Equivalence of differential equations of order one

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The notion of strict equivalence for order one differential equations of the form \( f(y', y, z) = 0 \) with coefficients in a finite extension \( K \) of \( \mathbb{C}(z) \) is introduced. The equation gives rise to a curve \( X \) over \( K \) and a derivation \( D \) on its function field \( K(X) \). Procedures are described for testing strict equivalence, strict equivalence to an autonomous equation, computing algebraic solutions and verifying the Painlevé property. These procedures use known algorithms for isomorphisms of curves over an algebraically closed field of characteristic zero, the Risch algorithm and computation of algebraic solutions. The most involved cases concern curves \( X \) of genus 0 or 1. This paper complements work of M. Matsuda and of G. Muntingh & M. van der Put.

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1. Introduction and summary

Let \( K \) denote a finite extension of \( \mathbb{C}(z) \) equipped with the \( \mathbb{C} \)-linear derivation \( \cdot' \) given by \( z' = 1 \). Let \( f \in K[S, T] \) be absolutely irreducible and assume that \( d := \frac{df}{dz} \mod (f) \) is non-zero. To the first order algebraic differential equation \( f(y', y) = 0 \) over \( K \) we associate the differential algebra \( R := K[s, t, \frac{1}{z}] := K[S, T]/(f)[\frac{1}{z}] \), its field of fractions \( K(s, t) \), and the pair \( (X, D) \), where \( X \) is the smooth

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projective algebraic curve over $K$ with function field $K(s, t)$ and $D$ is the derivation on $K(s, t)$ defined by $D(z) = 1$, $D(t) = s$. The genus of $X$ will also be called the genus of $f$.

By a solution of $f$ we mean a $K$-linear differential homomorphism $\phi : R \to \mathcal{F}$, where $\mathcal{F}$ is a differential extension of $K$ such that the field of constants of $\mathcal{F}$ is $\mathbb{C}$. More concretely, we may take for $\mathcal{F}$ a finite extension of the field of meromorphic functions on some open connected subset of the Riemann surface of $K$. Equivalently, a possible $\mathcal{F}$ could be the field of the convergent Laurent series $\mathbb{C}[[v^{1/m}]]$, where $v$ is a local parameter of a point of the Riemann surface of $K$ and $m$ is a positive integer.

Two first order algebraic differential equations $f_1, f_2$ over $K$, inducing pairs $(X_1, D_1)$ and $(X_2, D_2)$ are called strictly equivalent if there exists a finite extension $L \supset K$ such that $L \times_K (X_1, D_1) \cong L \times_K (X_2, D_2)$. If $f_1, f_2$ are strictly equivalent, then for any solution of $f_1$ we obtain finitely many solutions of $f_2$ and visa versa.

In the sequel we will rely on algorithms for testing and producing explicit answers to the following questions, briefly discussed in Appendix A. We note that such algorithms exist for the questions (Q1)–(Q4). This does not seem to be the case for question (Q5).

Q1. Are two given curves $X_1$ and $X_2$ (over $\overline{K}$ or over $\mathbb{C}$) isomorphic?
Q2. Given is a curve $X$ over $K$. Does there exist a curve $X_0$ over $\mathbb{C}$ with $\overline{K} \times_K X \cong \overline{K} \times_\mathbb{C} X_0$?
Q3. Does the differential equation $u' = a_0 + a_1 u + a_2 u^2$ with all $a_s \in K$ have solutions in $\overline{K}$? A well known special case is deciding whether $u' = a_1 u$ has a solution in $\overline{K}$, see Risch (1970) and Baldassarri and Dwork (1979, § 6).
Q4. Find a basis of $H^0(X, L)$ for a given line bundle $L$ on a given curve $X$.
Q5. Let $E$ be an elliptic curve over $\mathbb{C}$ and let $K$ be a finite extension of $\mathbb{C}(z)$. Find a basis of the Lang–Néron group (relative Mordell–Weil group) $\mathbb{Q} \otimes_\mathbb{Z} E(K)/E(\mathbb{C})$.

We will sketch procedures for strict equivalence of equations, for strict equivalence to an autonomous equation and for having many algebraic solutions. The algebraic treatment of the Painlevé property and related problems for first order equations in the work of M. Matsuda and G. Muntingh & M. van der Put is complemented with an algorithmic approach. Finally, the autonomous equations of any genus are classified.

2. The Painlevé property and algebraic solutions

We recall that a differential equation $f(y', y) = 0$ over a finite extension $K$ of $\mathbb{C}(z)$ is said to have the Painlevé property (PP) if the only ‘moving singularities’ for solutions on the Riemann surface of $K$ are poles. The next theorem provides the link between PP and the definition of ‘no moving singularities’ in Matsuda (1980).

**Theorem 2.1.** (See Muntingh and van der Put, 2007, Proposition 4.2.) Let $(X, D)$ denote the pair associated to a first order equation $f$ over $K$. Then $f$ has PP if and only if for every closed point $x \in X$, the local ring $O_{X,x}$ is invariant under $D$.

**Remarks 2.2.** (1). There exists an algorithm verifying PP for any $f$. Indeed, the differential algebra $R = K[s, t, \frac{d}{dt}]$ is the coordinate ring of an open affine subset of $X$. Since $R$ is invariant under $D$ one has to investigate the property $D(O_{X,x}) \subset O_{X,x}$ for the finitely many closed points $x \in X$ outside this open subset.

(2). Let $L \supset K$ be a finite extension. Then $f$ over $K$ has PP if and only if $f$ over $L$ has PP. \(\square\)

**Theorem 2.3.** (See Matsuda, 1980; Muntingh and van der Put, 2007.) Let $(X, D)$ denote the pair associated to $f$ over $K$ having PP. There are, after replacing $K$ by a finite extension, three cases for $(K, X, D)$ (where $K(X)$ denotes the function field of $X$), namely

(i). $(K(y), (a_0 + a_1 u + a_2 u^2) \frac{d}{du})$ with $a_0, a_1, a_2 \in K$, not all zero.
(ii). $(K(x, y), h \cdot y \frac{d}{dx})$ with $y^2 = x^3 + ax + b, a, b \in \mathbb{C}$ a non-singular elliptic curve and $h \in K^*$.
(iii). $(K(X_0), D_0)$ where $X_0$ is a smooth irreducible projective curve over $\mathbb{C}$ and $D_0$ is zero on its function field $\mathbb{C}(X_0)$ (which is a subfield of $K(X_0)$).
Remarks 2.4. (1) In Matsuda (1980), the base field $K$ is more general than a finite extension of $\mathbb{C}(z)$ and the three cases of $(X, D)$ are named Riccati fields, Poincaré fields and Clairaut fields.

(2) The proof of part (iii) in Muntingh and van der Put (2007) is not constructive. We present in Corollary 2.5 a constructive proof, using ideas from M. Matsuda, in particular his proof of Matsuda (1980, Thm. 14).

**Corollary 2.5.** Suppose that $(X, D)$ satisfies PP and $X$ is a curve of genus $\geq 2$. Let $L \supset K$ be a finite extension of $K$ such that all the Weierstrass points of $L \times_K X$ are rational over $L$. Then $(L \times_K X, D) \cong (L \times X_0, D_0)$, where $X_0$ is a smooth, irreducible, projective curve over $\mathbb{C}$ and the derivation $D_0$ of $L(X_0)$ is zero on the subfield $\mathbb{C}(X_0)$.

**Proof.** We assume that the Weierstrass points of $X$ are rational over $K$.

First, we consider a hyperelliptic curve $X$ and repeat here (for the convenience of the reader) a part of the proof of Proposition 4.3 of Muntingh and van der Put (2007).

The curve $X$ can be presented by the affine equation $y^2 = Q(x)$ with $Q = x(x - 1)(x - a_1) \cdots (x - a_m)$ where $m > 1$ is odd and $0, 1, a_1, \ldots, a_m$ are distinct elements of $K$. Then $D(x) = A + By$ with $A, B \in K[x]$. The completion of the local ring at $\infty$ can be written as $K[[u]]$ with $u^{-1} = u^2$ and $u^{m+2} = (1 - u) \cdots (1 - u^2a_1) \cdots (1 - u^2a_m)$. A small computation shows that $D(u) \in K[[u]]$ implies that $u = 0$ and that $A$ has degree at most one. The condition $D(y) \in K[x, y]$ implies $2yD(y) = D(Q) \in yK[x, y] \cap K[x] = QK[x]$. Now $\frac{DQ}{D} \in K[x]$ implies that $A = 0$ and all $a'_j = 0$. Thus $D(x) = D(y) = 0$ and $Q \in \mathbb{C}[x]$. This finishes this elementary case.

We suppose now that $X$ is not hyperelliptic. The canonical embedding of $X$ is given by the graded $K$-algebra $K := \bigoplus_{m \geq 0} H^0(X, \Omega_X^\otimes m)$. The derivation $D$ on $K$ extends to the sheaf $\Omega_X$ by the formula

$$
D(\sum f_i dg_i) = \sum_i D(f_i) dg_i + f_i D(g_i).
$$

Because of PP, the $K$-vector space of the holomorphic forms $H^0(X, \Omega_X)$ is invariant under $D$. Likewise $D$ extends to the sheaf $\Omega_X^\otimes m$ and the $K$-vector spaces $H^0(X, \Omega_X^\otimes m)$. We claim that $H^0(X, \Omega_X)$ is a trivial differential module, i.e., the canonical map $K \otimes_K (\ker(D, H^0(X, \Omega_X))) \rightarrow H^0(X, \Omega_X)$ is a bijection. The differential modules $H^0(X, \Omega_X^\otimes m)$ are images of symmetric powers of $H^0(X, \Omega_X)$ and therefore also trivial. Let $R_0 := \ker(D, R)$. This is a graded algebra and the canonical map $K \otimes_K R_0 \rightarrow R$ is bijective. Now $R_0$ defines a canonical embedded curve $X_0$ over $\mathbb{C}$ and $X \cong K \otimes_K X_0$ and $D$ is zero on $\mathbb{C}(X_0)$.

**Proof of the claim.**

(a) Let $x$ denote a Weierstrass point and let $\pi$ be a generator of the maximal ideal of $O_{X, x}$. Then $D(\pi) \in \pi O_{X, x}$.

**Proof.** Let $[x]$ denote the point $x \in X$ considered as a divisor. Also, $L(n[x])$ denotes the $K$-vector space of the elements $f$ of $K(X)$ with divisor $div(f) \geq -n[x]$. Since $x$ is a Weierstrass point there exists $n \geq 1$ such that $L((n - 1)[x]) \neq L(n[x]) = L((n + 1)[x])$. Take $f \in L(n[x]) \setminus L((n - 1)[x])$. The completion of the local ring $O_{X, x}$ has the form $K[\pi]$ and $f = a \pi^{-n} + \cdots$ with $a \in K^*$. Now $D(f)$ can only have a pole at $x$ because of PP. Further $D(f) = (a' \pi^{-n} + \cdots) + (-na \pi^{-n-1} + \cdots) D(\pi)$ and $D(f) \in L((n + 1)[x])$. Since $L(n[x]) = L((n + 1)[x])$ we have that $D(\pi) \in \pi O_{X, x}$.

(b) $H^0(X, \Omega_X)$ has a basis $\overline{\omega}_1, \ldots, \overline{\omega}_g$ such that $D(\overline{\omega}_j) = 0$ for all $j$.

**Proof.** We fix a Weierstrass point $x$ and consider a basis $\omega_1, \ldots, \omega_g$ of $H^0(X, \Omega_X)$ such that $ord_x(\omega_1) < ord_x(\omega_2) < \cdots < ord_x(\omega_g)$.

Let $x = x_1, \ldots, x_r$ be the set of all Weierstrass points and let $\pi_j$ be a local parameter at $x_j$. Since $D(\pi_j) \in \pi_j O_{X, x_j}$, there is an action of $D$ on $\Omega_{X, x_j} / (\pi_j) \cong K$ and this action coincides with the differentiation on $K$.

Step by step we will change the basis $\{\omega_j\}$ such that at the end $D(\overline{\omega}_j) = 0$ holds for all $j$.

Step 1. $D(\omega_g) = a \omega_g$ for some $a \in K$. There exists a $j$ such that the image of $b \omega_g$ in $\Omega_{X, x_j} / (\pi_j) = K$ is non-zero, since $r > 2g + 2$. Then $b' = ab$ holds for some $b \in K^*$. Now $\overline{\omega}_g := \frac{1}{b} \omega_g$ satisfies $D(\overline{\omega}_g) = 0$. PLEASE NOTE: THIS IS A PORTION OF THE ORIGINAL TEXT AND MAY NOT BE COMPLETE.

Step 2. $D(\omega_{g-1}) = a\omega_{g-1} + b\bar{\omega}_g$ for some $a, b \in \mathbb{K}$. If $b \neq 0$, then we replace $\omega_{g-1}$ by $b^{-1}\omega_{g-1} - z\bar{\omega}_g$ and arrive at a new $\omega_{g-1}$ satisfying $D(\omega_{g-1}) = a\omega_{g-1}$ for some (new) $a \in \mathbb{K}$. As in Step 1, there exists $b \in \mathbb{K}^*$ such that $b' = ab$. Now $\bar{\omega}_g := \frac{1}{b'}\omega_{g-1}$ satisfies $D(\bar{\omega}_g) = 0$.

Step 3. $D(\omega_{g-2}) = a\omega_{g-2} + b\bar{\omega}_{g-1} + c\bar{\omega}_g$ for some $a, b, c \in \mathbb{K}$ is ‘normalized’ in a similar way.

By the method of Step 2 one makes $b$ zero. Applying the method again one makes $c$ zero. Then $D(\omega_{g-2}) = a\omega_{g-2}$ for some (new) $a \in \mathbb{K}$. As in Step 1, there exists $b \in \mathbb{K}^*$ with $b' = ab$ and $\bar{\omega}_{g-2} := \frac{1}{b'}\omega_{g-2}$ satisfies $D(\bar{\omega}_{g-2}) = 0$. Induction finishes the proof. □

**Algorithm 2.6.** Given is a differential equation $f = 0$ and the corresponding pair $(K(X), D)$. Testing $PP$ for $f = 0$ is done by Remark 2.2.

For $f = 0$ corresponding to $(X, D)$ having $PP$ we indicate how 2.3 and 2.5 can be made explicit.

First one computes the genus $g$ of $X$. If $g = 0$, then $K(X) = K(u)$ since $K$ is a $C_1$-field (see van Hoeij’s paper [van Hoeij, 1997] for an explicit computation of $u$). Then $D(u)$ has the form $a_0 + a_1 u + a_2 u^2$ because of $PP$. If not all $a_s$ are zero, then we have arrived at case (i). One may extend the algorithm by computing a ‘standard form’ for $D$, as in the algorithm presented in Section 3.

For $g = 1$, one has to produce a rational point of $X$ after enlarging $K$. A standard computation using this point and possibly a further finite extension of $K$, yields the affine equation $y^2 = x(x-1)(x-\lambda)$ with $\lambda \in K^*$, $\lambda \neq 0, 1$. It follows from $PP$ that $\lambda \in \mathbb{C}$. Let $D(x) = h \cdot y$ with $h \in \mathbb{K}$. If $h = 0$, then we have arrived at (ii). If $h = 0$, then we are in case (iii).

There are algorithms computing Weierstrass points for the case $g \geq 2$. One replaces $K$ by a finite extension such that all Weierstrass points are rational. From the number (and/or nature) of Weierstrass points one can see whether the curve is hyperelliptic or not. For a hyperelliptic curve $X$ one can follow the procedure as for elliptic curves.

For $g > 2$ and a non-hyperelliptic curve $X$ one applies Coates’ algorithm to get a basis of $H^0(X, \Omega)$. After choosing a Weierstrass point $x$ one can transform this into a basis $\omega_1, \ldots, \omega_s$ such that $ord_x(\omega_1) < \cdots < ord_x(\omega_s)$. Then one applies the proof of 2.5 and finds a basis $[\bar{\omega}_j]$ with all $D(\bar{\omega}_j) = 0$ of $H^0(X, \Omega)$. Using this basis one finds that the canonical embedding $X \to \text{Proj}(H^0(X, \Omega))$ induces a curve $X_0$ over $\mathbb{C}$ and an isomorphism $X \cong K \times \mathbb{C} X_0$. We note that an affine open subset of $X_0$ has coordinate ring $\mathbb{C}[\bar{\omega}_1, \ldots, \bar{\omega}_s]$. As we know (and show in Theorem 2.8), all solutions of $f = 0$ are algebraic for the case that we are considering. These solutions can be expressed in terms of $C(X_0)$, the field of fractions of $\mathbb{C}[\bar{\omega}_1, \ldots, \bar{\omega}_s]$.

**Theorem 2.7.** The following properties of $f$ over $K$, corresponding to $(X, D)$, are equivalent.

1. All solutions of $f$ are algebraic and lie in a fixed finite extension of $K$.
2. There exists a finite extension $L \supset K$ such that $L \times \mathbb{C}(X_0, D_0)$ where $X_0$ is a smooth, irreducible, projective curve over $\mathbb{C}$ and $D_0$ is the zero derivation on the function field of $X_0$ over $\mathbb{C}$.

**Proof.** (1) $\Rightarrow$ (2). Then $f$ has $PP$ since the only singularities, apart from moving poles, are the fixed branch points of the extension $L \supset K$.

In case (i) of Theorem 2.3, $K(X) = K(u)$ with $u = a_2 u^2 + a_1 u + a_0$ and there are at least three algebraic solutions $a_s$, $s = 0, 1, \infty$ of $u = a_2 u^2 + a_1 u + a_0$. The automorphism $A$, given by $a_s \mapsto *$ for $s = 0, 1, \infty$ has the property that $v := A u$ satisfies $D(v) = 0$. Thus (2) holds in this case with $L = K(a_0, \alpha_1, \alpha_\infty)$.

In case (ii) of Theorem 2.3, one has $K(X) = K(x, y)$ with $y^2 = x^3 + ax + b, a, b \in \mathbb{C}, D = \frac{a}{x^2} + h \cdot y \frac{2}{x}$ for some $h \in \mathbb{K}$. Let $x_0, y_0 \in L$, where $L$ is a finite extension of $K$, satisfying $y_0^2 = x_0^3 + ax_0 + b$ and $x_0 = h y_0$. Define elements $x_1, y_1 \in L(X)$ by the identity $(x, y) = (x_0, y_0) \oplus (x_1, y_1)$, where $\oplus$ is the addition on the elliptic curve $X$. Then $L(X) = (x_1, y_1)$ with $y_1^2 = x_1^3 + ax_1 + b$. Let $A$ denote the translation on the elliptic curve over the element $(x_0, y_0)$. A computation shows that $A \frac{a}{x^2} A^{-1} = \frac{a}{x^2} + \frac{a}{x_0^2} \cdot y \frac{2}{x}$ and $A y \frac{2}{x} A^{-1} = y \frac{2}{x}$. It follows that $D(x_1) = D(y_1) = 0$.

In case (iii) of Theorem 2.3, the explicit proof of Corollary 2.5 produces the required result and $L \supset K$ is the smallest field such that all Weierstrass points are rational over $L$.

(2) $\Rightarrow$ (1) is easily seen. □
Theorem 2.8. All solutions of \( f \) are algebraic if and only if the kernel of \( D \) on the function field of \( X \) is greater than \( \mathbb{C} \).

Proof. Let \( f \) correspond to the differential algebra \( R = K[s, t, \frac{1}{s^2}] \) and the pair \( (X, D) \). Suppose that \( D(u) = 0 \) for some \( u \in K(X) \), \( u \notin \mathbb{C} \). Write \( u = \frac{x}{y} \) with \( a, b \in R, b \neq 0 \). Let \( \phi : R \to \mathcal{F} \) be a solution. If \( \phi(b) = 0 \), then \( \phi \) is an algebraic solution since \( R/bR \) has finite dimension over \( K \). If \( \phi(b) \neq 0 \), then \( \phi(a) - c\phi(b) = 0 \) for some \( c \in \mathbb{C} \), because \( \left( \frac{x}{y} \right)' = 0 \). Hence \( \phi \) is again an algebraic solution.

On the other hand, suppose that the differential field \((K(X), D)\) has constants \(\mathbb{C}\). Then the inclusion \(\phi : R \to K(X)\) is a transcendental solution. \(\square\)

Example 2.9. The equation \((y')^a = y^b z^c\) with \(\gcd(a, b) = 1, 0 \neq a \neq b \) and \(c \neq -a\) has a full set of algebraic solutions, namely \(y = \left(\frac{a-b}{b+c} z^\frac{a+c}{b} + \lambda\right)^\frac{1}{a-b}\) with \(\lambda \in \mathbb{C}\). The equation has genus zero and its function field can be written as \(K(t)\) with \(y = t^a, y' = t^b z^c\) and \(t' = \frac{1}{a} t^{b-a+1} z^c\) for suitable \(K\). Then \(s := t^{a-b} - \frac{a-b}{b+c} z^{\frac{a+c}{b}}\) satisfies \(s' = 0\). The following are equivalent: (a) the equation has \(PP\), (b) \(|a - b| = 1\), (c) all solutions are in a fixed finite extension of \(K\).

Corollary 2.10. Let the autonomous equation \( f \) correspond to the pair \((X_0, D_0)\) where \(X_0\) is a curve over \(\mathbb{C}\) and \(D_0 \neq 0\) is a derivation of the function field of \(X_0\). Then all solutions of \( f \) are algebraic over \(\mathbb{C}(z)\) if and only if there exists a \(T \in \mathbb{C}(X_0)\) with \(D_0(T) = 1\).

Proof. The equation \( f \) defines a differential algebra \(R_0 := \mathbb{C}[s, t, \frac{1}{s^2}]\) over \(\mathbb{C}\). The kernel of a (non-constant) solution \(\phi : R_0 \to K\) with \([K : \mathbb{C}(z)] < \infty\) is zero, since \(D_0 \neq 0\). Thus \(\phi\) extends to an embedding \(\phi : \mathbb{C}(X_0) \subset K\). The differential equation \(D_0(T) = 1\) has an algebraic solution, say \(z\), in \(K\). The differential Galois group of the equation \(D(T) = 1\) is a finite subgroup of the additive group \(\mathbb{G}_a\) and hence \(z \in \mathbb{C}(X_0)\).

On the other hand, suppose that there exists a \(T \in \mathbb{C}(X_0)\) with \(D(T) = 1\). Then any solution \(\phi\) satisfies \(\phi(T) = z + c\) for some \(c \in \mathbb{C}\). The solution \(\phi\) is algebraic since \([\mathbb{C}(X_0) : \mathbb{C}(T)] < \infty\). \(\square\)

Algorithm 2.11. Solving \(D(T) = 1\) for the autonomous equation \( f(y', y) = 0\).

Let the pair \((X, D)\) be induced by \(f\). According to Corollary 2.10, it suffices to produce an algorithm for finding a solution of \(D(T) = 1\) with \(T \in \mathbb{C}(X)\). Consider a closed point \(x \in X\) with local parameter \(p\). Then \(\tilde{O}_{X,x} = \mathbb{C}[p]\).

A local solution at \(x\) has the form \(T = a_0 k^{p^k} + a_{k+1} p^{k+1} + \cdots \in \mathbb{C}[[p]]\) and \(D(T) = (ka_0 p^{k-1} + \cdots)D(p) = 1\).

If \(D(p)\) has no pole or zero, then \(T = a_0 + a_1 p + \cdots\) with \(a_0 \neq 0\).

If \(D(p)\) has a zero, \(D(p) = b_k p^k + \cdots, b_k \neq 0, k \geq 1\), then \(k = 1\) is not possible and for \(k > 1\) one has that \(T\) has a pole of order \(k - 1\).

If \(D(p)\) has a pole of order \(-k\), then \(T\) has a zero of order \(k + 1\).

It follows that a possible \(T\) with \(D(T) = 1\) lies in \(H^0(X, L)\) for a known line bundle \(L\). The Coates algorithm (Coates, 1970) produces a basis \(\beta\) of \(H^0(X, L)\). Testing \(D(T) = 1\) now reduces to expressing 1 as a \(\mathbb{C}\) linear combination of \(D(\beta)\).

Remarks 2.12. (1) Corollary 2.10 is essentially present in Aroca et al. (2005) and 2.11 improves on the algorithm proposed in that paper.

(2) From the text one can deduce an algorithmic version of Theorem 2.7(1). However, we have not been able to do this for Theorem 2.8, i.e., we do not know how to verify \(\ker(D, K(X)) \neq \mathbb{C}\).

(3) Suppose that the equation \( f \) has \(PP\) and the genus of \( f \) is zero, then computing algebraic solutions amounts to computing algebraic solutions of a Riccati equation \(u' = a_2 u^2 + a_1 u + a_0\). There are algorithms for this problem.

In contrast to the above, we do not know whether a simple equation like \(y' = y^3 + z\) over \(\mathbb{C}(z)\) has an algebraic solution. The local solutions are:
For $z = a \neq \infty$ there is a holomorphic solution $y \in \mathbb{C}[z - a]$, depending on the initial value $y(a)$. Moreover there is a ramified meromorphic solution $y = \sum_{n \geq -1} a_n (z_a)^n/2$ in $\mathbb{C}((z - a))$, depending on $a_{-1}$ and $a_{-1}^2 = -\frac{1}{2}$.

For $z = \infty$ and with $t := \frac{1}{z}$ the equation reads $\frac{dy}{dt} = -t^2 y^3 - t^3$. The solutions are $y = \sum_{n \geq -1} c_n t^{n/3}$ in $C(t^{1/3})$, depending on $c_{-1}$ and $c_{-1}^2 = -1$.

An algebraic solution $y$ has to be ramified at $z = \infty$ of order 3 (and thus $y$ is not rational) and is ramified at some more points with ramification of order 2. However we have no idea what the other ramification points for $y$ could be and what the degree of $y$ over $\mathbb{C}(z)$ could be.

3. Testing strict equivalence

Given are two first order differential equations $f_1, f_2$ over a field $K$ with $[K : \mathbb{C}(z)] < \infty$. Let $(X_1, D_1), (X_2, D_2)$ correspond the ramified pairs of a curve over $K$ and a derivation of its function field extending the derivation of $K$. We present a procedure testing strict equivalence of $f_1$ and $f_2$. This amounts to testing whether there exist a finite extension $L \supset K$ and an isomorphism $L \times (X_1, D_1) \rightarrow L \times (X_2, D_2)$. As can be seen from the description below, the procedure can be made into an algorithm except for the case that the curves $X_j$ have genus 1 and moreover the derivations $D_j$ have no poles (equivalently, $f_1$ and $f_2$ have genus 1 and satisfy PP).

The first step is to compute the genus of $X_1$ and $X_2$ and to apply known algorithms testing whether $\overline{K} \times X_1$ is isomorphic to $\overline{K} \times X_2$. In case this question has a positive answer we may suppose (after extending $K$) that $X_1$ and $X_2$ are equal to a curve $X$ over $K$. The question is now whether there exists an automorphism $A$ of $\LX$ (with $[L : K] < \infty$) such that $AD_1A^{-1} = D_2$.

Suppose that the genus of $X$ is $\geq 2$. Then the automorphism group of $X$ is finite. Moreover, if $K$ is sufficiently large, then $\LX$ has the same group of automorphisms as $X$ for any $[L : K] < \infty$. There are algorithms computing the automorphism group of $X$, see Appendix A. Using these one can test whether $AD_1A^{-1} = D_2$.

Suppose that $X$ has genus one. Let $S_j$, for $j = 1, 2$, denote the set of poles of $D_j$, i.e., $S_j$ consists of the closed points $x \in X$ such that $D_j(Ox,x) \not\subset Ox_x$. We may suppose that $S_1, S_2 \subset X(K)$. An automorphism $A$ with $AD_1A^{-1} = D_2$ has the property $A(S_1) = S_2$. In particular we have to require that $\#S_1 = \#S_2$. If $S_j \not= \emptyset$, then there are only finitely many possibilities for $A$ (and these are computable). We are left with the case $S_1 = S_2 = \emptyset$. By Theorem 2.3, $K(X) = K(x, y)$, $y^2 = x^3 + ax + b$, $a, b \in \mathbb{C}$, $D_j = \partial / \partial x + h_j \cdot y \partial / \partial y$ with $h_j \in K$ for $j = 1, 2$. The group of the automorphism of the elliptic curve $\overline{K} \times X$ is a semi-direct product of the normal subgroup $X(\overline{K})$ of translations and the finite cyclic group $Aut(X, 0)$, i.e., the automorphism of $X$ which preserves the neutral element $0 \in X$. The action of $A \in Aut(X, 0)$ has the property $\partial / \partial x A^{-1} = \partial / \partial x$ and $A(y \partial / \partial x)A^{-1} = \zeta y \partial / \partial x$ with $\zeta^2 = 1$ and for special elliptic curves $X$ also $\zeta^2 = 1$ or $\zeta^3 = 1$. Thus the group $Aut(X, 0)$ poses no problems for the algorithm that we want to produce.

Let $A \in X(\overline{K})$ be translation over a point $(x_0, y_0) \in X(\overline{K})$. Then $A \partial / \partial x A^{-1} = \partial / \partial x + y_0 \partial / \partial y$ and $\partial / \partial x A^{-1} = y \partial / \partial x$. Here the second formula is obvious, the first one requires a computation. Thus the problem of strict equivalence amounts to testing whether for a given $h \in \overline{K}$ the equation $\frac{dy}{dx} = h$ has a solution $(x_0, y_0) \in X(\overline{K})$.

Define the map $X(\overline{K}) \rightarrow \Omega_{\overline{K} / \mathbb{C}}$ by $(x_0, y_0) \mapsto \frac{dy}{dx}$. This map is additive because of the above formulas, and its kernel is $X(\mathbb{C})$.

We start investigating the above problem by assuming that $h \in \overline{K}$, $h \not= 0$ satisfies $h \partial / \partial y$ for some $(x_0, y_0) \in X(\overline{K})$. Write $K = \mathbb{C}(x, h)$ and $L = \mathbb{C}(x_0, y_0)$. Then $K \subset L \subset \overline{K}$. For every $\sigma \in Gal(\overline{K}/K)$ one has $h \partial / \partial y = \sigma h \partial / \partial y \sigma$. Thus $(\sigma x_0, \sigma y_0) = (x_0, y_0) \oplus (\alpha, \beta)$, where $\oplus$ denotes the addition on the elliptic curve $(X, 0)$ and $(\alpha, \beta) \in X(\mathbb{C})$. It follows that $x_0, y_0 \in L$. Therefore $L \supset K$ is a Galois extension with Galois group $G \subset X(\mathbb{C})$. Now $(x_0, y_0) := \oplus_{\sigma \in G} (\sigma x_0, \sigma y_0)$ (again $\oplus$ denotes the addition on the elliptic curve) has the properties $(x_0, y_0) \in X(K)$ and $\frac{dy}{dx} = (\#G) \cdot h \partial / \partial y$.

The group $X(K)/X(\mathbb{C})$ is a Lang–Néron group (or a relative Mordell–Weil group) of the elliptic curve $X$. It is known that this group is finitely generated (and free in this special case). Given an
algorithm for finding a basis \((x_1, y_1), \ldots, (x_m, y_m)\) of \(\mathbb{Q} \otimes_{\mathbb{Z}} X(K)/X(\mathbb{C})\), testing and solving the equation \(h dz \in \mathbb{Q} \frac{dx_1}{y_1} + \cdots + \mathbb{Q} \frac{dx_m}{y_m}\) is easy.

Suppose that \(X\) has genus zero. \(S_j\) denotes, for \(j = 1, 2\), the set of poles of \(D_j\). We may assume that 
\(#S_1 = #S_2\) and \(S_1, S_2 \subset X(K)\). If \(#S_1 \geq 3\), then there are only finitely many automorphisms \(A\) with \(A(S_1) = S_2\) and we can test \(AD_1A^{-1} = D_2\).

Suppose \(#S_1 = #S_2 = 2\). Then we can suppose that \(S_1 = S_2 = [0, \infty), K(X) = K(u)\) and \(D(u) = \sum a_nu^n, D(v) = \sum b_nv^n\). The only automorphisms of \(K \times X\) that preserve \((0, \infty)\) are \(A(u) = hu^\pm 1\) with \(h \in \mathbb{K}^\times\). Write \(v = hu\), then \(D_1(v) = \frac{h^2}{u^2}v + \sum a_nh^{1-n}v^n\) and it is easy to test whether this equals \(\sum b_nv^n\) for a suitable \(h\). The same holds for the transformation \(u \mapsto hu^{-1}\).

Suppose that \(#S_1 = #S_2 = 1\). Now we may assume \(S_1 = S_2 = \{\infty\}\) and \(D(u), D_2(u) \in K[u]\) have degrees \(> 2\). The only transformations that we have to consider are \(A(u) = au + b\) with \(a, b \in \mathbb{K}, a \neq 0\). It is easy to test whether for suitable \(a, b\) the equality \(AD_1A^{-1} = D_2\) holds.

Suppose that \(S_1 = S_2 = \emptyset\). For a pair \((X, D)\) we define a closed point \(x \in X\) to be a zero of \(D\) if \(D\) maps the maximal ideal of \(O_{X, x}\) to itself. For the case \(K(X) = K(u)\), \(D(u) = a_0 + a_1u + a_2u^2\) one has: if \(x\) is a zero if and only if \(a_0 = 0\), \(\infty\) is a zero if and only if \(a_2 = 0\) and the point \(a \in \mathbb{K}^\times\), \(a \neq 0\).

Thus the zero's in \(\mathbb{K}\) of \(D\) are the algebraic solutions of the Riccati equation \(u' = a_0 + a_1u + a_2u^2\). Using known algorithms for testing and computing algebraic solutions of the Riccati equation (see Appendix A) one obtains a classification of the strict equivalence classes of the pairs \((K(u), D)\) with \(D(u) = a_0 + a_1u + a_2u^2\), namely:

1. If \(D\) has \(\geq 3\) zero's, then \(D\) is strict equivalent to \(\hat{D}\) with \(\hat{D}(u) = 0\) and moreover all solutions of the Riccati equation are algebraic.
2. If \(D\) has two zero's, then we may suppose that the zero’s are \(0, \infty\) and thus \(D(u) = a_1u\) and \(a_1 = \frac{b_1}{b_0}\) has no solution in \(\mathbb{K}^\times\).
3. If \(D\) has one zero, then we may suppose that this zero is \(\infty\) and \(D(u) = a_0 + a_1u\) and the equation \(b' = a_0 + a_1b\) has no solution in \(\mathbb{K}^\times\).
4. If \(D\) has no zero's, then the after changing \(u\) we obtain \(D(u) = a_0 - u^2\) and \(b' + b^2 = a_0\) has no solution in \(\mathbb{K}^\times\).

Given are two derivations \(D_1, D_2\) of \(K(u)\) without poles. For the question of strict equivalence we may suppose that they have the same number of zero's. We can skip case (a). For case (b) one has \(D_1(u) = a_1u, D_2(u) = a_0u\) and the only automorphisms \(A\) of \(K(u)\) that we have to consider are \(A(u) = hu \cdot u^{-1}\). This leads to the equations \(\frac{a_0}{a_1} = a_2 - a_1\) or \(b_0 = a_1 + a_2\) for \(h \in \mathbb{K}^\times\). There are algorithms testing and solving these equations (see the discussion of Question 5 in Appendix A). For case (c), \(D_1(u) = a_0 + a_1u, D_2(u) = b_0 + b_1u\) and the only automorphism to consider is \(A(u) = au + b\) with \(a, b \in \mathbb{K}\). For \(a = au + b\) one has \(D_1(v) = -a_1 + \frac{a_0}{b} - b' + a_0a + (\frac{a_0}{b} + a_1)v\). First we have to solve (if possible) \(\frac{a_0}{b} + a_1 = b_1\). If we obtain a solution \(a\), then we may suppose \(a_1 = b_1\) and we have to consider \(v = b + cv\) with \(c \in \mathbb{C}^\times\), \(b \in \mathbb{K}\) such that \(D_1(v) = ca_0 + b' - a_1b + a_1c = ca_0 + b_0 + a_1v\). We are left with an inhomogeneous equation \(b' - a_1b = -ca_0 + b_0\) which is called a Risch equation. We now present an algorithm finding all algebraic solutions for such an equation. Note that an extensive literature on the Risch equation exists, e.g. Singer (1991); Raab (2012). However we could not find an answer there for our specific question.

Let \(K = \mathbb{C}(z, a_0, b_0)\) and suppose that the equation \(b' - a_1b = -ca_0 + b_0\) has a solution \(b\) in a Galois extension of \(L\) of \(K\). For an element \(\sigma\) in the Galois group of \(L/K\), \(\sigma(b)\) is a solution of the same equation and thus \((\sigma(b) - b') = a_1(\sigma(b) - b)\). Suppose that the equation \(f' = a_1f\) has no solution in \(\mathbb{K}^\times\). Then \(b \in \mathbb{K}\). Let \(Z\) denote the curve over \(\mathbb{C}\) with function field \(K\). For the (possible) solution \(b\) one can compute for each point of \(Z\) a bound for the pole of \(b\) at that point. This produces a positive divisor \(D\) and \(\text{div}(b) \geq -D\). Let \(b_1, \ldots, b_t\) be a basis of the vector space \((f \in K \mid \text{div}(f) \geq -D)\). Then one has to verify whether the equation \(\sum c_j(b_j' - a_1b_j) = ca_0 + b_0\) with \(c_1, \ldots, c_t \in \mathbb{C}, c \in \mathbb{C}^\times\) has a solution and compute the solution if it exists.

Suppose that \(a_1 = \frac{f'}{f}\) for some \(f \in \mathbb{K}^\times\). Then one can change \(a_1\) to 0 and the equation becomes \(b' = ca_0 + b_0\) with \(c \in \mathbb{C}^\times\). Now a possible algebraic solution \(b\) lies in the field \(K = \mathbb{C}(z, a_0, b_0)\). Again
one computes a positive divisor $D$ on the curve of the field $K$ such that a possible solution $b$ satisfies $\text{div}(b) \geq -D$. As above one can decide whether a solution exists and compute $b$ in case $b$ exists.

Finally, we treat the complicated case (d): $D_1(u) + u^2 = a_0$, $D_2(u) + u^2 = b_0$. A direct computation relating $a_0$ and $b_0$ if $D_1$ is strictly equivalent to $D_2$ seems possible. We follow a more sophisticated approach. Consider the equations $Y''_1 = a_0 Y_1$ and $Y''_2 = b_0 Y_2$ over the differential field $R$. If the equations are equivalent then $\left( \begin{array}{c} a_0 \\ b_0 \end{array} \right) \in \text{GL}_2(R)$ exists such that $Y_2 = \alpha Y_1 + \beta Y'_1$ and $Y'_2 = \gamma Y_2 + \delta Y'_2$. Then $u_2 := \frac{Y'_2}{Y_2} = \frac{\gamma + \beta u_1}{\alpha + \beta u_1}$ with $u_1 := \frac{Y_1}{Y'}$. Conversely, if $u'_1 + u_1^2 = a_0$ and $u'_2 + u_2^2 = b_0$ are equivalent then the same holds for the linear equations.

Let $PV \supset R$ be its Picard–Vessiot field and let $G \subset \text{SL}_2$ be its differential Galois group. Since $G$ is connected and the Riccati equation $u' + u^2 = a_0$ has no algebraic solutions we have $G = \text{SL}_2$. Let $L \subset PV$ be the field of invariants under the subgroup $\{ \pm 1 \}$ of $G$. Then $L$ is the Picard–Vessiot field of the second symmetric power $M_1$ of the equation $Y'' = a_0 Y$ and the corresponding differential Galois group is $\text{PSL}_2$. Let $\text{Sol} \subset PV$ denote the solution space of $Y'' = a_0 Y$ and let $T := \{ \frac{Y'}{Y} \mid y \in \text{Sol}, \ y \neq 0 \}$. Then $T$ is invariant under $G$ and $L$ is the smallest differential subfield of $PV$ containing $T$, since the normal subgroups of $\text{PSL}_2$ are trivial. We note that $T$ is the solution space of the Riccati equation $u' + u^2 = a_0$.

Assume that $D_1$ and $D_2$ are strictly equivalent. Consider the equation $Y'' = b_0 Y$ over the differential field $R$, which also has differential Galois group $\text{SL}_2$. The same equation over the differential field $L$ has infinitely many solutions for the Riccati equation, since $D_1$ and $D_2$ are strictly equivalent. It follows that $L$ is also the Picard–Vessiot field of the second symmetric power $M_2$ of $Y'' = b_0 Y$. By a Tannakian argument, the existence of only one irreducible representation of $\text{PSL}_2$ of dimension 3 implies that the differential modules $R \otimes M_1$ and $R \otimes M_2$ are isomorphic. Let $N_1, N_2$ denote the two dimensional differential modules corresponding to $Y'' = a_0 Y$ and $Y'' = b_0 Y$. Using ideas from van Hoeij and van der Put (2006, Proposition 3.1), applied to the case of the differential field $R$, one can show the existence of an isomorphism $R \otimes N_1 \to R \otimes N_2$. For completeness we give details.

**Lemma 3.1.** Let $A_1$ and $A_2$ be two-dimensional differential modules over $R$ with differential Galois group $\text{SL}_2$. Suppose that the second symmetric powers $B_j = \text{sym}^2 A_j$, $j = 1, 2$, are isomorphic. Then $A_1$ and $A_2$ are isomorphic.

**Proof.** Choose a basis $v_1, v_2$ of $A_1$ such that $\partial(v_1 \wedge v_2) = 0$. Then $B_1$ has basis $w_1 = v_1^2$, $w_2 = v_2^2$, $w_3 = v_1 v_2$. Put $F_1 = w_3^2 - w_1 w_2, w_3 \in \text{sym}^2(B_1)$. Then $\partial F_1 = 0$, $R F_1$ is the only 1-dimensional submodule of $\text{sym}^2 B_1$, and $F_1$ is unique up to multiplication by an element in $C^\ast$. Further $R F_1$ is the kernel of the canonical surjective morphism of differential modules $\text{sym}^2 B_1 \to \text{sym}^4 A_1$.

Let $F_2 \in \text{sym}^2 B_2$ satisfy $\partial F_2 = 0$ and $F_2 \neq 0$. Let $\phi : B_1 \to B_2$ be an isomorphism of differential modules. Then $\phi$ induces an isomorphism $\text{sym}^2(\phi) : \text{sym}^2 B_1 \to \text{sym}^2 B_2$ which sends $F_1$ to $c F_2$ for some $c \in C^\ast$. Indeed, $F_1$ and $F_2$ are unique up to multiplication by a scalar. Since $R$ is algebraically closed there exists a $R$-linear bijection $\psi : A_1 \to A_2$ with $\text{sym}^2(\psi) = c \phi$ for some $c \in C^\ast$. From $\phi \circ \partial = \partial \circ \psi$ it easily follows that $\psi \circ \partial = \partial \circ \psi$. In other words $\psi : A_1 \to A_2$ is an isomorphism of differential modules. □

Testing whether $R \otimes N_1$ and $R \otimes N_2$ are isomorphic amounts to finding a non-trivial algebraic solution of the 4-dimensional differential module $M := \text{Hom}(N_1, N_2)$ over $K$. An explicit algorithm for this is the following.

Suppose that an isomorphism $f : R \otimes N_1 \to R \otimes N_2$ exists. Then $f$ is unique up multiplication by an element in $C^\ast$. Further, $R f \subset R \otimes M$ is a 1-dimensional submodule, invariant under $\text{Gal}(f/R)$. Hence $M$ has a 1-dimensional submodule $N$ and this 1-dimensional submodule is unique. Now $M$ is semi-simple, because $N_1$ and $N_2$ are simple. Then $M = N \oplus \bar{M}$. The projection $P : M \to N \subset M$ with kernel $M$ is an element of the $C$-algebra $\ker(\partial, \text{End}(M))$ and can be computed. If the 1-dimensional $N \subset M$ is found, then one writes $N = Ke$ with $\partial(e) = he$ and $h \in K$. Finally one has to solve (if possible) $\frac{e}{h} = h$ with $a \in R^\ast$. 

4. Strict equivalence with an autonomous equation

Given is an equation $f$ over $K$, its differential algebra $K[s, t, \frac{1}{2}]$ and the pair $(X, D)$. We present a procedure which investigates whether $(X, D)$ is strictly equivalent to an autonomous equation, i.e., a pair $(X_0, D_0)$ of a curve over $\mathbb{C}$ and a derivation $D_0$ of $\mathbb{C}(X_0)$. This is a complete algorithm in many cases; however, in one special case (namely, a curve $X$ of genus one with a derivation $D$ without poles) the problem is reduced to being able to answer Question 5 discussed in Appendix A.

The first step is to test whether $\overline{K} \times_K X$ is isomorphic to some $\overline{K} \times_{\mathbb{C}} X_0$. Algorithms for this question are known, see Appendix A, Question 2. Now we assume that $X = \mathbb{K} \times_{\mathbb{C}} X_0$ and we have to investigate whether there exists an automorphism $A$ of $L \times_{\mathbb{C}} X_0$ (where $L$ is a finite extension of $K$) such that the derivation $A D A^{-1}$ leaves the subfield $\mathbb{C}(X_0)$ of $L(X_0)$ invariant.

Suppose that the genus of $f$ is $\geq 2$. Then $X_0$ and $L \times_{\mathbb{C}} X_0$ have the same finite group of automorphisms. In particular, any automorphism $A$ leaves $\mathbb{C}(X_0)$ invariant and we have just to test whether $D$ leaves $\mathbb{C}(X_0)$ invariant.

Suppose that $f$ has genus one. Consider $X_0$ as an elliptic curve with neutral element 0. Let $S$ be the set of poles of $D$, i.e., the closed points $x \in X$ such that $D(O_{X,x}) \not\subset O_{X,x}$. After extending $K$ we may assume $S \subset X_0(K)$. We note that for an automorphism $A$ one has $A^{-1}(O_{X,x}) = O_{X,Ax}$. Thus the set of poles of $A D A^{-1}$ is $A(S)$. Therefore the first step to do is to produce an automorphism $A$ of $X$ such that $A(S) \subset X_0(\mathbb{C})$.

Suppose that $S$ has at least one point $x$. We may suppose that $x = 0 \in X_0(\mathbb{C})$. Further we have only to consider derivations $A D A^{-1}$ with $A(0) = 0$. Since $\mathbb{C}(X_0)$ is invariant under $A$ it suffices to test whether $\mathbb{C}(X_0)$ is invariant under $D$.

Suppose that $S = \emptyset$. Then $K(X) = K(x, y)$ with $y^2 = x^3 + ax + b$, $a, b \in \mathbb{C}$ and $D = \frac{\partial}{\partial x} h \cdot y \frac{\partial}{\partial y}$ with $h \in K$. In order to test strict equivalence with an autonomous equation one needs, according to Section 3, to solve an equation $h + c = \frac{X}{Y^2}$ with $c \in \mathbb{C}$ and $(x_0, y_0) \in X(\mathbb{K})$. Let $K = \mathbb{C}(z, h)$. A necessary condition is that $(h + c)dz$ is a holomorphic differential on the curve $Z$ with function field $K$. This determines $c$. As in Section 3, we have to solve $N(h + c)dz = \frac{dx}{Y^2}$ for some element $(x_*, y_*) \in X(\mathbb{K})$ and an integer $N \geq 1$. If an explicit basis of the Lang–Néron group $\mathbb{Q} \otimes_{\mathbb{Z}} X(\mathbb{K})/X(\mathbb{C})$ can be computed, then one can test and solve the equation $N(h + c)dz = \frac{dx}{Y^2}$.

Suppose that the genus of $f$ is zero. The function field of $X_0$ is $\mathbb{C}(u)$ and $D(u)$ is some element of $K(u)$. Let $S$ denote again the set of poles of $D$. We may suppose that $S \subset X_0(\mathbb{K})$.

Suppose that $\# S \geq 3$. After an automorphism of $X$ we may suppose that $S$ contains 0, 1, $\infty$. For strictly equivalence with an autonomous equation we have just to test whether $D(u) \in \mathbb{C}(u)$.

Suppose that $\# S = 2$. Then we may suppose that $S = \{0, \infty\}$. Thus $D(u) = \sum n_a h^n u^n$ in $K[u, u^{-1}]$ and we have to consider only the automorphisms $A$ with $A(0) = 0$, $A(\infty) = \infty$. Thus $Au = h u$ for some $h \in \overline{K}^*$. Write $v = hu$. Then $D(v) = h^2 v + \sum n_a h^{n-1} v^n$. We have to test whether for a suitable $h$ one has $D(v) \in \mathbb{C}[v, v^{-1}]$. Let $n_0$ be minimal with $a_0 \neq 0$ and $n_1$ maximal with $a_0 = 0$. Then $n_0 < n_1 > 2$. One obtains the conditions $a_0 n_i h^{-n_i+1}, a_0 n_i h^{-n_i+1} \in \mathbb{C}^*$. Substitution of, say, $h = c \cdot (a_0)^{1/(n_0-1)}$ with $c \in \mathbb{C}^*$ leads to a solution of the above question.

Suppose that $\# S = 1$. Then we may suppose that $S = \{\infty\}$ and $D(u) = \sum n_{a_i} u^n$. We have to search for $v = au + b$ with $a, b \in \overline{R}$, $a \neq 0$, such that $D(v) \in \mathbb{C}[v]$. Now $D(v) = b^2 + a (v-b) + a \sum n_a (\frac{v-b}{a})^n$. We note that the maximal $n_1$ with $a_0 \neq 0$ satisfies $n_1 > 2$. Then $a = c \cdot a_0^{1/(n_1-1)}$ for some $c \in \mathbb{C}^*$. The coefficient of $v^{n-1}$ in the expression for $D(v)$ yields a formula for $b$. Substitution leads to a solution of the question.

Suppose that $\# S = 0$. We have $D(u) = a_0 + a_1 u + a_2 u^2$. An autonomous $E$ can be normalized to $E(v) = \lambda v$ with $\lambda \in \mathbb{C}$. A necessary condition is that $D$ has at least two zero's. Thus one has to compute algebraic solutions of $u' = a_0 + a_1 u + a_2 u^2$. Suppose that we found two of these, then $D$ is strictly equivalent to a $D$ of the form $D(u) = a_1 u$. Finally we have to solve the equation $a_1 = \lambda + \frac{b}{a}$ for some $b \in \overline{K}^*$ and $\lambda \in \mathbb{C}$. This amounts to producing $\lambda \in \mathbb{C}$ such that $(a_1 - \lambda) dz$ has at most poles of order one. This determines $\lambda$. If this meets success, then we have to apply a known algorithm (see Risch, 1970 and Baldassarri and Dwork, 1979) to test and solve the equation $\frac{db}{\lambda} = (a_1 - \lambda) dz$ with $b \in \overline{K}^*$.
5. Autonomous equations

As before, we associate to an irreducible autonomous equation \( f(y', y) = 0 \) (it is assume that both \( y \) and \( y' \) are present in \( f \)) the pair \((X, D)\) where the complete, irreducible, smooth curve \( X \) over \( \mathbb{C} \) has function field \( \mathbb{C}(y_1, y_2) \) with equation \( f(y_1, y_2) = 0 \) and \( D \) is the meromorphic vector field on \( X \) determined by \( D(y_0) = y_1 \). Two pairs \((X_1, D_1), (X_2, D_2)\) are called isomorphic if there exists an isomorphism \( \phi : X_1 \to X_2 \) of complex curves such that \( \phi_* D_1 = D_2 \).

**Theorem 5.1.** The map \( f \mapsto (X, D) \) yields a bijection between strict equivalence classes of autonomous equations and isomorphism classes of pairs \((X, D)\), where \( X \) is a curve over \( \mathbb{C} \) and \( D \) a non-zero meromorphic vector field.

We note that the question whether two pairs \((X_1, D_1), (X_2, D_2)\) are isomorphic has an algorithmic answer. Indeed, according to Appendix A there is an algorithm testing whether \( X_1 \) and \( X_2 \) are isomorphic. If they are, the algorithm finds an explicit isomorphism. This reduces the problem to the case \( X = X_1 = X_2 \). Here we need to test whether \( \phi_* D_1 = D_2 \) for a suitable isomorphism \( \phi \). If the genus of \( X \) is at least two, the finitely many automorphisms of the curve are known explicitly. For smaller genus, the form of \( \phi \) is known and the test is equivalent to determining whether a finite system of polynomial equations over the complex numbers has a solution.

The proof of Theorem 5.1 will be given in 5.2 and 5.4.

**Lemma 5.2.** Every pair \((X, D)\), as in Theorem 5.1, is associated to some autonomous equation \( f(y', y) = 0 \).

**Proof.** Let \( g \in \mathbb{C}(X) \) satisfy \( D(g) \neq 0 \). Choose a closed point \( x \in X \) such that \( g \) has no pole at \( x \), \( \text{ord}_x(g - g(x)) = 1 \) and \( \text{ord}_x(D(g)) = 0 \). Let \( \pi \) denote a local parameter at \( x \). Then \( \partial_x = \mathbb{C}[\pi] \) and \( \text{ord}_x(D(\pi)) = 0 \). Let \( p \) be a prime number such that \( p > 2 \cdot \text{genus}(X) + 2 \) and let \( f \in \mathbb{C}(X) \) have a pole of order \( p \) at \( x \) and no further poles. Then \( [\mathbb{C}(X) : \mathbb{C}(f)] = p \). If \( D(f) \notin \mathbb{C}(f) \), then \( \mathbb{C}(f, D(f)) = \mathbb{C}(X) \) since \( p \) is a prime number.

Suppose that \( D(f) \in \mathbb{C}(f) \). Then \( \frac{D(f)}{f} \in \mathbb{C}(f) \subset \mathbb{C}(\mathbb{C}(f)) \). This contradicts the fact that \( \text{ord}_x(\frac{D(f)}{f}) = -1 \).

**Remark 5.3.** An irreducible order one equation \( f(y', y, z) = 0 \) over a finite field extension \( K \) of \( \mathbb{C}(z) \) induces a pair \((X, D)\) of a curve \( X \) over \( K \) and a derivation \( D \) of the function field of \( X/K \). The proof of Lemma 5.2 extends to this non-autonomous case. The statement is: For a given pair \((X, D)\) over \( K \), there exist a finite extension \( \overline{K} \) of \( K \) and an irreducible order one equation \( f(y', y, z) = 0 \) over \( \overline{K} \) which induces \( \overline{K} \times_K X \) equipped with the unique extension of \( D \).

Let \((X, D)\) be a pair as in Theorem 5.1 and let \( K \) be a finite extension of \( \mathbb{C}(z) \). We denote by \( K \times (X, D) \) the curve \( K \times \mathbb{C}(X) \) with function field \( K \otimes_{\mathbb{C}} \mathbb{C}(X) \) equipped with the derivation \( D^+ \) defined by \( D^+ = \frac{D}{z} \) on \( K \) and \( D^+ = D \) on \( \mathbb{C}(X) \). We note that \( D^+ \) is not a meromorphic vector field since it is not zero on \( K \).

**Lemma 5.4.** Let \( \phi : K \times (X_1, D_1) \to K \times (X_2, D_2) \) be an isomorphism. Then there exists an isomorphism \((X_1, D_1) \to (X_2, D_2)\).

**Proof.** The isomorphism \( \phi : K \times X_1 \to K \times X_2 \) induces an isomorphism \( \phi_1 : \text{spec}(R) \times X_1 \to \text{spec}(R) \times X_2 \) for some finitely generated \( \mathbb{C}\)-algebra \( R \) with field of fractions \( K \). After dividing by a maximal ideal of \( R \) we find an isomorphism \( X_1 \to X_2 \). In the sequel we identify \( X_1 \) and \( X_2 \) with some \( X \). It is given that some automorphism \( \phi \) of \( K \times X \) has the property \( D^+_2 = \phi^* D^+_1 \phi^{-1} \). We have to show that there exists an automorphism \( \psi \) of \( X \) with \( D^+_2 = \psi D^+_1 \psi^{-1} \).

If the genus of \( X \) is \( \geq 2 \), then \( \overline{K} \times X \) and \( X \) have the same finite group of automorphisms and there is nothing to prove.
Let the genus of $X$ be one. After choosing a point $0 \in X$ we regard $X$ as the elliptic curve with equation $y^2 = x^3 + ax + b$. Further, for $j = 1, 2$, one has $D_j = h_j \cdot y^{\partial h_j}$ with $h_1, h_2 \in \mathbb{C}(X)$. Suppose that $K \times X$ has an automorphism $A$ such that $AD_j^+A^{-1} = D_j^+$, where $D_j^+ = \frac{\partial}{\partial x} + h_j \cdot y^{\partial h_j}$ for $j = 1, 2$.

We may suppose that $A$ is a translation on $K \times X$ over a point $(x_0, y_0) \in X(K)$. Now $A\frac{\partial}{\partial x}A^{-1} = \frac{\partial}{\partial x} + \frac{x_0}{y_0}$ and $A(h_1 \cdot y^{\partial h_j})A^{-1} = A(h_1) \cdot y^{\partial h_j}$. Hence $A(h_1) + \frac{x_0}{y_0} = h_2$.

Suppose that $h_1$ has no poles, then the same holds for $h_2$ and $\frac{x_0}{y_0} = h_2 - h_1 = c \in \mathbb{C}$. Suppose $c \neq 0$. Then $(x_0)^2 = c^2y_0^2 = c^2(x_0^3 + ax_0 + b)$. The non-constant solutions of this Weierstrass equation are transcendental (since they are doubly periodic), contradicting the algebraicity of $x_0$.

Suppose that $X$ has genus zero. Write $C(X) = C(y)$ and $D_j(y) = h_j \in C(y)$ and $D_j^+ = \frac{\partial}{\partial x} + h_j \cdot y$ for $j = 1, 2$. We note that the poles of $D_j^+$ on $K \times X$ coincide with the poles of $D_j$ on $X$.

Suppose that $A$ is an automorphism of $K \times X$ with $AD_j^+A^{-1} = D_j^+$. Then $A$ maps the poles of $D_1$ to the poles of $D_2$. If $D_1$ has at least three poles, then $A$ is an automorphism of $X$. If $D_1$ has two poles, then the same holds for $D_2$ and we may suppose that these poles are $0, \infty$. Then $h_1, h_2 \in C[y, y^{-1}]$ and $A(y) = fy$ or $A(y) = f^{-1}y$ with $f \in K^*$. We may restrict to the first case and obtain then the equality $-\frac{\partial}{\partial y} + \frac{1}{\partial y}h_1(fy) = h_2(fy)$. This implies $f' = 0$ and $f \in \mathbb{C}^*$. The case where $D_1$ has only one singularity is similar.

Finally, suppose that $D_1$ has no poles. Then $C(y) = C(v)$ holds with $D_1v = \lambda_1v$ and $\lambda_1 \in \mathbb{C}^*$. Thus we may assume $D_1(y) = \lambda_1y$ with $\lambda_1, \lambda_2 \in \mathbb{C}^*$. We note that the only zero’s of $D_j^+$ on $K \times X$ are 0 and $\infty$. Hence the automorphism $A$ has the form $A(y) = fy$ or $A(y) = f^{-1}y$ with $f \in K^*$. We may restrict ourselves to the case $A(y) = fy$. Then one obtains the equality $\lambda_1 - \lambda_2 = \frac{f}{f'}$. This equation has no algebraic solution if $\lambda_1 \neq \lambda_2$. \qed

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Appendix A

Here we briefly discuss algorithmic aspects of the questions of Section 1.

Algorithms for Question Q1, i.e., testing whether two curves $X_1, X_2$ of genus $g \geq 2$ over an algebraically closed field are isomorphic, are known. Moreover there are algorithms computing the automorphism group of a curve of genus $\geq 2$ over an algebraically closed field. A description of such a procedure is given in Hess (2004, Algorithm 4); this has been implemented in Magma, see also Bosma et al. (1997). For hyperelliptic curves $X$ one can investigate the canonical morphism $X \to \mathbb{P}^1$ and for other curves the canonical embedding. For any curve of genus $\geq 2$ one can use its tricanonical embedding and also its set of Weierstrass points. The special case of smooth plane curves is described in Du (1994). We refer to Hess (2002), Mumford (1975), Poonen (2010) and the related manuscript Lercier et al. (2012).

A different approach to question (1) is provided by the coarse moduli space of curves of a genus $g \geq 2$ over an algebraically closed field. Generators of the function field of this moduli space can be computed and these can be used for testing whether two curves of genus $g$ are isomorphic (see e.g. van Rijswouw, 2001 for $g = 3$).

Question Q2, i.e., deciding whether a curve $X$, say, over $\overline{\mathbb{C}}(z)$ descends to $\mathbb{C}$, can be handled by the same type of algorithms. Indeed, let $X_0$ over $\mathbb{C}$ be a (smooth) specialization of $X$. The problem is reduced to testing whether $X$ and $X_0$ are isomorphic over $\mathbb{C}(z)$.
Question Q3, computing algebraic solutions of the Riccati equation \( u' + uu^2 = r \) where \( r \) lies in a finite extension \( K \) of \( \mathbb{C}(z) \), has been solved by J. Kovacic for the case of \( K = \mathbb{C}(z) \) (Kovacic, 1986). For a proper finite extension \( K \) of \( \mathbb{C}(z) \), the paper Ulmer and Weil (1996) provides some information. An algorithm is given in Singer (1991) for computing Liouvillian solutions in a very general situation. However, here we are interested in finding all algebraic solutions.

We present here an algorithm for finding all algebraic solutions to \( u' + uu^2 = r \).

Put \( K = \mathbb{C}(z) \) and let \( N/K \) be the two dimensional differential module corresponding to the equation \( y'' = ry \). Solutions of the Riccati equation \( u' + uu^2 = r \) correspond to 1-dimensional submodules of \( \mathcal{R} \otimes_K N \). We consider three cases:

(a). Suppose that there are at least three 1-dimensional submodules of \( \mathcal{R} \otimes_K N \). This is equivalent to \( N \) having a finite differential Galois group. The latter can be tested by trying to find factors of the symmetric powers \( \text{sym}^N \mathcal{A} \) for \( n = 1, 2, 4, 6, 12 \), as in the Kovacic algorithm. We may suppose that the module \( N \) (and its symmetric powers) is semisimple. Using the “eigenring” \( \text{ker}(\eta, \text{End}(\text{sym}^N \mathcal{A})) \), see Section 4.2 of van der Put and Singer (2003), one produces its irreducible submodules. For \( n = 1 \) one decides whether the possible 1-dimensional summands have a finite Galois group (i.e., whether the associated equation \( y' = ay \) has a non-trivial algebraic solution, Risch, 1970; Baldassarri and Dwork, 1979). If there are no 1-dimensional summands and if for \( n = 2 \) the eigenring is non-trivial, then one computes as in (b2) below the quadratic extension \( L/K \) such that \( L \otimes N \) decomposes and continues as in the case \( n = 1 \). In case the eigenring for \( n = 2 \) is trivial, one considers \( n = 4, 6, 12 \) to detect direct summands of \( \text{sym}^N \mathcal{A} \).

(b). Suppose that \( \mathcal{R} \otimes N \) has precisely two 1-dimensional submodules.

(b1). Suppose that both submodules are defined over \( K \), then one can find these factors by computing the eigenring \( \text{ker}(\eta, \text{End}(\text{sym}^N \mathcal{A})) \).

(b2). Suppose that both 1-dimensional submodules are defined over a quadratic extension \( K_2 = K(w) \) of \( K \) with \( w^2 \in K \). Let \( K_2 \otimes_K N \). Then one can find the 1-dimensional factors \( \text{ker}(\eta, \text{End}(\text{sym}^N \mathcal{A})) \).

(c). Suppose that \( \mathcal{R} \otimes_K N \) has only one 1-dimensional submodule. Then this submodule is defined over \( K \). Let \( A \subset N \) be this 1-dimensional submodule. Now \( N \) denotes \( N \), seen as a differential module of dimension \( 2d = 2 \cdot [K : \mathbb{C}(z)] \) over \( \mathbb{C}(z) \). Let \( \hat{A} \) denote the corresponding \( d \)-dimensional submodule of \( N \). By Beke’s algorithm (valid for a differential module over \( \mathbb{C}(z) \)) see Section 4.2.1 of van der Put and Singer (2003), one can find all \( d \)-dimensional submodules of \( N \) in parametrized form. For any submodule \( T \subset N \) of dimension \( d \) we test whether \( r \cdot T = T \). If \( T \) with this property is found, then \( T \) is the 1-dimensional submodule of \( N \) that we are looking for.

If (a)–(c) yield no results, then there are no 1-dimensional submodules of \( N \).

Question Q4 has a well-known answer by J. Coates (see Coates, 1970).

Question Q5. Unlike the earlier questions, we are not aware of any literature concerning an algorithm computing generators for the Lang–Néron group \( E(K)/E(\mathbb{C}) \). The essential case is to treat this problem with \( \mathbb{C} \) replaced by \( \overline{\mathbb{Q}} \). Moreover, the original problem to decide whether a given differential form \( h \) has the form \( N \cdot \frac{dx}{y} \) for some \( (x, y) \in E(K) \), seems somewhat easier.

The special case where \( K \) is the function field of an elliptic curve \( F \) is already non-trivial. In this case one asks for the existence of an isogeny between \( E \) and \( F \). We sketch what can be done in the case that the base field is a number field. An effective version of Faltings’ isogeny theorem may provide a key for solving the problem here. Indeed, a delicate reasoning using Faltings’ result determines whether the restriction to a subgroup of finite index of the \( \ell \)-adic Galois representations attached to \( E \) and \( F \) is isomorphic. If it is, then \( E \) and \( F \) are isogenous, and an explicit isogeny can be found (e.g., by calculating the cyclic subgroups \( \Gamma \) of \( E \) of increasing order \( n = 1, 2, 3, \ldots \), until \( E/\Gamma \) and \( F \) have equal \( j \)-invariant, then an explicit isogeny \( E \to F \) of order \( n \) is easily determined).

The more general situation, namely \( K = \overline{\mathbb{Q}}(X) \) for some algebraic curve \( X \), may be approached similarly. Here, the existence of a finite morphism \( X \to E \) is equivalent to the existence of a non-
trivial homomorphism $J \to E$ where $J$ denotes the Jacobian variety of $X$. Again, Faltings' isogeny theorem reduces this problem to determining whether some powers of eigenvalues of Frobenius on $E$ also appear as powers of eigenvalues of Frobenius on $J$. However, even if the existence of a finite morphism $X \to E$ can be settled in this way, we are not aware of any general method to construct such a morphism explicitly.

References


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