-1}]$ is isomorphic to $\mathbf{P}^1_{A^f[[x]][x^{-1}]}$.
(3) $M \otimes K^f[[x]]$ is isomorphic to M_0.
(4) $M \otimes k[[x]]$ is isomorphic to M_k for every maximal ideal of A^f.
In particular, for every geometric point ξ of $Spec(A^f[[x]])$ the fibre M_ξ is either an ordinary Drinfeld module of rank two with level f-structure and completed to a projective line or a generalized Drinfeld module over a field as defined in Lecture 8.

The existence of M can be proven as follows. Consider two sections f_1, f_2 of $\mathbf{P}^1_{A^f[[x]]}$. They are two-dimensional smooth subvarieties of the smooth three-dimensional space. If their intersection is not empty then they intersect transversely in a space isomorphic to $Spec(A^f)$. This is easily seen from the special form of the sections in F, namely for $f_* \in F, f_* \neq 0, \infty$ the leading coefficient is invertible in A^f. One blows up all the non-empty intersections $f_1 \cap f_2$. The result is the model M that has the required properties. We will call M the *Néron model of the Tate-Drinfeld module.*
It has a lot of structure induced by its fibre $\mathbf{P}^1_{A^f[[x]][x^{-1}]}$.

1. The open set M^* of M is defined by deleting the section ∞ and the closure of the irreducible components of $M \otimes K^f[[x]]/(x)$ on which no sections $\neq \infty$ of F lie. Then M^* has an induced structure of an algebraic group scheme.

2. The group law $M^* \times M^* \to M^*$ extends to an action of group schemes $M^* \times M \to M$. There is an A-action on the group scheme M^*.

3. Above $x = 0$ the group scheme M^* is isomorphic to $\mathbf{G}_{a,A^f} \times f^{-1}A/A$. The first term is the additive group over $A^f = A^f[[x]]/(x)$; it has the structure of the universal rank one Drinfeld module with level f. The second term is the constant group above A^f with as fibre the A-module $f^{-1}A/A$. The A-action on M^* above $x = 0$ coincides with the one given by this rank one Drinfeld module and the obvious action on $f^{-1}A/A$.

4. The kernel of $f\cdot$ on M^* is equal to the union of the section $f_* \in F, f_* \neq \infty$.

5. For every geometric point ξ of $Spec(A^f[[x]])$ the fibre M_ξ is either an ordinary Drinfeld module of rank two and level f completed with a section ∞, or a generalized Drinfeld module (rank two and level f) over the field corresponding to ξ.

5 The compactification of $M^2(f) \otimes A[1/f]$

In [D] ,§9 the compactification of M^2, this is the projective limit of all the Drinfeld modular curves M_I^2, is produced in an adelic way from M^1, which is the projective limit of the schemes M_I^1. A direct consequence is the existence of a morphism $M^2 \to M^1$. In particular one obtains, for any nontrivial ideal I of A, an embedding of the coordinate ring $O(M_I^1)$ into $O(M_I^2)$, the one of M_I^2.
Unfortunately we are unable to understand this construction. To the best of our knowledge, this construction of Drinfeld has not been developed in more detail.
One of our goals in this section is to give an explicit construction of the compactification of M_I^2 and the morphism $M_I^2 \to M_I^1$ in the special case $A = \mathbb{F}_q[T]$ and over the ring $A[1/f]$, where $I = (f)$.
The other aim of this section is to make the Néron structure above the cusps of the compactification $\bar{M}^2_{(f)}$ explicit and to formulate a moduli functor which is represented by $\bar{M}^2_{(f)}$.

Put $\mathcal{Z} := M^2(f) \otimes A[1/f]$. Our aim is to construct its compactification $\hat{\mathcal{Z}}$ together with all its Drinfeld data. Let us collect the information that we have about $\mathcal{Z}$.

According to lecture 2, the following structure lives above $\mathcal{Z}$: a line bundle L, a rank two Drinfeld module $\phi : A \to End(L)$ with $\phi_T = T + c_1\tau + c_2\tau^2$, $c_1 \in L^{1-q}$, $c_2 \in L^{1-q^2}$ and c_2 nowhere vanishing, plus a level structure $\lambda : (f^{-1}A/A)^2 \to ker(\phi_f, L)$. We also know that $\mathcal{Z}$ is affine and that $\mathcal{Z} \to Spec(A[1/f])$ is smooth of relative dimension 1. For every $\sigma \in Gl(2, A/fA)$ one can consider the Drinfeld module $(\mathcal{Z}, L, \phi, \lambda\sigma)$. From the universality of $(\mathcal{Z}, L, \phi, \lambda)$ one obtains an automorphism $\tilde{\sigma}$ of $\mathcal{Z}$ such that $(\mathcal{Z}, \tilde{\sigma}L, \phi, \tilde{\sigma}\lambda)$ is equivalent with $(\mathcal{Z}, L, \phi, \lambda\sigma)$. The kernel of the homomorphism $\sigma \mapsto \tilde{\sigma}$ is equal to $\mathbf{F}_q^* id$. In this way $Gl(2, A/fA)/\mathbf{F}_q^*$ acts as a group of automorphisms on $\mathcal{Z}$. The quotient of $\mathcal{Z}$ by this action is $\mathbf{A}^1_{A[1/f]}$. The morphism $\mathcal{Z} \to \mathbf{A}^1_{A[1/f]}$ is explicitly given by the map $j : \mathcal{Z} \to \mathbf{A}^1_{A[1/f]}$ with the formula $j = \frac{c_1^{q+1}}{c_2}$.
According to lecture 8, the curve $\mathcal{Z} \otimes K_\infty$ coincides with the analytic object $Gl(2, A)\backslash(\Omega \times Gl(2, A/fA))$. The analytic compactification Z and the Drinfeld data on it have been described in (4.4.4) of lecture 8. We recall that Z has $\#(A/fA)^*/\mathbf{F}_q^*$ components.

An explicit description of $\mathcal{Z}$.
The section $\lambda(1/f, 0)$ of L vanishes nowhere and we can therefore identify L with $O_\mathcal{Z}$ and this section with $1 \in O(\mathcal{Z}) = O_\mathcal{Z}(\mathcal{Z})$. The section $\lambda(0, 1/f)$ is then an element $y \in O(\mathcal{Z})$ and also c_1, c_2 are elements of $O(\mathcal{Z})$. Further, for

any non-zero element $\xi \in (f^{-1}A/A)^2$ the element $\lambda(\xi) \in O(\mathcal{Z})$ is invertible. This easily leads to the following description of the ring $O(\mathcal{Z})$:
$O(\mathcal{Z}) = A[1/f][c_1, c_2, c_2^{-1}, y]_{loc}$ with ϕ given by $\phi_T = t + c_1\tau + c_2\tau^2$. The only relations (between the c_1, c_2, y) in this ring are given by $\phi_f(1) = 0$ and $\phi_f(y) = 0$. The ring is localized at the finite set consisting of all elements $\phi_{g_1}(1) + \phi_{g_2}(y)$ where $g_1, g_2 \in A$ are polynomials of degree $< n$, not both zero.

Definition of the scheme $\mathcal{Y}$
The next thing that we want to construct is an $A[1/f]$-algebra R such that its associated affine scheme $\mathcal{Y}$ over $A[1/f]$ will later be "an affine neighbourhood of a cusp of $\hat{\mathcal{Z}}$ ".
Consider the ring $R = A[1/f][c, c^{-1}, e, x]_{loc}$ and ϕ given by $\phi_T = T + c\tau + ex^{q^n(q-1)}\tau^2$, having the only relations $\phi_f(1) = 0$ and $x^{q^n}\phi_f(x^{-1}) = 0$. Note that this makes sense since by Lemma 2.3 $x^{q^n}\phi_f(x^{-1})$ is indeed an element of $A[1/f][c, c^{-1}, e, x]$. The localization is taken with respect to the finite set $x^{q^{m_2}}(\phi_{g_1}(1) + \phi_{g_2}(x^{-1}))$, where $g_1, g_2 \in A$ are polynomials (not both zero) of degrees $m_1, m_2 < n$. In case $g_2 = 0$ the expression $x^{q^{m_2}}$ is by definition 1. Again, all the expressions are well defined.

Lemma 5.1
(1) *R/xR with its Drinfeld data is isomorphic to $A^f[[x]]/(x)$ with the Drinfeld data induced by the Tate-Drinfeld module $A^f[[x]]$ defined in* (2.3).
(2) *The completion of R with respect to the ideal xR with its Drinfeld data is isomorphic to the Tate-Drinfeld module $A^f[[x]]$.*
(3) *e is invertible in R.*
(4) *$R[x^{-1}]$ with its Drinfeld data is isomorphic to $O(\mathcal{Z})[c_1^{-1}]$.*
(5) *R is smooth over $A[1/f]$ of relative dimension one.*

Proof. (1) According to the explicit expression for $x^{q^n}\phi_f(x^{-1})$ in (2.2), R/xR is isomorphic to $A[1/f][c, c^{-1}, e]_{loc}/(\phi_f(1), e + c^{1+q-q^n})$. Thus R/xR is isomorphic to the ring $A^f[[x]]/(x) = A^f$ (including its Drinfeld data).
(2) The same thing holds for R/x^kR for any $k \geq 1$, i.e. R/x^kR is isomorphic to $A^f[[x]]/(x^k)$ (including the Drinfeld data). We conclude that the completion of R with respect to the ideal xR (i.e. the projective limit of the R/x^kR) is isomorphic to $A^f[[x]]$ (including the Drinfeld data). We note further that R/xR is smooth over $A[1/f]$ and that e is invertible in R/xR.
(3) Suppose that e is not invertible in R and let $M \subset R$ be a maximal ideal which contains e. Put $k = R/M$. From (2) above it follows that x is invertible in k. Since $e = 0$ in k, the Drinfeld structure on k is that of a rank one Drinfeld module. The f-torsion of this module contains the images in k of the $\phi_{g_1}(1) + \phi_{g_2}(x^{-1}) \in R[x^{-1}]$. This module is still isomorphic to $(f^{-1}A/A)^2$,

which contradicts the observation that the Drinfeld module over k has rank 1. Thus e is invertible in R.
(4) Consider the $A[1/f]$-homomorphism $O(\mathcal{Z})[c_1^{-1}] \to R[x^{-1}]$, given by $c_1 \mapsto c,\ c_2 \mapsto ex^{q^n(q-1)},\ y \mapsto x^{-1}$. This has an inverse given by $c \mapsto c_1,\ x \mapsto y^{-1},\ e \mapsto c_2 y^{q^n(q-1)}$. It is clear that this isomorphism of $A[1/f]$-algebras preserves the Drinfeld data.
(5) From (4) it follows that $R[x^{-1}]$ is smooth over $A[1/f]$. Combining this with the isomorphism $R/xR \cong A^f$ found in (1) and the fact (proved in section 2.1) that A^f is unramified over $A[1/f]$, we conclude that R is smooth over $A[1/f]$. □

Lemma 5.2
(1) *$\mathcal{Y} \otimes K_\infty$ has $\#(A/fA)^*/\mathbf{F}_q^*$ components, as is shown in lecture 8.*
(2) *$\mathcal{Y} \otimes K_\infty$ has an embedding as an open subset of Z. The closed subset, given by $x = 0$, is then identified with a set C consisting of $\#(A/fA)^*/\mathbf{F}_q^*$ cusps of Z. On each component of $\mathcal{Y} \otimes K_\infty$ there lies one point of C.*
(3) *The stabilizer $H \subset Gl(2, A/fA)/\mathbf{F}_q^*$ of $\mathcal{Y} \otimes K_\infty$ as subset of Z is (up to conjugation) the subgroup of the matrices $(h_{i,j})$ with*

$$h_{1,1} \in (A/fA)^*,\ h_{1,2} = 0,\ h_{2,1} \in A/fA,\ h_{2,2} \in \mathbf{F}_q^*.$$

(4) *$\mathcal{Y}$ is connected.*
(5) *$\mathcal{Z}$ is connected.*

Proof. (1) The open subset of $\mathcal{Y} \otimes K_\infty$, given by $x \neq 0$, is identified with the open subset of $\mathcal{Z} \otimes K_\infty$ defined by $c_1 \neq 0$. The latter is known to have this number of components.
(2) The open subset $x \neq 0$ of $\mathcal{Y} \otimes K_\infty$ is identified with the open subset $c_1 \neq 0$ of $\mathcal{Z} \otimes K_\infty$ by lemma (5.1). Moreover one chooses an isomorphism of $\mathcal{Z} \otimes K_\infty$ with the affine part of Z. In this way $\mathcal{Y} \otimes K_\infty$ is embedded in Z. The closed subset $x = 0$ of $\mathcal{Y} \otimes K_\infty$ has affine ring $A^f \otimes K_\infty$ and thus consists of $\#(A/fA)^*/\mathbf{F}_q^*$ points which are rational over K_∞. Clearly this is identified with a set C of cusps of Z. We note that the group $G := Gl(2, A/fA)/\mathbf{F}_q^*$ acts on Z and $\mathcal{Z}$ in a compatible way. The subgroup which preserves the components of Z is equal to the image of $\{M \in Gl(2, A/fA)|\ det(M) \in \mathbf{F}_q^*\}$ in G.

In order to determine the nature of this set C we study the localization with respect to x of the formal completion at $x = 0$ of $\mathcal{Y} \otimes K_\infty$. This is the Drinfeld module $\phi : A \to (A^f \otimes K_\infty)[[x]][x^{-1}]\{\tau\}$ with $\phi_T = T + c\tau + ex^{q^n(q-1)}\tau^2$ and with level structure given by $(1/f, 0)$ maps to the unique element in $ker(\phi_f, (A^f \otimes K_\infty)[[x]])$ with image 1 in $(A^f \otimes K_\infty)[[x]]/(x)$ and $(0, 1/f)$ maps to x^{-1}. On this structure the Galois group of $A^f/A[1/f]$ acts. This action is trivial on the

level structure and is nontrivial on ϕ_T. We will change ϕ into an equivalent ψ such that the action of the Galois group is trivial on ψ and acts nontrivially on the level structure. There is a $u \in (A^f \otimes K_\infty)^*$ such that $u^{1-q}c = -T$. Then $\psi := u\phi u^{-1}$ has the form $\psi_T = T - T\tau + ax^{q^n(q-1)}\tau^2$ with $a \in (A^f \otimes K_\infty)[[x]]^*$. This ψ is invariant under the Galois group of $A^f/A[1/f]$ because the coefficient $-T$ of τ is invariant. We note that this Galois group is isomorphic to the group $(A/fA)^*/\mathbf{F}_q^*$. The action of this Galois group on the level permutes the points of C.
For a fixed point c_0 of C its stabilizer in G can be identified with the subgroup H_0 of G consisting of the matrices $(h_{i,j})$ with $h_{1,1}, h_{2,2} \in \mathbf{F}_q^*,\ h_{1,2} = 0,\ h_{2,1} \in A/fA$.
It can be calculated that every element in the Galois group of $A^f/A[1/f]$ can be lifted to an element of H, defined in the lemma. This yields a homomorphism of this Galois group to H/H_0, which can be shown to be an isomorphism. As a consequence one finds that there lies a point of C on every component of Z.
(3) The group H stabilizes the set C. A counting of the cusps shows that H is in fact the stabilizer of C. The open subset $\mathcal{Y} \otimes K_\infty \setminus C$ is invariant under the whole group G. Thus H is also the stabilizer of $\mathcal{Y} \otimes K_\infty$.
(4) and (5). The element $x \in O(\mathcal{Y}) = R$ cannot be identical zero on any component of $\mathcal{Y}$ since the closed subset $x = 0$ of $\mathcal{Y}$ has relative dimension 0. Similarly, $c_1 \in O(\mathcal{Z})$ cannot be identical zero on a component of $\mathcal{Z}$. Moreover the open subsets $x \neq 0$ of $\mathcal{Y}$ and $c_1 \neq 0$ of $\mathcal{Z}$ are isomorphic. Therefore (4) and (5) are equivalent.
In order to prove (4) we remark that it suffices to show that R has no idempotents $\neq 0, 1$. Let $h \in R$ be an idempotent. Then the image of h in $R/xR = A^f$ is 0 or 1. We may suppose that the image is 0 and so $h \in xR$.
The morphism $R \to R \otimes_{A[1/f]} K_\infty$ is injective, since R is a flat $A[1/f]$-module. Therefore h can be considered as an idempotent function on $\mathcal{Y} \otimes K_\infty$. Since $h \in xR$, the function h has a zero on each component of $\mathcal{Y} \otimes K_\infty$. Thus h is identically 0. □

Construction of $\hat{\mathcal{Z}}$.
This space is obtained by gluing copies of $\mathcal{Y}$ to $\mathcal{Z}$. Let $\mathcal{U}$ denote the open subset of $\mathcal{Z}$, defined by $c_1 \neq 0$. From the proof of lemma (5.1) we obtain an explicit isomorphism h of the open subset $x \neq 0$ of $\mathcal{Y}$ with $\mathcal{U}$. For the moment we denote the result of gluing $\mathcal{Y}$ and $\mathcal{Z}$ over $\mathcal{U}$ (by means of h) by $\tilde{\mathcal{Z}}$. We note that $\tilde{\mathcal{Z}}$ is separated. For this we have to verify that the map $O(\mathcal{Y}) \otimes_{A[1/f]} O(\mathcal{Z}) \to O(\mathcal{U})$ is surjective. This is a consequence of $O(\mathcal{Y})[x^{-1}] = O(\mathcal{U})$ and $x^{-1} \in O(\mathcal{Z})$. It is further clear that $\tilde{\mathcal{Z}}$ is smooth over $A[1/f]$ of relative dimension one.
The (chosen) isomorphism of $\mathcal{Z} \otimes K_\infty$ with the affine part of Z extends in a unique way to an embedding of $\tilde{\mathcal{Z}} \otimes K_\infty$ in Z.

From (5.3) one concludes that the subgroup H of G which stabilizes $\tilde{\mathcal{Z}} \otimes K_\infty$ consists of the triangular matrices $(g_{i,j}) \in G$ with diagonal entries $g_{1,1} \in (A/fA)^*$, $g_{2,2} \in \mathbf{F}_q^*$ and further $g_{1,2} = 0$, $g_{2,1} \in A/fA$. Let S denote a set of representatives of G/H. Then $\cup_{\sigma \in S} \sigma(C)$ is precisely the set of all cusp of Z. As a consequence Z is the union of the $\sigma(\tilde{\mathcal{Z}} \otimes K_\infty)$ taken over all $\sigma \in S$. Further, any two of those open subsets (with distinct σ's) intersect in $\mathcal{Z} \otimes K_\infty$. This is used in the construction of $\hat{\mathcal{Z}}$.

For each $\sigma \in S$ we consider a copy $\mathcal{Y}_\sigma$ of $\mathcal{Y}$. Every $\mathcal{Y}_\sigma$ is glued to $\mathcal{Z}$ by using the isomorphism σh between the open subset $x \neq 0$ and $\mathcal{U}$. The resulting space is called $\hat{\mathcal{Z}}$. Again, by gluing one obtains a line bundle, again denoted by L, on $\hat{\mathcal{Z}}$ and a homomorphism $\phi : A \to End(L)$. The finite morphism $j : \mathcal{Z} \to \mathbf{A}^1_{A[1/f]}$ extends to a morphism $j : \hat{\mathcal{Z}} \to \mathbf{P}^1_{A[1/f]}$.

Theorem 5.3
(1) *$\hat{\mathcal{Z}}$ is separated, connected and smooth of relative dimension one over $A[1/f]$.*
(2) *$j : \hat{\mathcal{Z}} \to \mathbf{P}^1_{A[1/f]}$ is a finite morphism. In particular $\hat{\mathcal{Z}}$ is the integral closure of $\mathbf{P}^1_{A[1/f]}$ in the field of fractions of $O(\mathcal{Z})$. The group $Gl(2, A/fA)/\mathbf{F}_q^*$ acts on $\hat{\mathcal{Z}}$ and the quotient of $\hat{\mathcal{Z}}$ by this action is $\mathbf{P}^1_{A[1/f]}$.*
(3) *$\hat{\mathcal{Z}}$ is proper over $A[1/f]$.*
(4) *$\hat{\mathcal{Z}} \otimes K_\infty$ is isomorphic to Z.*
(5) *The ring of global functions $O(\hat{\mathcal{Z}})$ on $\hat{\mathcal{Z}}$ is isomorphic to A^f. In particular, there is a morphism $\hat{\mathcal{Z}} \to M^1_{(f)}$.*
(6) *The connected components of $\hat{\mathcal{Z}} \otimes_{A[1/f]} A^f$ are absolutely irreducible and remain irreducible after tensoring with K_∞ over the subring A^f of K_∞.*

Proof. (1) The definition of the space $\hat{\mathcal{Z}}$ yields an obvious affine covering $\{U_i\}$, i.e. $\mathcal{Z}$ and the $\mathcal{Y}_\sigma$ with $\sigma \in S$. All the intersections $U_i \cap U_j$ (with $i \neq j$) are equal to the affine space $\mathcal{U}$. For $i \neq j$ one has to verify that the map $m_{i,j} : O(U_i) \otimes_{A[1/f]} O(U_j) \to O(\mathcal{U})$ is surjective. In the case of $\mathcal{Y}_\sigma$ and $\mathcal{Z}$ this follows from $O(\mathcal{U}) = O(\mathcal{Y}_\sigma)[x^{-1}]$ and $x^{-1} \in O(\mathcal{U})$. In the case of $\mathcal{Y}_{\sigma_1}$ and $\mathcal{Y}_{\sigma_2}$ with $\sigma_1 \neq \sigma_2$ one has again $O(\mathcal{U}) = O(\mathcal{Y}_{\sigma_1})[x^{-1}]$. From $\sigma_1 H \neq \sigma_2 H$ one can deduce that the images of $ker(\phi_f, O(\mathcal{Y}_{\sigma_1})$ and $ker(\phi_f, O(\mathcal{Y}_{\sigma_2})$ in $O(\mathcal{U})$ generate $ker(\phi_f, O(\mathcal{U}))$. It follows that x^{-1} lies in the image of $m_{i,j}$ and thus $m_{i,j}$ is again surjective. This proves that $\hat{\mathcal{Z}}$ is separated. Since each U_i is smooth and of relative dimension one over $A[1/f]$, also $\hat{\mathcal{Z}}$ is smooth over $A[1/f]$ of relative dimension one.
(2) The preimage $\mathcal{F}$ of the infinite section of $\mathbf{P}^1_{A[1/f]}$ is equal to the disjoint union of the closed subspaces $x = 0$ of the $\mathcal{Y}_\sigma$ and thus isomorphic to $\#S$ copies of $Spec(A^f)$. The ramification is everywhere $q^n(q-1)$. For every geometric

point ξ of $Spec(A^f)$ the fibre $\mathcal{F}_\xi$ has the correct number of points. From this one concludes that j is finite. The remaining part of (2) is obvious.
(3) follows from (2).
(4) is obvious from the construction.
(5) and (6) From $\hat{\mathcal{Z}}$ connected, smooth and proper over $A[1/f]$, one draws the conclusion that the domain $O(\hat{\mathcal{Z}})$ is finite and unramified over $A[1/f]$. The group $G := Gl(2, A/fA)/\mathbf{F}_q^*$ acts on $\hat{\mathcal{Z}}$ and on $O(\hat{\mathcal{Z}})$. Define the normal subgroup H of G by $H := \{B \in Gl(2, A/fA)| \ det(B) \in \mathbb{F}_q^*\}/\mathbb{F}_q^*$. We claim that H acts trivially on $O(\hat{\mathcal{Z}})$. Indeed, the analytic theory tells us that $O(\hat{\mathcal{Z}}) \otimes K_\infty$ is the ring of the global sections of $\hat{\mathcal{Z}} \otimes K_\infty$. The group G also acts on the analytic curve and one easily sees that the group H stabilizes every component of $\hat{\mathcal{Z}} \otimes K_\infty$. Thus H acts trivially on $O(\hat{\mathcal{Z}}) \otimes K_\infty$ and also on $O(\hat{\mathcal{Z}})$. The quotient group G/H is isomorphic to $A^*/\mathbb{F}_q^*$ and the invariants under this group acting upon $O(\hat{\mathcal{Z}})$ equals $A[1/f]$ by (2). Finally, the analytic theory also tells us that $O(\hat{\mathcal{Z}}) \otimes K_\infty$ is a product of copies of the field K_∞. Thus the field $O(\hat{\mathcal{Z}}) \otimes K$ is an abelian extension of K, totally split at the place ∞, and the ring $O(\hat{\mathcal{Z}})$ is a finite unramified extension of $A[1/f]$. The class field theory, explained in section 2.1, implies that $O(\hat{\mathcal{Z}})$ is isomorphic to A^f. The remaining statements of (5) and (6) are easy consequences of the proof given above. □

The line bundle L above $\hat{\mathcal{Z}}$ is completed in the obvious way to a bundle of projective lines which we will denote by PL. Let F denote the set $(f^{-1}A/A)^2 \cup \{\infty\}$. Outside the cusps there is for every $\xi \in F$ a section of PL. Each section has an obvious extension to $\hat{\mathcal{Z}}$. Those sections still intersect above the cusps. As in section 4, one can blow up the (non-empty) intersections of sections. Let us call $pr : M \to \hat{\mathcal{Z}}$ the result. M has a lot of structure induced by the Drinfeld data above the open part $\mathcal{Z}$.

1. $pr : M \to \hat{\mathcal{Z}}$ is a stable F-pointed tree, where F is the set $(f^{-1}A/A)^2 \cup \{\infty\}$ and where $\lambda : F \to \{$ sections of $pr\}$ is fixed.

2. Let M^* denote the open subset of M obtained by deleting the infinite section and above every cusp of $\hat{\mathcal{Z}}$ the irreducible components which do not carry points of $\lambda((f^{-1}A/A)^2)$. Then $pr^* : M^* \to \hat{\mathcal{Z}}$ has the structure of a group scheme with an A-action on it.

3. The group law $M^* \times M^* \to M^*$ extends to an action of a group scheme $M^* \times M \to M$.

4. The restriction of λ to the map $\lambda^* : (f^{-1}A/A)^2 \to \{$ the sections of $pr^*\}$ is an isomorphism between the constant sheaf of A-modules on $\hat{\mathcal{Z}}$ and the kernel of the multiplication by f on M^*.

5. Above $\mathcal{Z}$, one has that M^* with its group structure and A-action is identical with the Drinfeld data above $\mathcal{Z}$.

6. The formal fibre of M above any cusp is isomorphic to the Néron model of the Tate-Drinfeld module studied in section 4.

6 Generalized Drinfeld modules over a scheme

It is possible to produce a more or less natural functor $\mathcal{F}$ from the collection of $A[1/f]$-schemes to the category of sets which is represented by the scheme $\hat{\mathcal{Z}}$ (with its Drinfeld data). For a scheme S over $A[1/f]$, $\mathcal{F}(S)$ is the set of isomorphy classes of generalized Drinfeld modules of rank two and level f above S. We propose the following definition of a *generalized Drinfeld module of rank two and with level f over S*:

1. A stable F-pointed tree $pr : C \to S$, where F is the set $(f^{-1}A/A)^2 \cup \{\infty\}$ and where $\lambda : F \to \{$ sections of $pr\}$ is fixed.

2. Let C^* denote the open subset of C such that every geometric fibre of $pr^* : C^* \to S$ is equal to the geometric fibre of pr where $\lambda(\infty)$ and the irreducible components which do not carry points of $\lambda((f^{-1}A/A)^2)$ are deleted. Then $pr^* : C^* \to S$ is supposed to have a structure of group scheme with an A-action.

3. The group law $C^* \times C^* \to C^*$ extends to an action of a group scheme $C^* \times C \to C$.

4. The restriction of λ to the map $\lambda^* : (f^{-1}A/A)^2 \to \{$ the sections of $pr^*\}$ is an isomorphism between the constant sheaf of A-modules on S and the kernel of the multiplication by f on C^*.

5. For every geometric point ξ of S the fibre C_ξ is either an ordinary Drinfeld module of rank two and level f completed with a section ∞, or a generalized Drinfeld module (rank two and level f) over the field corresponding to ξ.

6. Let the closed point $s \in S$ have a fibre C_s which is a not a projective line. Then there exists a homomorphism $\alpha : A^f[[x]] \to \hat{O}_{S,s}$ such that $M \otimes_{A^f[[x]]} \hat{O}_{S,s}$ is isomorphic as scheme to $C \times_S Spec(\hat{O}_{S,s})$.

The last condition is not very natural. It seems possible to replace this by a more natural one, at the cost of rather technical statements and proofs.
It is obvious what is meant by an isomorphism of generalized Drinfeld modules over S. Thus the functor $\mathcal{F}$ is well defined.

The definition above is composed such that $M \to \hat{\mathcal{Z}}$ is the universal generalized Drinfeld module of rank two and with level f. We will not give a proof of this.

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