Hamiltonian and variational linear distributed systems

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SUMMARY

We use the formalism of bilinear- and quadratic differential forms in order to study Hamiltonian and variational linear distributed systems. It was shown in [7] that a system described by ordinary linear constant-coefficient differential equations is Hamiltonian if and only if it is variational. In this paper we extend this result to systems described by linear, constant-coefficient partial differential equations. It is shown that any variational system is Hamiltonian, and that any scalar Hamiltonian system is contained (in general, properly) in a variational system that is Hamiltonian with respect to the same vector bilinear differential form.

Keywords: Linear Hamiltonian systems, linear variational systems, multi-variable polynomial matrices, bilinear- and quadratic differential forms.

1. INTRODUCTION

The objective of this paper is to present some recent results in the application of quadratic- and bilinear differential forms, introduced in [9], to the modeling and analysis of systems described by linear, constant-coefficient partial differential equations (in the following also called "nD systems"). We focus on the relationship between Hamiltonian and variational linear distributed behaviors, which we now define. A linear distributed behavior is called Hamiltonian if there exists a nondegenerate, skew-symmetric bilinear functional of the system variables and their partial derivatives up to some finite order, whose divergence is zero along the trajectories of the behavior. A behavior is called variational if it consists of all trajectories which are stationary with respect to some quadratic functional of the variables and their partial derivatives up to a given order.

In [7] Hamiltonian systems described by linear, constant-coefficient ordinary dif-

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Differential equations (in the following “Hamiltonian 1D systems”) were studied using the formalism of quadratic- and bilinear differential forms, and various structural- and representational properties of this class of behaviors were investigated. In that work it was shown that a 1D-system is Hamiltonian if and only if it is variational. In particular, it was shown that a Hamiltonian 1D system admits an interpretation as a mechanical system, in that it consists of the set of trajectories stationary with respect to a higher-order “Lagrangian” functional depending on a “generalized position” and a “generalized velocity”.

In the present paper we attempt to generalize the representation-free approach of [7] to distributed linear systems: we do not assume any special type of representation of a system as starting point, and concentrate instead on the interplay of system dynamics and bilinear- or quadratic differential forms. By adopting such a point of view, considerable results have been obtained in the investigation of physical properties such as losslessness and dissipativity for systems described by linear, constant-coefficient, partial differential equations (see [4], [5]). The main results presented in this paper can be summarized as follows. In Proposition 14 we prove that every linear, variational nD system is also Hamiltonian; as for the converse, in Proposition 16 we prove that if a behavior $\mathcal{B}$ with one external variable (“scalar”) is Hamiltonian with respect to a given vector bilinear differential form, then there exists a variational behavior $\mathcal{B}'$ that contains $\mathcal{B}$ and that is Hamiltonian with respect to the same vector bilinear differential form.

In writing this paper, we have concentrated primarily on presenting and illustrating the basic concepts of our approach in a manner as simple as possible. Consequently we decided to emphasize physical examples and we tried to appeal to the readers’ intuition: no attempt is made to enter into detailed proofs of the mathematical results, which will be published elsewhere. Because of space limitations, we limit our exposition to closed (“autonomous” in behavioral parlance) systems; the case of controllable systems will be discussed elsewhere.

The paper is organized as follows: in section 2 we discuss the basics of multidimensional (nD) behavioral systems. In section 3 we discuss the basics of bilinear- and quadratic differential forms for multidimensional systems, limiting the exposition to the notions necessary in order to understand the material of this paper. In section 4 we review the main results of [7] for Hamiltonian 1D systems. In section 5 we present our results on the relationship of Hamiltonian variational systems described by partial differential equations. A final section contains comments on the result presented and some indications of the directions for future research in this area.
2. ND BEHAVIORS

The purpose of this section is to introduce the reader to those concepts of multidimensional behavioral system theory which are most relevant for the purposes of the paper; see [5] for a thorough treatment of the subject.

In behavioral system theory, the behavior is a subset of the space \( \mathcal{W}^T \) consisting of all trajectories from \( T \), the indexing set, to \( \mathcal{W} \), the signal space. In this paper we consider systems with \( T = \mathbb{R}^n \) (from which the terminology “\( nD \)-system” derives) and \( \mathcal{W} = \mathbb{R}^r \). We call \( \mathcal{B} \) a linear differential \( nD \) behavior if it is the solution set of a system of linear, constant-coefficient partial differential equations (in the following PDEs); more precisely, if \( \mathcal{B} \) is the subset of \( \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R}^r) \) consisting of all solutions to

\[
R \left( \frac{d}{dx} \right) w = 0
\]

where \( R \) is a polynomial matrix in \( n \) indeterminates \( \xi_i \), \( i = 1, \ldots, n \), and \( \frac{\partial}{\partial x} = (\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}) \). We call (1) a kernel representation of \( \mathcal{B} \).

Obviously, any linear differential \( nD \) behavior \( \mathcal{B} \) is a linear subspace of \( \mathcal{W}^T \). Also, any such behavior is shift-invariant in the sense that \( \mathcal{B} = \sigma^x \mathcal{B} \), where for \( x' = (x'_1, \ldots, x'_n) \), the \( x' \)-shift \( \sigma^x \) is defined as

\[
\sigma^x : (\mathbb{R}^r)^\mathbb{R}^n \to (\mathbb{R}^r)^\mathbb{R}^n
\]

\[
(\sigma^x w)(x_1, \ldots, x_n) := w(x_1 + x'_1, \ldots, x_n + x'_n)
\]

We denote the set consisting of all linear, shift-invariant differential \( nD \)-systems with \( w \) external variables with \( \mathcal{L}^n \); when \( n = 1 \), we write simply \( \mathcal{L}^r \).

The following are examples of elements of \( \mathcal{L}^2 \) and \( \mathcal{L}^1 \) respectively.

Example 1 Consider the one-dimensional wave equation describing the displacement \( w(t, x) \) from equilibrium of a homogeneous elastic medium:

\[
\rho \frac{\partial^2 w}{\partial t^2} - \tau^2 \frac{\partial^2 w}{\partial x^2} = 0
\]

where \( \rho \) and \( \tau \) are physical constants related to the mass density and the elasticity of the medium, respectively. Such equation defines a linear, shift-invariant, differential system with indexing set \( T = \mathbb{R}^2 \), signal space \( \mathcal{W} = \mathbb{R} \), and behavior

\[
\mathcal{B} = \{ w \in \mathcal{C}^\infty(\mathbb{R}^2, \mathbb{R}) \mid w \text{ satisfies } (2) \}
\]

The polynomial associated with the representation (2) is \( R(\xi_1, \xi_2) = \rho^2 \xi_1^2 - \tau^2 \xi_2^2 \). In order to stress that the we are dealing with variables \( t \) and \( x \), we often write \( \xi_t \) instead of \( \xi_1 \), and \( \xi_x \) instead of \( \xi_2 \).
Example 2 Let $w(t,x,y)$ be the displacement of an infinite vibrating plate in the position $(x,y)$ at time $t$; then it can be shown that $w$ satisfies the PDE

$$\rho \frac{\partial^2 w}{\partial t^2} + \frac{\partial^4 w}{\partial x^4} + 2 \frac{\partial^2 w}{\partial x \partial y} + \frac{\partial^4 w}{\partial y^4} = 0$$

(3)

where $\rho$ is a constant depending on the physical properties of the plate. Such equation defines a linear, shift-invariant, differential system with indexing set $\mathbb{T} = \mathbb{R}^3$, signal space $\mathcal{W} = \mathbb{R}$, and behavior $\mathcal{B} = \{ w \in C^\infty(\mathbb{R}^3,\mathbb{R}) \mid w \text{ satisfies (3)} \}$. The polynomial associated with the representation (3) is $R(\xi_1,\xi_2,\xi_y) = \rho \xi_1^2 + \xi_2^4 + 2\xi_1\xi_2 + \xi_y^4$.

Finally, we introduce the notion of (weakly) autonomous $nD$-behavior. Intuitively, an autonomous behavior consists of trajectories which are completely determined by their “initial conditions”, i.e. systems on which no external influence in the form of “input” is exerted. In order to formalize such intuition, we need to define the characteristic ideal and characteristic variety associated with a kernel representation (1). Let $\mathbb{R}^{w\times w}[\xi_1,\ldots,\xi_n]$ be the set of all $w \times w$ matrices with components in the polynomial ring $\mathbb{R}[\xi_1,\ldots,\xi_n]$ of polynomials in $n$ indeterminates, with real coefficients. For $R \in \mathbb{R}^{w\times w}[\xi_1,\ldots,\xi_n]$, the characteristic ideal is the ideal of $\mathbb{R}[\xi_1,\ldots,\xi_n]$ generated by the determinants of all $w \times w$ minors of $R$, and the characteristic variety is the set of roots common to all polynomials in the ideal. The behavior $\mathcal{B}$ represented in kernel form by (1) is said to be (weakly) autonomous if its characteristic ideal is not the zero ideal; or equivalently, if its characteristic variety is not all of $\mathbb{C}^n$. Observe that if an $nD$-behavior is represented by a single equation (1) with a nonzero polynomial, such as the ones considered in Example 1 and Example 2, then it is (weakly) autonomous.

3. BILINEAR- AND QUADRATIC DIFFERENTIAL FORMS

In many modeling and control problems for linear systems it is necessary to study bilinear- and quadratic functionals of the system variables and their derivatives. For finite-dimensional linear systems, an efficient representation for such functionals by means of two-variable polynomial matrices was introduced in [9]; in order to represent bilinear- and quadratic functionals of the variables of $nD$-systems, $2n$-variable polynomial matrices are used (see [5]).

In order to simplify the notation, define the vector $\mathbf{x} := (x_1,\ldots,x_n)$, the multi-indices $\mathbf{k} := (k_1,\ldots,k_n)$ and $\mathbf{l} := (l_1,\ldots,l_n)$, and the notation $\zeta := (\zeta_1,\ldots,\zeta_n)$ and $\eta := (\eta_1,\ldots,\eta_n)$. Let $\mathbb{R}^{w_1\times w_2}[\zeta,\eta]$ denote the set of real polynomial $w_1 \times w_2$ matrices in the $2n$ indeterminates $\zeta$ and $\eta$; that is, an element of $\mathbb{R}^{w_1\times w_2}[\zeta,\eta]$ is of the
form
\[ \Phi(\zeta, \eta) = \sum_{k,l} \Phi_{k,l} \zeta^k \eta^l \]

where \( \Phi_{k,l} \in \mathbb{R}^{n \times n} \): the sum ranges over the nonnegative multi-indices \( k \) and \( l \), and is assumed to be finite. Such matrix induces a bilinear differential form (BDF) in the following

\[ L_\Phi : C^\infty(\mathbb{R}^n, \mathbb{R}^n) \times C^\infty(\mathbb{R}^n, \mathbb{R}^n) \rightarrow C^\infty(\mathbb{R}^n, \mathbb{R}) \]

\[ L_\Phi(v, w) := \sum_{k,l} (\frac{\partial^k v}{\partial \zeta^k})^T \Phi_{k,l} (\frac{\partial^l w}{\partial \eta^l}) \]

where the \( k \)-th derivative operator \( \frac{\partial^k}{\partial \zeta^k} \) is defined as \( \frac{\partial^k}{\partial \zeta^k} := \frac{\partial^1}{\partial \zeta_1^1} \cdots \frac{\partial^k}{\partial \zeta_k^k} \) (similarly for \( \frac{\partial^l}{\partial \eta^l} \)).

We call \( L_\Phi \) skew-symmetric if \( L_\Phi(w_1, w_2) = -L_\Phi(w_2, w_1) \) for all infinitely differentiable trajectories \( w_1, w_2 \). It can be shown that this is the case if and only if \( \Phi \) is a skew-symmetric \( 2n \)-variable polynomial matrix, i.e., if \( w_1 = w_2 \) and \( \Phi(\zeta, \eta) = -\Phi(\eta, \zeta)^T \). The \( 2n \)-variable polynomial matrix \( \Phi(\zeta, \eta) \) is called symmetric if \( w_1 = w_2 = w \) and \( \Phi(\zeta, \eta) = \Phi(\eta, \zeta)^T \). In such case, \( \Phi \) induces also a quadratic functional

\[ Q_\Phi : C^\infty(\mathbb{R}^n, \mathbb{R}) \rightarrow C^\infty(\mathbb{R}^n, \mathbb{R}) \]

\[ Q_\Phi(w) := L_\Phi(w, w) \]

We will call \( Q_\Phi \) the quadratic differential form (in the following abbreviated with QDF) associated with \( \Phi \).

In this paper we also consider vectors \( \Psi \in (\mathbb{R}^{n \times n}[\zeta, \eta])^n \), i.e., \( \Psi = (\Psi_1^T, \ldots, \Psi_n^T)^T \) with \( \Psi_i \in \mathbb{R}^{n \times n}[\zeta, \eta] \). Such \( \Psi \) induces a vector bilinear differential form, in short a VBDVF, defined as

\[ L_\Psi : C^\infty(\mathbb{R}^n, \mathbb{R}^n) \times C^\infty(\mathbb{R}^n, \mathbb{R}^n) \rightarrow (C^\infty(\mathbb{R}^n, \mathbb{R}))^n \]

\[ L_\Psi(v, w) := (L_{\Psi_1}(v, w), L_{\Psi_2}(v, w), \ldots, L_{\Psi_n}(v, w))^T. \]

Finally, we introduce the notion of divergence of a VBDVF. Given a VBDVF \( L_\Psi = (L_{\Psi_1}, L_{\Psi_2}, \ldots, L_{\Psi_n})^T \), we define its divergence as the BDF defined by

\[ (\text{div } L_\Psi)(w_1, w_2) := (\frac{\partial}{\partial \zeta_1} L_{\Psi_1})(w_1, w_2) + \cdots + (\frac{\partial}{\partial \zeta_n} L_{\Psi_n})(w_1, w_2) \quad \text{(4)} \]

for all infinitely differentiable trajectories \( w_1, w_2 \). In terms of the \( 2n \)-variable polynomial matrices associated with the BDF's, the relationship between a VBDVF and its divergence is expressed as

\[ \Phi(\zeta, \eta) = (\zeta_1 + \eta_1) \Psi_1(\zeta, \eta) + \cdots + (\zeta_n + \eta_n) \Psi_n(\zeta, \eta) \]
In order to characterize those BDFs which are the divergence of some VBDF, we need to introduce some notation. The “del” operator is defined as
\[
\delta : \mathbb{R}^{w_1 \times w_2}[\xi_1, \ldots, \xi_n, \eta_1, \ldots, \eta_n] \longrightarrow \mathbb{R}^{w_1 \times w_2}[\xi_1, \ldots, \xi_n]
\]
\[
\delta \Phi(\xi_1, \ldots, \xi_n) = \Phi(-\xi_1, \ldots, -\xi_n, \xi_1, \ldots, \xi_n)
\]
Observe that the matrix \(\delta \Phi(\xi_1, \ldots, \xi_n)\) is para-Hermitian, meaning \(\delta \Phi(\xi_1, \ldots, \xi_n) = \Phi(-\xi_1, \ldots, -\xi_n, \xi_1, \ldots, \xi_n)\). Observe also that by means of the “del” operator, a differential operator \(\delta \Phi(\frac{\partial}{\partial x})\) can be assigned to a QDF; this has important applications when considering the problem of which BDFs are the divergence of some VBDF, as we now show. Indeed, it can be shown that \(L_{\Phi}\) is the divergence of some VBDF \(L_{\Phi}\) if and only if \(\delta \Phi(\xi) = 0\) (see Th. 4 p. 1411 of [5]).

Example 3 Consider the behavior described by (2). On the basis of physical considerations, the total energy of the system trajectory \(w\) at time \(t\) can be shown to be \(\int_{\mathbb{R}} Q_{\Phi}(w)dx\), where \(Q_{\Phi}(w) = \frac{1}{2}\rho(\frac{\partial w}{\partial t})^2 + \frac{1}{2}\tau(\frac{\partial w}{\partial x})^2\) is associated with the 4-variable polynomial
\[
\Phi(\xi, \zeta, \eta, \eta_x) = \frac{1}{2}\xi \eta + \frac{1}{2}\zeta \eta_x
\]

Example 4 Consider the behavior defined by the transverse motion of a homogeneous flexible sheet (“membrane”) with surface mass density \(v\). It can be shown that if \(w(t, x, y)\) is the displacement from equilibrium of point \((x, y)\) of the membrane at time \(t\), then \(w\) satisfies the PDE (see Section 7.36.3 [2]):
\[
\rho \frac{\partial^2 w}{\partial t^2} - \tau \frac{\partial^2 w}{\partial x^2} - \tau \frac{\partial^2 w}{\partial y^2} = 0
\]
(5)
This defines the behavior
\[
\mathcal{B} = \{w \in C^\infty(\mathbb{R}^3, \mathbb{R}) \mid (5) \text{ is satisfied} \}
\]
(6)
On the basis of physical considerations (see [2]) it can be shown that the Lagrangian (i.e. the difference between the kinetic and potential energy) of the membrane at time \(t\) is \(\int_{\mathbb{R}^2} Q_{\Phi}(w)dx\), where
\[
Q_{\Phi}(w) = \frac{1}{2}\rho(\frac{\partial w}{\partial t})^2 - \frac{1}{2}\tau((\frac{\partial w}{\partial x})^2 + (\frac{\partial w}{\partial y})^2)
\]
(7)
is associated with the 6-variable polynomial
\[
\Phi(\zeta, \zeta_x, \zeta_y, \eta, \eta_x, \eta_y) := \frac{1}{2}\rho\zeta \eta - \frac{1}{2}\tau(\zeta_x \eta_x + \zeta_y \eta_y)
\]
4. AUTONOMOUS HAMILTONIAN \(1D\) SYSTEMS

In this section we review some of the results of [7] about autonomous Hamiltonian \(1D\) behaviors; in order to do this, a number of preliminary remarks are in order.

The trajectories of a \(1D\) behavior map \(\mathbb{R}\) into a signal space \(\mathbb{R}^n\), and normally are considered to be time-signals; for this reason we denote the “independent variable” of which such a trajectory is function with \(t\). It can be shown (see [6]) that an autonomous \(1D\) behavior corresponds to a kernel representation (1) associated with a one-variable polynomial matrix \(H\) having full column rank \(o\); in particular, such a matrix can be chosen to be square and nonsingular. It can also be shown that the behavior of a \(1D\) autonomous system is finite-dimensional, and consists of vector-polynomial exponential trajectories \(BCt\rangle A;B\), where \(C\) is even. The polynomial \(\lambda\langle\xi : = \Pi_{i=1}^{n}(\xi - \lambda_i)\rangle\) associated with \(C\) is called the characteristic polynomial of \(C\).

The definition of autonomous Hamiltonian \(1D\) system is as follows.

**Definition 5** Let \(C \in \mathcal{L}^n\) be autonomous. \(C\) is called Hamiltonian if there exists a bilinear differential form \(L_\Phi\), such that

(i) \(\frac{d}{dt} L_\Phi (w_1, w_2) = 0 \) for all \(w_1, w_2 \in C\);
(ii) \(L_\Phi\) is skew-symmetric;
(iii) \(L_\Phi (v, w) (0) = 0 \) for all \(v \in C \iff w = 0\) (nondegeneracy).

Observe that in Definition 5 no assumption on the number \(w\) of external variables of \(C\) is made, in contrast with the usual definition, in which an even number of such variables is assumed.

**Example 6** Consider the autonomous behavior \(C\) represented by the first order differential equation \(\frac{d}{dt}w = Aw\), where \(A \in \mathbb{R}^{n \times n}\), and \(w\) is even. Such system is often called Hamiltonian if the matrix \(A\) is a Hamiltonian matrix, i.e. if \(ATJ + JA = 0\), where \(J\) is equal to the nonsingular skew-symmetric matrix

\[
J = \begin{pmatrix}
0 & I_n/2 \\
-I_n/2 & 0
\end{pmatrix}
\]

This behavior \(C\) is also Hamiltonian in the sense of Definition 5: the bilinear differential form \(L_\Phi (v, w) := v^T J w\) is easily seen to satisfy the conditions (i), (ii) and (iii).

**Example 7** Consider two masses \(m_1\) and \(m_2\) attached to springs with constants \(k_1\) and \(k_2\). The first mass is connected to the second one via the first spring, and the
second mass is connected to a “wall” with the second spring. Denote the positions of
the masses with \( L \) and \( L \). The system equations are

\[
\begin{align*}
    m_1 \ddot{w}_1 + k_1 w_1 - k_1 w_2 &= 0 \\
    m_2 \ddot{w}_2 - k_1 w_1 + (k_1 + k_2) w_2 &= 0
\end{align*}
\]

and it can be shown that such equations describe an autonomous system. Now assume
that we are only interested in modeling the position \( L \) of the first mass; manipulating
the equations we can then eliminate \( L \) obtaining a higher-order model for \( L \) as

\[
\begin{align*}
    m_1 m_2 \dddot{w}_1 + (k_1 m_1 + k_2 m_1 + k_1 m_2) \dot{w}_1 + k_1 k_2 w_1 &= 0
\end{align*}
\]

(see Ch. 6 of [6] for a thorough discussion of the issue of variable elimination). For
simplicity, denote \( r_0 := k_1 k_2, r_2 := k_1 m_1 + k_2 m_1 + k_1 m_2 \) and \( r_4 := m_1 m_2 \). Now
define the skew-symmetric two-variable polynomial

\[
\Psi(\zeta, \eta) := r_2 (\zeta - \eta) + r_4 (\zeta^2 - \eta^2) + r_4 (\zeta \eta - \zeta^2 \eta).
\]

The corresponding BDF \( L_\Psi(v, w) \) is skew-symmetric and \( \frac{d}{dv} L_\Psi(v, w) = 0 \) for all
\( v, w \in \mathcal{B} \), as can be easily verified. It can also be easily proved that \( L_\Psi \) is nonde-
generate on \( \mathcal{B} \). Hence the behavior \( \mathcal{B} \) is a Hamiltonian system.

In order to state the main result of this section, that is the equivalence of Hamiltoni-
nity and variationality for \( 1D \)-systems, we need to review the concept of stationarity
of a trajectory with respect to a QDF. Let \( \Phi \in \mathbb{R}^{2 \times 2} \) be symmetric and consider
the corresponding QDF \( Q_\Phi(w) \) on \( \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^2) \). For a given \( w \) we define the cost degrada-
tion of adding the compact-support function \( \delta \) to \( w \) as

\[
J_w(\delta) := \int_{-\infty}^{+\infty} (Q_\Phi(w + \delta) - Q_\Phi(w)) \, dt = \int_{-\infty}^{+\infty} Q_\Phi(\delta) \, dt + 2 \int_{-\infty}^{+\infty} L_\Phi(w, \delta) \, dt
\]

The second term on the right in this equation is called the variation associated with
\( w \): it is a functional associating to \( \delta \) the real number \( 2 \int_{-\infty}^{+\infty} L_\Phi(w, \delta) \, dt \). We call \( w \) a
stationary trajectory of \( Q_\Phi \) if the variation associated with it is the zero functional.
It can be shown (by repeated partial integration of the integral defining the variation)
that \( w \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^2) \) is a stationary trajectory with respect to the QDF \( Q_\Phi \) if and only
if \( w \) satisfies the differential equation

\[
\partial \Phi \left( \frac{d}{dt} \right) w = 0
\]

(Recall: \( \partial \Phi(\xi) \) is defined as the one-variable polynomial matrix \( \Phi(-\xi, \xi) \).) A behavior
\( \mathcal{B} \) consisting of all trajectories that are stationary with respect to a given QDF \( Q_\Phi \)
is called variational with respect to \( \mathcal{B} \). It follows from the definition of stationary
trajectory that \( \mathcal{B} \) is variational with respect to \( Q_\Phi \) if and only if \( \mathcal{B} = \ker \partial \Phi \left( \frac{d}{dt} \right) \).

The following theorem is an extension of [7], Theorem 10.
Theorem 8 Let \( \mathcal{B} \in \mathcal{L}\) be autonomous, \( \chi_{\mathcal{B}}(0) \neq 0 \). Then \( \mathcal{B} \) is Hamiltonian if and only if there exists a symmetric \( \Phi(\zeta, \eta) \in \mathbb{R}^{n \times n}[\zeta, \eta] \) such that \( \mathcal{B} = \ker \partial \Phi(\frac{\partial}{\partial t}) \). In that case the number \( n(\mathcal{B}) \) of state variables in any minimal state representation of \( \mathcal{B} \) is even, and there exists a full column rank matrix \( P(\xi) \in \mathbb{R}^{n \times q} \), \( M = M^T, K = K^T \in \mathbb{R}^{n \times n} \) nonsingular, with \( q := \frac{n(\mathcal{B})}{2} \), such that \( \mathcal{B} \) is equal to \( \ker \partial \Phi(\frac{\partial}{\partial t}) \), with \( \Phi(\zeta, \eta) \) the two variable polynomial matrix

\[
\Phi(\zeta, \eta) := P(\zeta)^T(\zeta M - K)P(\eta),
\]

(10)
corresponding to the QDF

\[
Q_\Phi(w) = \left( \frac{d}{dt}P(\frac{d}{dt})w \right)^2_M - \left| P(\frac{d}{dt})w \right|^2_K,
\]

If one interprets the latent variable \( q = P(\frac{\partial}{\partial t})w \) as a generalized position, then \( \dot{q} = \frac{d}{dt}P(\frac{\partial}{\partial t})w \) is a generalized velocity; consequently the expressions \( \left| \frac{d}{dt}P(\frac{\partial}{\partial t})w \right|^2_M = \left| \dot{q} \right|^2_M \) and \( \left| P(\frac{d}{dt})w \right|^2_K = \left| q \right|^2_K \) can be interpreted respectively as kinetic and potential energy. From this point of view, the QDF \( Q_\Phi(w) \) can be interpreted as a Lagrangian of the system. The system of differential equations \( \partial \Phi(\frac{\partial}{\partial t})w = 0 \) (representing the stationary trajectories with respect to this Lagrangian) coincide with the Euler-Poisson equations associated with the Lagrangian.

Remark 9 In [7] algebraic procedures are stated, which compute the “generalized Lagrangian”, and the symplectic BDF \( L_\Phi \) in Definition 5 starting from a representation of a system.

The following example illustrates the result of Theorem 8.

Example 10 Consider the configuration of Example 7. As shown in that Example, the behavior \( \mathcal{B} \) of the position \( w_1 \) of the first mass is represented by equation (8). Define the latent variable \( q = \text{co}(w_1, \dot{w}_1) \). It can be shown that a generalized Lagrangian for \( \mathcal{B} \) is given by \( Q_\Phi(w_1) = \dot{q}^T M \dot{q} - q^T K q \), with

\[
M := \begin{pmatrix} r_2 & r_4 \\ r_4 & 0 \end{pmatrix}, \quad K := \begin{pmatrix} r_0 & 0 \\ 0 & -r_4 \end{pmatrix}
\]

where the \( r_i \) are defined as in Example 7. Observe that such generalized Lagrangian does not correspond to a difference of energies, as can be readily seen checking the physical dimensions of the terms \( \dot{q}^T M \dot{q} \) and \( q^T K q \). By choosing the latent variable \( q \) as before, and the matrices

\[
\tilde{K} := \begin{pmatrix} k_2 & k_1 n \frac{m_2}{k_1} \\ k_1 n \frac{m_1}{k_2} & \frac{m_1 m_2}{k_1 k_2} \end{pmatrix}, \quad \tilde{M} := \begin{pmatrix} m_1 + m_2 & \frac{m_1 m_2}{k_1} \\ \frac{m_1 m_2}{k_2} & \frac{m_1 m_2}{k_1 k_2} \end{pmatrix},
\]

(11)
we obtain a new generalized Lagrangian for $\mathcal{B}$. With this second choice of $L$, the
generalized kinetic energy $q^T \dot{M} q$ and generalized potential energy $q^T \dot{K} q$ (which are
a function of the position and the acceleration of the first mass) coincide with the
physical kinetic energy and potential energy of the system with the two masses:

$$E_{\text{kin}}(w_1, w_2) = \frac{1}{2}(m_1 w_1^2 + m_2 w_2^2)$$

$$E_{\text{pot}}(w_1, w_2) = \frac{1}{2}(k_1 w_1^2 - 2k_1 w_1 w_2 + (k_1 + k_2) w_2^2)$$

This can be verified easily, since the generalized position $\dot{q}$ is related to
the actual position $(w_1, w_2)$ by the nonsingular linear map

$$\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & \frac{1}{k_1} \end{pmatrix} \begin{pmatrix} \dot{w} \\ \dot{\bar{w}} \end{pmatrix}$$

5. AUTONOMOUS HAMILTONIAN $nD$ SYSTEMS

In this section we attempt to generalize the result of Theorem 8 in Section 4 to linear
differential $nD$ systems, i.e., systems described by linear, constant-coefficient PDEs.
We first define Hamiltonian $nD$ systems, give a couple of examples in order to illustrate
the definition, and then we show that every autonomous variational $nD$ system is
Hamiltonian. Finally, we show that every scalar (i.e. $\omega = 1$), autonomous Hamiltonian
$nD$ system is a sub-system of an autonomous variational system that is Hamiltonian
with respect to the same VBDF.

The definition of autonomous Hamiltonian $nD$ system is as follows.

**Definition 11** Let $\mathcal{B} \in L^\omega_n$ be autonomous. $\mathcal{B}$ is called Hamiltonian if there exists a
VBDF $L_\omega$, with $\text{div } L_\omega \neq 0$, such that

(i) $\text{div} L_\omega (w_1, w_2) = 0$ for all $w_1, w_2 \in \mathcal{B}$;

(ii) $L_\omega$ is skew-symmetric;

(iii) $[L_\omega(v, w)(0) = 0 \text{ for all } v \in \mathcal{B}] \iff [w = 0]$ (nondegeneracy);

Note that in this definition it is required that the divergence of $L_\omega$ is unequal to zero.
In the $1D$ case this condition reduces to $\frac{d}{dt} L_\omega \neq 0$, which in that case in implied by
the nondegeneracy condition (iii). In the general $nD$ case, nondegeneracy does not
imply $\text{div } L_\omega \neq 0$. We now consider a couple of examples of Hamiltonian systems.

**Example 12** Consider the behavior of the membrane system illustrated in Example 4.
Define the 6-variable polynomial vector

\[ \Psi(\zeta_t, \zeta_x, \zeta_y, \eta_t, \eta_x, \eta_y) := \begin{pmatrix} \zeta_t - \eta_t \\ -\frac{\tau}{\rho}(\zeta_x - \eta_x) \\ -\frac{\tau}{\rho}(\zeta_y - \eta_y) \end{pmatrix} \]

and observe that \( L_\Psi(v, w) = -L_\Psi(w, v) \) for every pair of trajectories \( v, w \), so that \( L_\Psi \) is skew-symmetric. It is also easily seen that \( \text{div}\ L_\Psi \neq 0 \). Moreover, along every pair of trajectories \( v, w \) belonging to the behavior \( \mathfrak{B} \) defined in (6), we have

\[
(\text{div} \ L_\Psi)(v, w) = \frac{\partial}{\partial t}(\frac{\partial v}{\partial t} w - v \frac{\partial w}{\partial t}) + \frac{\partial}{\partial x}[-\frac{\tau}{\rho}(\frac{\partial v}{\partial x} w - v \frac{\partial w}{\partial x})] + \frac{\partial}{\partial y}[-\frac{\tau}{\rho}(\frac{\partial v}{\partial y} w - v \frac{\partial w}{\partial y})]
\]

\[
= \left( \frac{\partial^2 v}{\partial t^2} - \frac{\tau^2}{\rho} \frac{\partial^2 v}{\partial x^2} \right)w - v\left( \frac{\partial^2 w}{\partial t^2} - \frac{\tau^2}{\rho} \frac{\partial^2 w}{\partial x^2} \right) = 0
\]

so that \( L_\Psi \) also satisfies property (i) of Definition 11. It can also be shown that \( L_\Psi \) is nondegenerate, and consequently that \( \mathfrak{B} \) defined in (6) is a symplectic behavior.

**Example 13** Let \( w(t, x) \) be the position at time \( t \) of point \( x \) of an infinitely long stiff beam subject to vibrations. It can be shown (see for example [2]) that the behavior consisting of all possible motions \( w(\cdot, \cdot) \) is described by

\[
\mathfrak{B} = \{ w \in C(\mathbb{R}^2, \mathbb{R}) | \frac{\partial^2 w}{\partial t^2} + a^2 \frac{\partial^4 w}{\partial x^4} = 0 \} \quad (12)
\]

where \( a \) is a constant which depends on the physical properties of the beam. We claim that the system is symplectic. Indeed, the 2-dimensional VBDF associated with

\[
\Psi(\zeta_t, \zeta_x, \eta_t, \eta_x) := \left( a^2(\zeta_x^3 - \eta_x^3 - C_x^2 \eta_x + \zeta_x \eta_x^2) \right) \quad (13)
\]

i.e.

\[
L_\Psi(v, w) = \left( a^2(\frac{\partial^3}{\partial x^3} w - v \frac{\partial^3}{\partial x^3} w - \frac{\partial^3}{\partial x^3} w \frac{\partial^3}{\partial x^3} w + \frac{\partial^3}{\partial x^3} w \frac{\partial^3}{\partial x^3} w) \right)
\]
is skew-symmetric, as it is easy to verify. Also, $\text{div} L_\Phi \neq 0$. Moreover, since
\[
(\text{div} L_\Phi)(v, w) = \frac{\partial}{\partial t} \left( \frac{\partial v}{\partial t} - \frac{\partial w}{\partial t} \right) + \frac{\partial}{\partial x} \left[ a^2 \left( \frac{\partial^3 v}{\partial x^3} w - \frac{\partial^3 w}{\partial x^3} v - \frac{\partial^2 v}{\partial x^2} \frac{\partial w}{\partial x} + \frac{\partial v}{\partial x} \frac{\partial^2 w}{\partial x} \right) \right]
\]
\[
= \frac{\partial^2 v}{\partial t^2} w - \frac{\partial^2 w}{\partial t^2} v + a^2 \left( \frac{\partial^4 v}{\partial x^4} w - \frac{\partial^4 w}{\partial x^4} v \right)
\]
\[
= \left( \frac{\partial^2 v}{\partial t^2} + a^2 \frac{\partial^4 v}{\partial x^4} \right) w + \left( \frac{\partial^2 w}{\partial t^2} + a^2 \frac{\partial^4 w}{\partial x^4} \right) v
\]
we conclude that $(\text{div} L_\Phi)(v, w) = 0$ for all $v, w \in \mathcal{B}$. It is a matter of tedious verification to show that $L_\Phi$ is also non-degenerate, and consequently that $\mathcal{B}$ defined in (12) is Hamiltonian.

Similar as in the 1D case, for systems described by partial differential equations we have the notion of stationarity with respect to a given QDF. Let $\Phi(\zeta, \eta)$ be a symmetric $2n$-variable polynomial matrix, and consider the corresponding QDF $Q_\Phi(w)$ on $C^\infty(\mathbb{R}^n, \mathbb{R}^n)$. For a given $w$ the cost degradation of adding the compact-support function $\delta \in \mathcal{D}(\mathbb{R}^n, \mathbb{R}^n)$ to $w$ is defined as
\[
J_w(\delta) := \int_{\mathbb{R}^n} Q_\Phi(w + \delta) - Q_\Phi(w) d\mathbf{x} = \int_{\mathbb{R}^n} Q_\Phi(\delta) d\mathbf{x} + 2 \int_{\mathbb{R}^n} L_\Phi(w, \delta) d\mathbf{x}
\]
The second term on the right in this equation is called the variation associated with $w$: it is a functional associating to $\delta$ the real number $2 \int_{\mathbb{R}^n} L_\Phi(w, \delta) d\mathbf{x}$. We call $w$ a stationary trajectory of $Q_\Phi$ if the variation associated with it is the zero functional. It can be shown (by repeated application of the Gauss divergence theorem) that $w \in C^\infty(\mathbb{R}^n, \mathbb{R}^n)$ is a stationary trajectory with respect to the QDF $Q_\Phi$ if and only if $w$ satisfies the system of linear partial differential equations
\[
\partial \Phi \left( \frac{d}{d\mathbf{x}} \right) w = 0
\]
(Recall: $\partial \Phi(\xi)$ is defined as the $n$-variable polynomial matrix $\Phi(-\xi, \xi)$.) As in the 1D case, a behavior $\mathcal{B} \in \mathcal{L}^n_n$ consisting of all trajectories that are stationary with respect to a given QDF $Q_\Phi$ is called variational with respect to $Q_\Phi$. It follows from the definition of stationary trajectory that $\mathcal{B}$ is variational with respect to $Q_\Phi$ if and only if $\mathcal{B} = \ker \partial \Phi \left( \frac{d}{d\mathbf{x}} \right)$.

We now show that every $nD$ variational behavior is Hamiltonian.

**Proposition 14** Let $\mathcal{B} \in \mathcal{L}^n_n$ be an autonomous $nD$ behavior. Assume that $\mathcal{B}$ is variational with respect to some QDF $Q_\Phi$, i.e. $\mathcal{B} = \ker \partial \Phi \left( \frac{d}{d\mathbf{x}} \right)$; then $\mathcal{B}$ is Hamiltonian.
Proof. Define a 2n-variable \( w \times w \) polynomial matrix \( \Phi'(\zeta, \eta) := \partial \Phi(-\zeta) - \partial \Phi(\eta) \). Since \( \Phi(\zeta, \eta) \) is symmetric, \( \partial \Phi \) is para-Hermitian, i.e., \( \partial \Phi(\xi) = \partial \Phi(-\xi)^T \). From this we conclude that for all \( v, w \) we have

\[
L_{\Phi}(v, w) = (\partial \Phi(\frac{d}{dx})v)^T w - v^T (\partial \Phi(\frac{d}{dx})w).
\]

As a consequence, since \( \mathcal{B} = \ker \partial \Phi(\frac{d}{dx}) \), for all \( v, w \in \mathcal{B} \) we have \( L_{\Phi}(v, w) = 0 \).

Since also

\[
\partial \Phi'(\zeta) = \partial \Phi(\xi) - \partial \Phi(\xi) = 0
\]

we conclude (see Th.4 of [5]) that there exists a \( \Psi = \text{col}(\Psi_1, \Psi_2, \ldots, \Psi_n) \in (\mathbb{R}^{2n \times n})^n \) such that \( \text{div} L_{\Phi} = L_{\Phi} \Psi \). We will prove that \( \Psi \) can be chosen to be skew-symmetric, i.e., \( \Psi_i(\zeta, \eta)^T = -\Psi_i(\eta, \zeta), i = 1, 2, \ldots, n \). Indeed, we have

\[
\Phi'(\zeta, \eta) = \sum_{i=1}^{n} (\zeta_i + \eta_i) \Psi_i(\zeta, \eta)
\]

so

\[
\Phi'(\eta, \zeta)^T = \sum_{i=1}^{n} (\zeta_i + \eta_i) \Psi_i(\eta, \zeta)^T
\]

Note that from the definition of \( \Phi'(\zeta, \eta) \) we have \( \Phi'(\eta, \zeta)^T = -\Phi'(\zeta, \eta) \). Thus we get

\[
\Phi'(\zeta, \eta) = \frac{1}{2}(\Phi'(\zeta, \eta) - \Phi'(\eta, \zeta)^T)
\]

\[
= \sum_{i=1}^{n} (\zeta_i + \eta_i) \frac{1}{2}(\Psi_i(\zeta, \eta) - \Psi_i(\eta, \zeta)^T)
\]

By redefining \( \Psi_i(\zeta, \eta) \) as \( \frac{1}{2}(\Psi_i(\zeta, \eta) - \Psi_i(\eta, \zeta)^T) \) we thus get a skew-symmetric matrix \( \Gamma \). Observe that \( \text{div} L_{\Phi}(v, w) = L_{\Phi}(v, w) \neq 0 \), but that \( \text{div} L_{\Phi}(v, w) = 0 \) for all \( v, w \in \mathcal{B} \). The proof of the nondegeneracy of \( L_{\Phi} \) is rather technical and laborious, and is omitted.

We illustrate the content of Proposition 14 with an example.

Example 15 Consider the behavior \( \mathcal{B} \) of the vibrating membrane of Example 4. It can be shown that \( \mathcal{B} \) defined in (6) is stationary with respect to the quadratic functional defined in equation (7) and associated with the 6-variable polynomial

\[
\Phi(\xi, \xi_x, \xi_y, \eta, \eta_x, \eta_y) = \mu \xi \eta - \tau \xi_x \eta_x - \tau \xi_y \eta_y
\]

Observe that \( \partial \Phi(\xi, \xi_x, \xi_y) = -\rho \xi_x^2 + \tau \xi_x^2 + \tau \xi_y^2 \). We proceed as in the proof of
Proposition 14, and define the $\Psi$-variable polynomial

$$\Psi(\zeta_1, \ldots, \zeta_n, \eta_1, \ldots, \eta_n) = \mathcal{L}(\zeta_1, \ldots, \zeta_n, \eta_1, \ldots, \eta_n)$$

Observe that $L_\Psi = \text{div} L_\Psi$, where

$$L_\Psi = \Psi(\zeta_1, \ldots, \zeta_n, \eta_1, \ldots, \eta_n)$$

Such $\Psi$-variable polynomial vector is proportional to that considered in Example 12.

We proceed to show that in the scalar case, i.e. when the number $w$ of external variables equals 1, every Hamiltonian system is contained in a variational one, which is Hamiltonian with respect to the same VBDF.

**Proposition 16** Let $\mathcal{B} \in \mathcal{L}_1^n$ be an autonomous $n$D behavior with 1 external variable. Assume that $\mathcal{B}$ is Hamiltonian with respect to the VBDF $L_\Psi$. Then there exists a QDF $Q_\Psi$ and an autonomous behavior $\mathcal{B}' \in \mathcal{L}_1^n$, such that $\mathcal{B} \subseteq \mathcal{B}'$, $\mathcal{B}' = \ker \partial_\Phi(L_\Psi)$ and $\mathcal{B}'$ is Hamiltonian with respect to the VBDF $L_\Psi$.

**Proof.** In order to prove the claim, we need to characterize the property of a skew-symmetric BDF being zero along a behavior, in terms of properties of the polynomial matrices involved in the representation of the BDF and of the system itself. The following result can be proven using Gröbner basis techniques (see [1]) and the results of [3].

**Lemma 17** Let $\mathcal{B} \in \mathcal{L}_1^n$ be represented in kernel form by a polynomial $R(\xi_1, \ldots, \xi_n)$, and let $\Phi$ be a $2n$-variable skew-symmetric polynomial. Then $L_\Phi(v, w) = 0$ for all $v, w \in \mathcal{B}$ if and only if there exists an $n$-variable polynomial $X$ such that

$$\Phi(\zeta, \eta) = R(\zeta)X(\eta) - X(\zeta)R(\eta)$$

We conclude from this lemma that if $\mathcal{B}$ is Hamiltonian with respect to VBDF $L_\Psi = \text{col}(L_{\Psi_1}, L_{\Psi_2}, \ldots, L_{\Psi_n})$, then

$$\sum_{i=1}^n (\zeta_i + \eta_i)\Psi_i(\zeta, \eta) = R(\zeta)X(\eta) - X(\zeta)R(\eta)$$

for some non-zero $n$-variable polynomial $X(\zeta)$. Now substitute $\zeta_i$ and $\eta_i$ with $-\xi_i$ and $\xi_i$ respectively, obtaining $R(-\zeta)X(\xi) - X(-\zeta)R(\xi) = 0$. Conclude from this
that the $n$-variable polynomial $R'(\xi) = X(-\xi)R(\xi)$ satisfies $R'(\xi) = R'(-\xi)$. Define $\mathfrak{B}' := \ker X(-\frac{\partial}{\partial \xi})R(\frac{\partial}{\partial \eta})$. Then $\mathfrak{B}'$ is autonomous, and obviously $\mathfrak{B} \subseteq \mathfrak{B}'$. Define a $2n$-variable polynomial $\Phi(\zeta, \eta)$ by

$$\Phi(\zeta, \eta) := \frac{1}{2} R'(\zeta) + \frac{1}{2} R'(\eta).$$

Then $\Phi(\zeta, \eta)$ satisfies $\partial \Phi = R'$ so $\mathfrak{B}' = \ker \partial \Phi (\frac{\partial}{\partial \zeta})$. Consequently, $\mathfrak{B}'$ is stationary w.r.t. $Q_\Phi$. Finally, $\mathfrak{B}'$.

We illustrate the result of Proposition 16 with an example.

**Example 18** Consider the model for the vibrations of an infinitely long beam illustrated in Example 13. We proceed as in the proof of Proposition 16, observing that with $\Psi(\zeta, \eta)$ defined as in (13), the following equation holds:

$$\sum_{i=1}^{n} (\zeta_i + \eta_i) \Psi_i(\zeta, \eta) = (\zeta_2^4 + \alpha^2 \zeta_2^2) - (\eta_2^4 + \alpha^2 \eta_2^2) = R(\zeta) - R(\eta)$$

i.e. $X(\zeta, \eta) = 1$ in equation (15). Conclude from this, substituting $-\xi$ in place of $\zeta$ and $\xi$ in place of $\eta$, that $R(-\xi) = R(\xi)$. Now define

$$\Phi(\zeta, \eta) := \frac{1}{2}(R(\zeta) + R(\eta)) = \frac{1}{2}(\zeta_2^4 + \alpha^2 \zeta_2^2 + \eta_2^4 + \alpha^2 \eta_2^2)$$

and observe that since $\partial \Phi = R$, $\mathfrak{B}$ is the set of stationary trajectories with respect to $Q_\Phi$.

**Example 19** Consider the behavior of an infinite vibrating plate illustrated in Example 2. It is a matter of straightforward (though tedious) verification to see that the behavior described by equation (3) is symplectic with respect to the VBDFs induced by

$$\Psi(\zeta, \eta) = \begin{pmatrix} \zeta_x^3 - \eta_x^3 - \zeta_y^3 + \eta_y^3 - \zeta_{xx} \eta_x + \zeta_{yy} \eta_y - \eta_{xx} \eta_x + \eta_{yy} \eta_y - \zeta_{x} \eta_{x} + \zeta_{y} \eta_{y} \\ \zeta_y^3 - \eta_y^3 - \zeta_x^3 + \eta_x^3 - \eta_x \zeta_{x} + \eta_y \zeta_{y} + \eta_{x} \zeta_{x} + \eta_{y} \zeta_{y} - \eta_{x} \eta_{x} - \eta_{y} \eta_{y} \end{pmatrix}$$

With easy calculations it can be shown that

$$\sum_{i=1}^{n} (\zeta_i + \eta_i) \Psi_i(\zeta, \eta) = (\rho \zeta_x^2 + \zeta_x^2 + 2z_{xx} \zeta_{xx} + \zeta_x^4) - (\rho \eta_x^2 + \eta_x^2 + 2z_{xx} \eta_{xx} + \eta_x^4)$$

Substituting $-\xi$ in place of $\zeta$ and $\xi$ in place of $\eta$, we obtain that $R(-\xi) = R(\xi)$. Now
define

\[ \Phi(\zeta, \eta) := \frac{1}{2}(R(\zeta) + R(\eta)) = \frac{1}{2} \rho(\zeta_i^2 + \eta_i^2) + \frac{1}{2}(\zeta_v^2 + \eta_v^2) + (\zeta_w^2 \zeta_y^2 + \eta_w^2 \eta_y^2) + \frac{1}{2}(\zeta_y^4 + \eta_y^4) \]

and observe that since \( \partial \Phi = R \), \( \mathcal{B} \) is the set of stationary trajectories with respect to \( Q_\Phi \).

6. CONCLUSIONS AND FURTHER WORK

We have used the formalism of bilinear- and quadratic differential forms in order to study symplectic and variational linear systems. We have shown that for systems described by ordinary linear constant-coefficient differential equations, a system is symplectic if and only if it is variational (see Proposition 8). The main results regarding systems described by linear, constant-coefficient PDEs are Proposition 14, in which we state that a variational system is also symplectic; and Proposition 16, showing that a scalar symplectic system is contained in a variational system.

Due to space limitations and to the need to concentrate on those aspect of our work which are more closely related to modeling, we have been forced to omit several interesting results which we briefly consider in the following.

Most notable among the results of [7] are the definition of controllable symplectic system and the characterization of symplecticity for autonomous- and controllable systems in terms of various representations (kernel, image, state-space). Other important results regarding 1D systems will be treated elsewhere: for example, the relationship of our definition of symplecticity with LQ-optimal control, which leads to a generalization of the Hamiltonian system as it is commonly intended in the state-space setting, is the subject of a forthcoming paper.

A great deal of work remains to be done for the construction of a representation-free theory of symplectic- and of variational systems described by linear, constant-coefficient PDEs. In particular, issues such as the equivalence- or lack of it- of symplectic and variational system with more than one external variable; the characterization of the symplectic and the variational property of a behavior in terms of properties of its representations; and the design of effective algorithms which, starting from a representation of the system, test the system for symplecticity, variationality, construct the quadratic functionals with respect to which the system is stationary, etc., need to be addressed.
REFERENCES