INTERCONNECTION AND DECOMPOSITION PROBLEMS WITH
MULTIDIMENSIONAL BEHAVIORS

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Abstract. In the behavioral approach, the notion of interconnection is the basis of control. In
this setting, feedback interconnection of systems is based on the still more fundamental concept of
regular interconnection, which has been introduced by J.C. Willems. This paper deals with multi-di-
dimensional systems, in particular systems described by linear, constant coefficient partial differential
equations. The following problem is addressed: given a plant, under what conditions does there exist
a controller such that their interconnection is regular, and has finite codimension with respect to a
certain desired system. If it exists, provide a constructive solution to the problem. The second part
of this paper treats the related problem of decomposition of systems. First, we study the problem
decomposing the system into its controllable part and an autonomous subsystem, and finally we
study decompositions of the controllable part.

Key words. multidimensional systems, module theory, duality, behavioral approach, regular
almost implementability, controllable-autonomous decomposition

AMS subject classifications. 13B10, 13C10, 13C12, 13N10, 33B37, 93B05, 93B25, 93C20.

1. Introduction. The behavioral approach relies on the idea that systems are
described by equations, but their properties are naturally described in terms of the set
of all solutions to the equations. This idea is formalized by the notion of system behav-
ior due to J.C. Willems. In this setting, a new perspective to control has been given,
which is based on interconnection of systems, and where no a priori input/output
partition is considered, see [22]. The act of controlling a system is simply viewed as
interconnecting its behavior $B$ with a controller behavior $B_c$ in order to achieve a desired
behavior $B_d = B \cap B_c$. Of particular interest is the kind of interconnection that is
called regular interconnection. In such interconnection, the restrictions imposed on
the plant by the controller are not redundant, i.e. the restrictions of the controller are
independent of the restrictions already present in the plant. Hence the notion of feed-
back control, which is of significant interest in modern control theory, is based on the
still more fundamental concept of regular interconnection. Feedback interconnection
is, indeed, a simple example of regular interconnection since the controller imposes
restrictions only on the plant input, which is not restricted by the plant.

The regular implementability problem can be formulated as follows: given a plant
behavior together with a desired behavior, find, if possible, another behavior (a con-
troller) such that the interconnection is regular and equal to the given desired behav-
ior.

Willems in [22, 23] stated and solved this problem for 1D behaviors. The multi-di-
ensional counterpart was treated by Rocha and Wood in [14, 15], Zerz and Lomadze
in [28], Shankar in [17], and Trentelman and Napp Avelli in [18]. Conditions and an
algorithm were given for solving the problem. Actually, for multidimensional behav-
iors, these conditions are very seldomly satisfied and strong properties are required of
the plant and the desired system.

Due to the importance of the regular implementability problem, it is worthwhile to
further study the limits of achievability by regular interconnection. Therefore, in this
paper we propose a new version of the regular implementability problem, with weaker

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requirements. We will treat the following problem: given a multidimensional plant behavior $\mathcal{B}$ and a certain desired behavior $\mathcal{B}_d$, find, if possible, another behavior (a controller) such that the interconnection is regular and is contained in the given desired behavior $\mathcal{B}_d$ with finite codimension, i.e. find a behavior $\mathcal{B}_c$ such that the interconnection is regular and such that the quotient behavior $\mathcal{B}_d/(\mathcal{B} \cap \mathcal{B}_c)$ is an autonomous behavior that is finite-dimensional as a vector space over the field $k$, see [6, 10]. In this paper, the field $k$ will be either $\mathbb{R}$ or $\mathbb{C}$. We also use the notation $\dim_k(\mathcal{B}_d/(\mathcal{B} \cap \mathcal{B}_c)) < \infty$ to stress that it is finite-dimensional as a vector space over the field $k$. Of course, in the 1D case, all autonomous behaviors form a finite-dimensional vector space. For general multidimensional variable behaviors this is no longer true. In fact, an autonomous $nD$ behavior that is finite-dimensional is called strongly autonomous in [12].

If a controller $\mathcal{B}_c$, satisfying the specifications of the previous paragraph exists, then we say that $\mathcal{B}_d$ is almost implementable by regular interconnection from $\mathcal{B}$. This constitutes a generalization of the regular interconnection problem, as it represents the ‘closest’ achievability one can get through regular interconnection in the sense of finite codimension.

In this paper we also investigate in some detail the related problem of decomposing a given behavior into the sum of finer components. It is immediately apparent that decomposition is a powerful tool for the analysis of the system properties. Decomposition is, indeed, of particular interest in the case of multidimensional systems, where a description of the $nD$ system trajectories can be complicated, and where decomposing the original behavior into smaller components seems to be an effective way for simplifying the systems analysis.

The controllable-autonomous decomposition has played a significant role in the theory of linear time-invariant systems. This kind of decomposition expresses the idea that every trajectory of the behavior can be thought of as the sum of two components: a free evolution, only depending on the set of initial conditions, and a forced evolution, due to the presence of inputs. In the case of 1D systems, this sum can be chosen to be direct, i.e. $\mathcal{B} = \mathcal{B}_{\text{cont}} + \mathcal{B}_{\text{aut}}$ and $\mathcal{B}_{\text{cont}} \cap \mathcal{B}_{\text{aut}} = 0$. Here $\mathcal{B}_{\text{cont}}$ denotes the controllable part of $\mathcal{B}$, and $\mathcal{B}_{\text{aut}}$ an autonomous subbehavior of $\mathcal{B}$.

It was proven in [25, 6, 26] that an $nD$ system can still be decomposed into the sum of the controllable part of the system and an autonomous subbehavior. However, for $n \geq 2$ such decomposition can, in general, no longer be chosen to be a direct sum decomposition, and we may have that the controllable part of $\mathcal{B}$ (which is uniquely defined for a given $\mathcal{B}$) intersects all possible autonomous subbehaviors involved in a controllable-autonomous decomposition, see [25, 20, 4].

Finally, in our quest to completely decompose a behavior into smaller components, we address the problem of decomposing the controllable part itself. The following problem is studied: given a controllable $nD$ behavior $\mathcal{B}$ and a subbehavior $\mathcal{B}_1 \subset \mathcal{B}$, find a behavior $\mathcal{B}_2 \subset \mathcal{B}$ such that $\mathcal{B}_1 + \mathcal{B}_2 = \mathcal{B}$ and the intersection $\mathcal{B}_1 \cap \mathcal{B}_2$ has finite dimension as a vectorspace over $k$. If such $\mathcal{B}_2$ exists, then we say that $\mathcal{B}_1$ is an almost direct summand of $\mathcal{B}$. This constitutes a generalization of the direct sum decomposition as it represents a decomposition with ”minimal” intersection.

In this paper we denote the polynomial ring $k[x_1, x_2, \ldots, x_n]$ of polynomials in $n$ indeterminates with coefficients in the field $k = \mathbb{R}$ or $\mathbb{C}$, by $\mathcal{D}$. We identify this ring with the ring of partial differential operators with constant coefficients $k[\partial_1, \partial_2, \ldots, \partial_n]$.

We mainly investigate the problems outlined above for the case $n = 2$ (i.e. $\mathcal{D} = \mathbb{R}[x, y]$ or $\mathcal{D} = \mathbb{C}[x, y]$).
$k[x_1, x_2])$, even though some results are still valid for any $n$.

The outline of this paper is as follows. In section 2 we review the basic material on multidimensional behaviors and the relation with homological algebra. Section 3 forms the core of this paper. We review the notion of localization for rings, and via a series of lemmas arrive at the main technical result of this paper. The result states that every finitely generated, torsion free module over the polynomial ring in two variables can be extended to a free module, with finite codimension. We show that this free module is unique, and we also explain how it can be computed using MAPLE. In section 4, we apply our main result to the problem of regular almost implementability as introduced above. In section 5 we study the autonomous-controllable decomposition. We show that every 2D behavior can be written as the sum of its controllable part and some autonomous subbehavior, while their intersection is finite dimensional. We also provide a counterexample showing that this result does not hold for $n=3$. Finally, in section 6 we study the problem of decomposing the controllable part of the system behavior.

2. Multidimensional behaviors. In this section we review some concepts of $n$D behavioral systems. For a nice overview we refer to, for example, [12], [27] or [25].

In the behavioral approach to $n$D systems, a system is defined by a triple $(\mathcal{A}, q, \mathcal{B})$, where $\mathcal{A}$ is the signal space, $q \in \mathbb{Z}^+$ is the number of components and $\mathcal{B} \subset \mathcal{A}^q$ is the behavior. In this paper, we assume $\mathcal{A}$ to be the space of all infinitely often differentiable functions from $\mathbb{R}^n$ to $k$ (denoted by $C^\infty(\mathbb{R}^n, k)$), or all $k$-valued distributions on $\mathbb{R}^n$ (denoted by $\mathcal{D}'(\mathbb{R}^n, k)$).

We call $\mathcal{B}$ a linear differential $n$D behavior or simply $n$D behavior if it is the solution set of a system of linear, constant-coefficient partial differential equations, more precisely, if $\mathcal{B}$ is the subset of $\mathcal{A}^q$ consisting of all solutions to

$$R(\frac{d}{dx})w = 0 \quad (2.1)$$

where $R$ is a polynomial matrix in $n$ indeterminates $x_i, i = 1, \ldots, n$, and where we use the shorthand notation $\frac{d}{dx}$ for $(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n})$. The elements of $\mathcal{B}$ are called trajectories. We call (2.1) a kernel representation of $\mathcal{B}$ and we write $\mathcal{B} = \ker(R)$. Obviously, any linear differential $n$D behavior $\mathcal{B}$ is a $k$-linear subspace of $\mathcal{A}^q$. Furthermore, it has the structure of a module over the ring of partial differential operators $\mathcal{D}$.

Since the theory of discrete linear shift-invariant system as discussed in [9] is completely analogous to that of the present paper, the same tools and similar conclusions will apply in the discrete case $\mathcal{A} = k\mathbb{N}^n$. For the sake of simplicity we will however focus on the continuous case $\mathcal{A} = C^\infty(\mathbb{R}^n, k)$.

It was shown in [9] that there is a one-to-one correspondence between $n$D behaviors and submodules of $\mathcal{D}^q$. With any $n$D behavior $\mathcal{B} \subset \mathcal{A}^q$ we associate the submodule $\mathcal{B}^\perp$ of $\mathcal{D}^q$ defined by

$$\mathcal{B}^\perp := \{ r \in \mathcal{D}^q \mid r(\frac{d}{dx})w = 0 \text{ for all } w \in \mathcal{B} \}.$$

Conversely, for any submodule $M$ of $\mathcal{D}^q$ we have that

$$M^\perp := \{ w \in \mathcal{A}^q \mid r(\frac{d}{dx})w = 0 \text{ for all } r \in M \}$$

is an $n$D behavior. Indeed one has that $(\mathcal{B}^\perp)^\perp = \mathcal{B}$ and $(M^\perp)^\perp = M$. With this bijection, we have $(\mathcal{B}_1 \cap \mathcal{B}_2)^\perp = \mathcal{B}_1^\perp + \mathcal{B}_2^\perp$ and $(M_1 \cap M_2)^\perp = M_1^\perp + M_2^\perp$. If
\( \mathcal{B} = \ker(R) \) then \( \mathcal{B}^\perp \) is the submodule of \( \mathcal{D}^q \) of all \( \mathcal{D} \)-linear combinations of the rows of \( R \), which is denoted by \( (R) \).

For any \( \mathcal{D} \)-module \( M \), let \( \text{Hom}_\mathcal{D}(M, \mathcal{A}) := \{ \ell \mid \ell \text{ is a } \mathcal{D} \text{-linear map from } M \text{ to } \mathcal{A} \} \).

It was proven by Malgrange in [8] that any \( n \mathcal{D} \) behavior \( \mathcal{B} \subset \mathcal{A}^q \) is isomorphic to \( \text{Hom}(\mathcal{D}^q/\mathcal{B}^\perp, \mathcal{A}) \). The isomorphism is given by

\[
\chi : \text{Hom}(\mathcal{D}^q/\mathcal{B}^\perp, \mathcal{A}) \xrightarrow{\sim} \mathcal{B} \subset \mathcal{A}^q,  \tag{2.2}
\]

where \( \ell \in \text{Hom}(\mathcal{D}^q/\mathcal{B}^\perp, \mathcal{A}) \), \( \{ e_1, e_2, \ldots, e_q \} \) is the standard basis of \( \mathcal{D}^q \) and \( \{ \bar{e}_1, \bar{e}_2, \ldots, \bar{e}_q \} \) their images in \( \mathcal{D}^q/\mathcal{B}^\perp \). Because of this, we identify any \( n \mathcal{D} \) behavior \( \mathcal{B} \subset \mathcal{A}^q \) with \( \text{Hom}(\mathcal{D}^q/\mathcal{B}^\perp, \mathcal{A}) \).

If \( M \) is a finitely generated \( \mathcal{D} \)-module, then we use the notation \( D(M) := \text{Hom}_\mathcal{D}(M, \mathcal{A}) \) and \( M^* := \text{Hom}_\mathcal{D}(\mathcal{A}, \mathcal{D}) \). Often we will omit an explicit reference to the ring \( \mathcal{D} \), as there will be no ambiguity, and write \( \text{Hom}(-, -) \) instead of \( \text{Hom}_\mathcal{D}(-, -) \).

We note that for any finitely generated \( \mathcal{D} \)-module there exists a positive integer \( q \) and a finitely generated \( \mathcal{D} \)-module \( N \) such that \( M = \mathcal{D}^q/N \). Thus, \( D(M) \) can be interpreted as the \( n \mathcal{D} \) behavior \( \mathcal{B} \subset \mathcal{A}^q \) given by \( \mathcal{B} = N^\perp \).

We now introduce some basic definitions, mathematical tools and known results that will be needed in the rest of the paper.

Given \( \mathcal{D} \)-modules \( B, C \), and \( E \), and a \( \mathcal{D} \)-linear map \( \alpha : B \to C \), we define the map

\[
\text{Hom}(\alpha, E) : \text{Hom}(C, E) \to \text{Hom}(B, E) \quad \text{by} \quad \varphi \mapsto \varphi \circ \alpha.
\]

We denote \( \bar{\alpha} := \text{Hom}(\alpha, E) \).

**Definition 1.** A sequence \( \cdots \to A_{j-1} \xrightarrow{d_j} A_j \xrightarrow{d_{j+1}} A_{j+1} \to \cdots \) of \( \mathcal{D} \)-modules and homomorphisms \( d_j \) is called exact if for every \( j \) we have \( \ker(d_{j+1}) = \text{im}(d_j) \) and is called complex if for every \( j \) we have \( \ker(d_{j+1}) \supset \text{im}(d_j) \). For example, the sequence

\[
0 \to A_1 \xrightarrow{\alpha} A_2 \xrightarrow{\beta} A_3 \to 0
\]

is exact if and only if \( \alpha \) is injective, \( \beta \) surjective and \( \ker(\beta) = \text{im}(\alpha) \). In other words, \( A_1 \) can be identified with a submodule of \( A_2 \), and \( A_3 \) with the module \( A_2/A_1 \). Exact sequences are an easy way to express algebraic and system-theoretic properties.

Based on the work of Malgrange [8] and Palamodov [11], Oberst in [9] proved the following:

**Proposition 2.** Let \( B, C \) and \( D \) be finitely generated \( \mathcal{D} \)-modules, and \( \alpha : B \to C \) and \( \beta : C \to D \) be \( \mathcal{D} \)-linear maps. If the sequence

\[
0 \to B \xrightarrow{\alpha} C \xrightarrow{\beta} D \to 0  \tag{2.3}
\]

is complex, then the dual sequence

\[
0 \leftarrow \text{Hom}(B, \mathcal{A}) \xleftarrow{\bar{\alpha}} \text{Hom}(C, \mathcal{A}) \xleftarrow{\bar{\beta}} \text{Hom}(D, \mathcal{A}) \leftarrow 0  \tag{2.4}
\]

is complex. Moreover, we have: (2.3) is exact if and only if (2.4) is exact. The last theorem amounts to saying that the signal space \( \mathcal{A} \) is an injective cogenerator. It is important to note that many other signal spaces, e.g. the set of smooth functions with compact support, are not injective cogenerators [16].
Given a $\mathcal{D}$-module $M$, an element $m \in M$ is called a torsion element if there exists $0 \neq d \in \mathcal{D}$ such that $dm = 0$. The subset of torsion elements is a submodule of $M$. If this submodule is the 0-module, then $M$ is called torsion free. If every element of $M$ is a torsion element, we say that $M$ is torsion.

We now review some basic material on interconnection of behaviors. In the behavioral approach, interconnection of systems is defined by intersection of the corresponding behaviors. Thus the interconnected behavior consists of those trajectories that satisfy the equations of both systems, i.e., if $\mathcal{B}_1 = \ker(R_1)$ and $\mathcal{B}_2 = \ker(R_2)$ then $\mathcal{B}_1 \cap \mathcal{B}_2 = ((R_1) + (R_2))^{\perp} = \ker(\frac{R_1}{R_2})$.

**Definition 3.** Let $\mathcal{B}_1, \mathcal{B}_2 \subset \mathcal{A}^q$ be nD behaviors. The interconnection $\mathcal{B}_1 \cap \mathcal{B}_2$ is called regular if $\mathcal{B}_1^{\perp} \cap \mathcal{B}_2^{\perp} = 0$. Thus, regular interconnection expresses the idea that the controller imposes new constraints on the plant which are not already present, i.e., there is no redundancy between the laws of the plant and the controller. For example, any feedback interconnection is a regular interconnection since the controller imposes restrictions only on the input of the plant, which is unconstrained by the plant.

**Definition 4.** Let $\mathcal{B} \subset \mathcal{A}^q$ be an nD behavior. The $i$-th component $w_i$ of $w$ is called a free variable if $\pi_i : \mathcal{B} \rightarrow \mathcal{A}$ given by $w \mapsto w_i$ is surjective. The behavior $\mathcal{B}$ is called autonomous if it has no free variables.

**Proposition 5.** (see [12, 25]) Let $\mathcal{B} \subset \mathcal{A}^q$ be an nD behavior. The following statements are equivalent:
1. $\mathcal{B}$ is autonomous,
2. $\mathcal{D}^q/\mathcal{B}^\perp$ is torsion.

A strong form of autonomy is studied next.

**Definition 6.** An nD behavior $\mathcal{B} \subset \mathcal{A}^q$ is said to be strongly autonomous if it is finite dimensional as a vector space over $k$.

**Proposition 7.** (see [12]) Given a strongly autonomous nD behavior $\mathcal{B} \subset \mathcal{A}^q$, then for every open non-empty subset $U \subset \mathbb{R}^n$ the restriction map $\pi_U : \mathcal{A}^q \rightarrow (\mathcal{A}|_U)^q$ is injective on $\mathcal{B}$. Thus, a trajectory of a strongly autonomous behavior is determined by its values on any open subset of $\mathbb{R}^n$.

**Definition 8.** An nD behavior $\mathcal{B} \subset \mathcal{A}^q$ is said to be controllable if for all $w_1, w_2 \in \mathcal{B}$ and all subsets $U_1, U_2 \subset \mathbb{R}^n$ with disjoint closure, there exist a $w \in \mathcal{B}$ such that $w|_{U_1} = w_1|_{U_1}$ and $w|_{U_2} = w_2|_{U_2}$.

**Proposition 9.** (see [12, 25]) Given an nD behavior $\mathcal{B} \subset \mathcal{A}^q$, the following conditions are equivalent:
1. $\mathcal{B}$ is controllable,
2. $\mathcal{D}^q/\mathcal{B}^\perp$ is torsion free.

It is well-known (see e.g., [25, 24, 6]) that every nD behavior $\mathcal{B}$ has a 'controllable-autonomous decomposition', in the sense that it can be decomposed as $\mathcal{B} = \mathcal{B}_{cont} + \mathcal{B}_{aut}$ for subbehaviors $\mathcal{B}_{cont}$ and $\mathcal{B}_{aut}$ of $\mathcal{B}$, with $\mathcal{B}_{cont}$ controllable and $\mathcal{B}_{aut}$ autonomous. Furthermore, $\mathcal{B}_{cont}$ is uniquely determined by these conditions, and is called the controllable part of $\mathcal{B}$. It was shown in [24] to be the largest controllable subbehavior of $\mathcal{B}$.

Note that if $\mathcal{B} = D(M)$ and $\tilde{M}$ is a submodule of $M$ then $D(M/\tilde{M})$ is a subbehavior of $\mathcal{B}$, see [24]. If $\mathcal{B} = D(M)$, then the controllable part of $\mathcal{B}$ is equal to $\mathcal{B}_{cont} = D(M/M_t)$ in $\mathcal{B}$ with $M_t$ the torsion part of $M$. If $A \subset M$ and $M/A$ is torsion, then $D(M/A)$ is an autonomous subbehavior of $\mathcal{B}$. In contrast with the controllable part, the set of autonomous subbehaviors of $\mathcal{B}$ does in general not have a maximal element. Thus, in general there does not exist a unique 'autonomous part' of $\mathcal{B}$.

The following definition was first introduced in [14], see also [15].
DEFINITION 10. The nD behavior $\mathcal{B} \subset \mathcal{A}^n$ is said to be strongly controllable if $\mathcal{D}^\theta/\mathcal{B}^\perp$ is a free module.

DEFINITION 11. An nD behavior $\mathcal{B}$ is said to be regular if $\mathcal{B}^\perp$ is a free module, equivalently, $\mathcal{B}$ has a full row rank kernel representation.

For the case $n = 1$, all behaviors are regular. This is not longer true for $n \geq 2$, as can be seen by taking for example the 2D differential behavior $\mathcal{B} = \ker(R)$ with $R(x_1, x_2) = (\frac{x_1}{x_2})$, consisting of all constant functions of two variables. This behavior cannot be described as the kernel of a single polynomial operator.

3. From torsion free to free with finite codimension. In this section we will provide most of the technical results of this paper. In particular, we will show that, for the case that $\mathcal{D} = k[x_1, x_2]$ (the polynomial ring in two indeterminates) any finitely generated torsion free $\mathcal{D}$-module is contained in a unique, free $\mathcal{D}$-module with finite codimension. This result will enable us to establish several results on 2D behaviors. We will also explain how this unique free module can be computed using MAPLE, and give an example. The proof of the main result of this section hinges on a number of lemmas that apply to the polynomial ring $\mathcal{D}[x_1, x_2]$.

The main algebraic tool that we will use is localization. Localization is a systematic method of adding multiplicative inverses to a ring in order to construct local rings out of a ring. This notion allows us to reduce many questions concerning arbitrary rings to local rings. A ring is called local if it has exactly one maximal ideal. The unique maximal ideal consists precisely of the non-invertible elements of the ring.

Let $R$ be a ring (always commutative with identity element 1), and let $S \subset R$ be a multiplicative set (i.e. $1 \in S$, and $s_1, s_2 \in S$ implies $s_1s_2 \in S$). We introduce the following equivalence relation $\sim$ on $R \times S$:

$$(a, s) \sim (b, t) \iff \exists u \in S \text{ such that } u(at - bs) = 0.$$ 

We will write $a/s$ for the equivalence class of $(a, s)$. Then the ring of fractions of $R$ with respect to $S$, denoted by $S^{-1}R$, is $(R \times S)/\sim$ with ring operations defined by the usual arithmetic operations on fractions:

$\frac{a}{s} \pm \frac{b}{t} = \frac{at \pm bs}{st}$ and $\frac{a}{s} \cdot \frac{b}{t} = \frac{ab}{st}$.

PROPOSITION 12. The following statements hold:

1. The ring operations are well defined, and $S^{-1}R$ is a ring.
2. $\varphi : R \longrightarrow S^{-1}R$ defined by $a \longrightarrow a/1$ is a ring homomorphism.

Given a ring $R$, there are two popular and useful choices of multiplicative subsets $S \subset R$:

1. $S = \{1, z, z^2, z^3, \ldots\}$, for a given element $z \in R$. In this case we write $R_z := S^{-1}R$.
2. $S = R \setminus m$ where $m$ is a maximal ideal of the ring $R$. Then $S^{-1}R$ is a local ring with unique maximal ideal $m \cdot S^{-1}R$. The local ring $S^{-1}R$ is called the localization of $R$ at $m$, and denoted by $R_m := S^{-1}R$.

Note that if $0 \in S$ then $S^{-1}R = 0$.

We now proceed in a similar way with modules instead of ideals. Let $M$ be an $R$-module and $S \subset R$ a multiplicative set. Define the equivalence relation $\sim$ on $M \times S$ as follows:

$$(m, s) \sim (n, t) \iff \exists u \in S \text{ such that } u(tm - sn) = 0.$$
Denote \((M \times S)/{\sim}\) by \(S^{-1}M\). This is again a module, this time over the ring \(S^{-1}R\), with operations defined by:

\[
\frac{m}{s} \pm \frac{m}{t} = \frac{tm \pm sn}{st} \quad \text{and} \quad \frac{a}{s} \cdot \frac{n}{t} = \frac{an}{st}.
\]

If \(S = R \setminus m\), with \(m\) a maximal ideal, then \(S^{-1}M\) is a module over the local ring \(S^{-1}R = R_m\). This \(S^{-1}R\)-module is denoted by \(M_m\). If \(S = \{1, z, z^2, \ldots\}\) for a given element \(z\) then we denote \(S^{-1}M = M_z\).

If \(D = k[x_1, x_2, \ldots, x_n]\), \(M\) a finitely generated \(D\)-module and \(S = D \setminus \{0\}\), then \(S^{-1}D\), denoted by \(DS\), is the set of rational functions. Furthermore, \(S^{-1}M = \{\frac{m}{s} \mid m \in M, 0 \neq p \in D\}\) is a vector space over the field \(S^{-1}D\), which is denoted by \(MS\).

We will now introduce the notion of \emph{codimension}. Codimension is a term used to indicate the difference between the dimensions of a certain set and one of its subsets.

**Definition 13.** Let \(A \subseteq B\) be finitely generated \(D\)-modules. We say that \(A\) has \emph{finite codimension} in \(B\) if the dimension of \(B/A\) as a vector space over \(k\) is finite i.e. \(\dim_k(B/A) < \infty\). If this condition holds, then we write \(A \subset B\). It is easily seen that if \(A \subseteq B\) and if \(A \subset C \subseteq B\), then \(C \subset B\). Also, if \(A \subset B \subset C\) then \(A \subset C\).

**Lemma 14.** Let \(M\) be a finitely generated \(D\)-module. The following statements are equivalent:

1. \(\dim_k(M) < \infty\),
2. for all \(i = 1, 2, \ldots, n\) there exists a nonzero polynomial \(p_i(x_i) \in k[x_i]\) such that \(p_i(x_i)M = 0\),
3. there exists an ideal \(I \subset D\) with \(I \cdot M = 0\),
4. \(\dim(\Hom(M, A)) < \infty\).

Furthermore, for any pair of \(nD\) behaviors \(\mathcal{B}_1, \mathcal{B}_2 \subset A^q\) we have: \(\mathcal{B}_2 \subset \mathcal{B}_1\) if and only if \(\mathcal{B}_1^+ \subset \mathcal{B}_2^+\).

**Proof.** First note that any finitely generated \(D\)-module can be written in the form \(D^q/N\) for some \(q\) and some submodule \(N\) of \(D^q\).

(1) \(\Rightarrow\) (2): Let \(\{f_1, f_2, \ldots, f_s\}\) be a set of generators of \(M = D^q/N\). For \(1 \leq i \leq n\) and \(1 \leq j \leq s\) consider the sequence \(\{f_j, x_j f_j, x_j^2 f_j, \ldots\}\) in \(M\). Since \(\dim_k(M) < \infty\) there exists \(t \in \mathbb{N}\) and \(a_0, a_1, a_2, \ldots, a_{t-1} \in k\) such that \(x_j^t f_j = a_0 + a_1 x_j + a_2 x_j^2 + \cdots + a_{t-1} x_j^{t-1} + a_t x_j^t\). Define a polynomial \(p_j(x) := -x_j^t + a_{t-1} x_j^{t-1} + a_t x_j^t + \cdots + a_0\). Then clearly \(p_j(x) f_j = 0\). In this way, one may construct \(p_j\) for all \(j = 1, 2, \ldots, s\), and their product \(p_i(x) := p_1(x) p_2(x) \cdots p_s(x)\) clearly satisfies \(p_i(x)M = 0\). Since this holds for any \(i = 1, \ldots, n\), this proves statement 2.

(2) \(\Rightarrow\) (3): Let \(t_i\) be the degree of \(p_i\). A basis of \(D/(p_1, p_2, \ldots, p_n)\) as a vector space over \(k\) is given by \(\{1, x_1, \ldots, x_1^{t_1-1}, x_2, \ldots, x_2^{t_2-1}, \ldots, x_n, \ldots\}\). Thus \(I := (p_1, p_2, \ldots, p_n) \subset D\) and \(I \cdot M = 0\).

(3) \(\Rightarrow\) (4): Fix \(1 \leq i \leq n\). Consider the equivalence classes \(1, \bar{x}_i, \bar{x}_i^2, \ldots\) in \(D/I\). Since \(I \subset D\), there exists an integer \(s\) and coefficients \(a_0, a_1, a_2, \ldots, a_s \in k\) such that \(a_0 x_i^s + a_{s-1} x_i^{s-1} + a_{s-2} x_i^{s-2} + \cdots + a_0 = 0\) in \(D/I\). Therefore there exists a nonzero polynomial \(a_0 x_i^s + a_{s-1} x_i^{s-1} + a_{s-2} x_i^{s-2} + \cdots + a_0 \in I\). Denote it by \(p_i(x)\). Now, let \(\{f_1, f_2, \ldots, f_s\}\) be a set of generators of \(M = D^q/N\). Every \(\ell \in \Hom(M, A)\) can be identified with \(\ell(f_1), \ell(f_2), \ldots, \ell(f_s) \in A\). Hence it is sufficient to show that
\{ \ell(f_i) \in \mathcal{A} \mid \ell \in \text{Hom}(M, \mathcal{A}) \} \text{ is finite dimensional. Since } I f_1 = 0 \Rightarrow I f = 0 \text{ one has that } \{ \ell(f_i) \in \mathcal{A} \mid \ell \in \text{Hom}(M, \mathcal{A}) \} \subset \{ g \in \mathcal{A} \mid I g = 0 \} \subset \{ g \in \mathcal{A} \mid p_i(x)g = 0, \ i = 1, 2, \ldots, n \}. \text{ Thus it suffices to show that } \{ g \in \mathcal{A} \mid p_i(x)g = 0, \ i = 1, 2, \ldots, n \} \text{ is finite dimensional. This follows from the fact that } \{ g \in \mathcal{A} \mid p_i(x)g = 0, i = 1, 2, \ldots, n \} = \text{span}_\mathbb{K} \{ x_1^{a_1} x_2^{a_2} \ldots x_n^{a_n} \exp(\lambda_1 x_1 + \lambda_2 x_2 + \ldots + \lambda_n x_n) \mid p_i(\lambda_i) = 0, a_i = 0, 1, \ldots, \deg(\lambda_i) - 1 \}, \text{ which has finite dimension over } k. \text{ Here, } \deg(\lambda_i) \text{ denotes the multiplicity of } \lambda_i \text{ as a root of } p_i(x_i).

(4) \Leftrightarrow (1): A proof of this can be found in [10], theorem 2.

Finally, by theorem 2, exactness of the sequence

\[ 0 \longrightarrow \mathcal{B}_2^+ / \mathcal{B}_1^+ \longrightarrow \mathcal{D}^\prime / \mathcal{B}_1^+ \longrightarrow \mathcal{D}^\prime / \mathcal{B}_2^+ \longrightarrow 0 \tag{3.1} \]

implies that

\[ 0 \longrightarrow \text{Hom}(\mathcal{B}_2^+ / \mathcal{B}_1^+, \mathcal{A}) \longrightarrow \mathcal{B}_1 \longrightarrow \mathcal{B}_2 \longrightarrow 0 \tag{3.2} \]

is exact. This implies that \( \text{Hom}(\mathcal{B}_2^+ / \mathcal{B}_1^+, \mathcal{A}) \approx \mathcal{B}_1 / \mathcal{B}_2 \). Hence, using the equivalence of statements 1 and 4, \( \mathcal{B}_2^+ / \mathcal{B}_1^+ \) is finite dimensional if and only if \( \text{Hom}(\mathcal{B}_2^+ / \mathcal{B}_1^+, \mathcal{A}) \approx \mathcal{B}_1 / \mathcal{B}_2 \) is finite dimensional.

Throughout the rest of this section we will take \( n = 2 \) and consider the ring \( \mathcal{D} = \mathbb{k}[x_1, x_2] \). The results we obtain are valid for this particular ring.

**Lemma 15.** Let \( \mathcal{D} = \mathbb{k}[x_1, x_2] \). Let \( I \subset \mathcal{D} \) be an ideal, and let \( \{ f_1, \ldots, f_m \} \) be a generating set for \( I \). Then the following three statements are equivalent:

1. \( I \subset \mathcal{D} \).
2. The greatest common divisor (g.c.d) of \( \{ f_1, \ldots, f_m \} \) is equal to 1.
3. \( Z(I) := \{(a_1, a_2) \in \mathbb{C}^2 \mid f(a_1, a_2) = 0 \forall f \in I \} \) is finite.

**Proof.** Let \( \sqrt{I} = \{ d \in \mathcal{D} \mid d^m \in I \text{ for some positive integer } m \} \) be the radical of \( I \).

(1) \( \Rightarrow \) (2): Assume that the g.c.d of \( f_1, \ldots, f_m \) is equal to \( f \), with \( f \) nonconstant. Then clearly \( I \subset \langle f \rangle \). As a consequence of the Noether normalization (see [13]), after a linear change of variables \( f \) can be written as \( f = x_2^j + a_{d-1}(x_1)x_2^{d-1} + \cdots + a_0(x_1) \), and therefore \( \mathcal{D} / \langle f \rangle \) is a free \( \mathbb{k}[x_1] \)-module of rank \( d \) with infinite dimension over \( k \).

(2) \( \Rightarrow \) (3): Every \( f_i \) can be decomposed as \( f_i = g_1g_2 \ldots g_t \) with \( g_i \) irreducible and hence \( Z(g_i) \) is an irreducible curve in \( \mathbb{C}^2 \). Hence \( Z(\langle f_i \rangle) \) is a finite union of irreducible curves, and therefore \( Z(I) \) is the intersection of all \( \Gamma_i \). The assumption g.c.d(\( f_1, \ldots, f_m \)) = 1 means that the curves \( \Gamma_i \) do not coincide anywhere and therefore they intersect just in points, i.e. the set \( Z(I) \) contains just points and it is finite because \( \Gamma_1, \Gamma_2, \ldots, \Gamma_r \) can not intersect infinitely many times.

(3) \( \Rightarrow \) (1): Since \( Z(I) = Z(\sqrt{I}) \) one has that \( Z(\sqrt{I}) \) is finite. We use that \( \sqrt{I} \) is equal to the intersection of all prime ideals \( P_i \) containing \( I \) (see [1]). For all \( P_i \supset I \) one has that \( Z(\sqrt{I}) \supset Z(P_i) \) which implies \( P_i \) is a maximal ideal (since \( Z(\sqrt{I}) \) finite).

For any two maximal ideals \( m_1, m_2 \subset \mathcal{D} \) one has that \( Z(m_1) = Z(m_2) \Leftrightarrow m_1 = m_2 \), so there exists finite number of \( P_i \) containing \( I \) such that \( \sqrt{I} = \cap P_i \). Hence, \( \sqrt{I} = \cap_{i=1}^r m_i \) for some \( t \in \mathbb{N} \), with \( m_i \) maximal ideals in \( \mathcal{D} \).

Consider first \( k = \mathbb{R} \). Each maximal ideal of \( \mathbb{R}[x_1, x_2] \) is of the form \( \mathfrak{m} = (p, g) \subset \mathcal{D} \) where \( p \in \mathbb{R}[x_1] \) irreducible and \( g \in \mathcal{D} \) irreducible in \( \mathbb{R}[x_1]/(p)[x_2] \), see [13].

We claim that there exists \( h(x_1, x_2) = x_2^r + q_{r-1}(x_1) + \cdots + q_0(x_1) \in \mathfrak{m} \) for some \( r \in \mathbb{N} \) and \( q_{r-1}, \ldots, q_0 \in \mathbb{R}[x_1] \). Proof of the claim: \( g(x_1, x_2) \in \mathbb{R}[x_1]/(p)[x_2] \), with not all coefficients in \( p \) (otherwise \( \mathfrak{m} = (p) \) which is not maximal). One can always write \( g(x_1, x_2) = x_2^r g_0(x_1) + x_2^{r-1} g_1(x_1) + \cdots + g_0(x_1) \). Take \( h \equiv g(x_1, x_2) \mod(p) \).
Since \( p \) is a non zero prime in the principal ideal domain \( \mathbb{R}[x_1] \), \( p \) is actually maximal and therefore \( \mathbb{R}[x_1]/(p) \) is a field. Hence there exists \( f \in \mathbb{R}[x_1] \) such that \( fh = x_2^2 + x_2^{-1}q_{x_1-1}(x_1) + \cdots + q_0(x_1) \mod (p) \) for some \( s' \leq s \). Then \( x_2^2 + x_2^{-1}q_{x_1-1}(x_1) + \cdots + q_0(x_1) \mod (p) \) is in \( \mathfrak{m} \).

By the above, each of \( \mathfrak{m}_i \) \( 0 \leq i \leq t \) contains a \( p_i \in \mathbb{R}[x_1] \) and \( h_i(x_1, x_2) = x_2^N + lower \) order terms in \( x_2 \) (with coefficient in \( \mathbb{R}[x_1] \)). Then \( p' := p_1 \cdots p_t = x_1^{N_1} + f_1, f_1 \in \mathbb{R}[x_1] \) with degree \( (f_1) < N_1 \) and \( h' := h_1 \cdots h_t = x_2^{N_2} + f_2, f_2 \in \mathbb{R}[x_1][x_2], \) degree \( (f_2) < N_2 \). By construction, \( p' \) and \( h' \) are in the product of the \( \mathfrak{m}_i \), and hence in \( \sqrt{I} \) so that there exists an \( N \in \mathbb{N} \) such that \( p^N \) and \( h^N \) are in \( I \). This implies that \( \mathfrak{D}/I \) has finite dimension over \( k \).

Similar arguments can be used for the case \( k = \mathbb{C} \), where each maximal ideal of \( \mathbb{C}[x_1, x_2] \) is of the form \( \mathfrak{m} = (x_1 - \alpha_1, x_2 - \alpha_2) \), where \( (\alpha_1, \alpha_2) \in \mathbb{C}^2 \).

**Lemma 16.** Let \( \mathfrak{D} = k[x_1, x_2] \). Let \( \mathfrak{m} \) be a maximal ideal of \( \mathfrak{D} \). Let \( \mathfrak{D}_m \) be the localized local ring of \( \mathfrak{D} \) at \( \mathfrak{m} \), and let \( z_1, z_2 \) be generators of the unique maximal ideal in \( \mathfrak{D}_m \). Let \( N \) be a finitely generated \( \mathfrak{D}_m \)-module \( N \) with the properties: \( N \) is torsion free, and \( N = N_1 \cap N_2 \). Then \( N \) is free.

**Proof.** We claim that the \( \mathfrak{D}_m/(z_2) \)-module \( \overline{N} := N/z_2N \) is torsion free. Indeed, if this module has torsion then there exists a non zero element \( \pi \in \overline{N} \), the image of \( n \in N \), with \( \pi \overline{n} = 0 \), and with \( \pi \in \mathfrak{D}_m/(z_2) \). Since \( \pi = cz_1 \) with \( c \) in \( \mathfrak{D}_m \), we may take \( a = z_1 \). Thus \( z_1n = x_2N \) and we may write \( z_1n = z_2\overline{n} \) and \( \xi := z_2n = z_1n^2 \). Then \( \xi \in N_1 \cap N_2 \) and \( n = z_2\xi \) is in contradiction with \( \pi \neq 0 \).

Since \( \mathfrak{D}_m/z_2 \mathfrak{D}_m \) is a principal ideal domain (it is in fact a discrete valuation ring, see [1]) every torsion free \( (\mathfrak{D}_m/z_2 \mathfrak{D}_m) \)-module is free. Hence the module \( \overline{N} \) is free and we can choose elements \( n_1, \ldots, n_s \in N \) such that their images in \( \overline{N} \) form a free basis. In particular, their images \( \{\overline{n_i}\} \) in \( \overline{N} = N/(x_1, x_2)N \) form a basis over the residue field of \( \mathfrak{D}_m \). It follows (by Nakayama’s lemma) that the \( \{n_i\} \) generate \( N \). Suppose now that there is a non trivial relation \( f_1n_1 + \cdots + f_sn_s = 0 \). Then all \( f_i \) lie in the maximal ideal of \( \mathfrak{D}_m \). Since \( N \) has no zero divisors, one may divide by the g.c.d. of all \( f_i \) and find a relation, again written as \( f_1n_1 + \cdots + f_sn_s = 0 \), where the g.c.d. of all \( f_i \) is 1.

Write \( f_i = z_1g_i + z_2h_i \), with \( g_i, h_i \in \mathfrak{D}_m \). Then \( z_1(\sum g_i n_i) + z_2(\sum h_i n_i) = 0 \). Thus \( z_1(\sum g_i n_i) \in z_2N \) and, since \( \overline{N} \) has no torsion, \( \sum g_i n_i \) has image 0 in \( \overline{N} \). Since \( \{\overline{n_i}\} \) forms a free basis, one finds that all \( g_i \in z_2 \mathfrak{D}_m \). The latter leads to the contradiction that all \( f_i \) are divisible by \( z_2 \).

The following theorem and its corollaries will be essential for the rest of the paper and will be used in most of the results. First, we will recall a lemma (see [1]) that will allow us to check the freeness of a module over an arbitrary ring by checking the freeness of certain modules over a local ring.

**Lemma 17.** Let \( R \) be a ring (commutative with 1, the identity element). Let \( N \) be an \( R \)-module. Then \( N \) is projective if and only if for all maximal ideals \( \mathfrak{m} \) of \( R \), \( N_\mathfrak{m} \) is free over \( R_\mathfrak{m} \).

We now arrive at the main technical result of this paper:

**Theorem 18.** Let \( M \) be a finitely generated torsion free \( \mathfrak{D} \)-module. Then there exists a free \( \mathfrak{D} \)-module \( N \) such that \( M \subset N \).

**Proof.** There exists a finitely generated, free module \( F \) containing \( M \) (see [7] p.44). Let \( N \subset F \) be the subset defined by: \( f \in F \) belongs to \( N \) if the ideal \( I_f := \{ r \in \mathfrak{D} \mid rf \in M \} \) has finite codimension in \( \mathfrak{D} \).

For \( f_1, f_2 \in N \) and \( r_1 \in \mathfrak{D} \) the element \( r_1f_1 + f_2 \) also lies in \( N \). Indeed, the ideal \( I := I_{f_1} \cap I_{f_2} \) has again finite codimension in \( \mathfrak{D} \) and for any \( r \in I \) one has \( r(r_1f_1 + f_2) \in M \). We conclude that \( N \) is a \( \mathfrak{D} \)-submodule of \( F \) and thus \( N \) is also
finitely generated.

Moreover, if \( f \in F \) is such that the ideal \( J_f := \{ r \in \mathcal{D} \mid rf \in N \} \subseteq \mathcal{D} \), then \( f \in N \). Indeed, take coprime elements \( p, q \in J_f \) (recall that the ideal \( (p, q) \) has also finite codimension). Then \( pf, qf \in N \) implies that there exists an ideal \( I \subseteq \mathcal{D} \), such that \( Ipf \) and \( Iqf \) belong to \( M \). The ideal \( J := I \cdot (p, q) \) has also finite codimension and \( Jf \in M \). Thus \( f \in N \).

Now \( \text{dim}_k(N/M) < \infty \). Indeed, this \( \mathcal{D} \)-module is finitely generated, say by \( \pi_1, \ldots, \pi_r \). There exists an ideal \( I \subseteq \mathcal{D} \) such that \( I\pi_i = 0 \) for all \( i \). This implies \( \text{dim}_k(N/M) < \infty \).

Consider a maximal ideal \( m \) of \( \mathcal{D} \). The ideal \( m \) is generated by two elements, say \( y_1, y_2 \) (this is valid if \( k \) has characteristic 0 and thus certainly in our case. If \( k \) has positive characteristic one has to change the arguments slightly). These are also generators of the regular local ring \( \mathcal{D}_m \). Consider the localizations \( A := N_m \subseteq B := F_m \). Then \( A, B \) are finitely generated modules over the regular local ring \( \mathcal{D}_m \). The last module is free and thus the first module is obviously torsion free.

We claim that \( A_{y_1} \cap A_{y_2} = A \). Indeed, take an element \( \xi \in A_{y_1} \cap A_{y_2} \). There is an \( s \in \mathcal{D} \setminus m \) such that \( \eta := ss' \in N \). Then also \( \eta \in A_{y_1} \cap A_{y_2} \) and there is an integer \( k \) such that \( y_1^k \eta, y_2^k \eta \in A \). There is an element \( s' \in \mathcal{D} \setminus m \) such that \( s'y_1^k \eta, s'y_2^k \eta \in N \). The ideal \( I := (y_1^k, y_2^k) \subseteq \mathcal{D} \) and \( Is' \eta \subseteq N \). Hence \( s' \eta \in N \) (by the definition of \( N \)). Then \( ss' \xi \in N \) and thus \( \xi \in A \). Lemma 16 then yields that \( A \) is free.

Since all localizations \( N_m \) at the maximal ideals of \( \mathcal{D} \) are free, the module \( N \) is projective. By the theorem of Quillen and Suslin, any finitely generated projective module over a polynomial ring over a field is free. Since our polynomial ring is \( \mathcal{D} = k[x_1, x_2] \), this proves the result. \( \square \)

**Remark 19.** In [5], theorem 8.3 and remark 8.4, it was shown that lemma 16 (in a slightly different formulation) does not hold for regular local rings of dimension \( d > 2 \). Since for \( \mathcal{D} = k[x_1, x_2, \ldots, x_n] \), \( \mathcal{D}_m \) is a regular local ring of dimension \( n \), lemma 16 does not hold for \( n > 2 \). Therefore the proof of theorem 18 given above is not valid for \( n > 2 \).

**Remark 20.** We note that theorem 18 and lemma 16 are also valid if \( \mathcal{D} = \mathbb{Z}[x] \). We omit the details here since to our knowledge this ring is not relevant in the theory of \( n \mathcal{D} \) behaviors.

As an immediate consequence of theorem 18 we obtain the following result for 2D behaviors. Recall that an \( n \mathcal{D} \) behavior is called regular if it has a full row rank kernel representation.

**Corollary 21.** Let \( \mathcal{B} \subseteq \mathcal{A}^q \) be a 2D behavior. There exists a regular 2D behavior \( \mathcal{B}' \subseteq \mathcal{B} \). Moreover if \( \mathcal{B} \) is controllable then there exists a strongly controllable 2D behavior \( \mathcal{B}' \), and a 2D behavior \( \hat{\mathcal{B}} \) that is finite dimensional over \( k \), such that \( \mathcal{B} = \mathcal{B}'/\hat{\mathcal{B}} \).

**Proof.** The \( \mathcal{D} \)-module \( \mathcal{B} \perp \subseteq \mathcal{A}^q \) is torsion free. By theorem 18 there exists a free \( \mathcal{D} \)-module \( F \subseteq \mathcal{D}^q \) such that \( \mathcal{B} \perp \subseteq F \), so with \( \mathcal{B}' := F \perp \) we have \( \mathcal{B}' \subseteq \mathcal{B} \) by lemma 14.

If the \( \mathcal{D} \)-module \( M = \mathcal{D}^q/\mathcal{B} \perp \) is torsion free, then by theorem 18 there exists a free \( \mathcal{D} \)-module \( F \) such that \( M \subseteq F \). Obviously \( \mathcal{B}' := \mathcal{D}(F) \) is strongly controllable since \( F \) is free. Consider the following exact sequence:

\[
0 \rightarrow M \rightarrow F \rightarrow F/M \rightarrow 0 \quad (3.3)
\]
Using the injective and cogenerator properties of $A$, we have that

$$0 \to \text{Hom}(F/M, A) \to \text{Hom}(F, A) \to \text{Hom}(M, A) \to 0 \quad (3.4)$$

is exact. Define the 2D behavior $B$ by $B := \mathcal{D}(F/M)$. Then, by lemma 14, $B$ is finite dimensional since $F/M$ is finite-dimensional. Moreover, using exactness of the dual sequence, from $B = \text{Hom}(M, A)$ and $B' = \text{Hom}(F, A)$ we may conclude that $B = B'/B$. $\square$

Whereas in Theorem 18 we established, for any finitely generated torsion free $\mathcal{D}$-module $M$, the existence of a free $\mathcal{D}$ module $N$ such that $M \subseteq N$ with finite codimension, in the following we will show that for any given $M$ this free module $N$ is actually unique. We will also give a characterization of $N$, and outline how it can be computed using MAPLE.

**Lemma 22.** Let $M$ be a finitely generated torsion free $\mathcal{D}$-module, and $N$ a free $\mathcal{D}$-module such that $M \subseteq N$. Let $S = \mathcal{D} \setminus \{0\}$. The following holds:

1. $N_S = M_S$
2. If $\xi \in N$ then the ideal $I = \{ p \in \mathcal{D} \mid p\xi \in M \} \subseteq \mathcal{D}$. There exists an ideal $\tilde{I} \subseteq S$ such that $\tilde{I} \cdot N \subseteq M$.
3. If $\xi \in N_S = M_S$, but does not belong to $N$, then the ideal $I = \{ p \in \mathcal{D} \mid p\xi \in N \}$ (and hence also the ideal $J = \{ f \in \mathcal{D} \mid f\xi \in M \}$) does not have finite codimension in $\mathcal{D}$.

**Proof.** Note that such an $N$ always exists from lemma 18.

(1) : Since $M \subseteq N$, using an argument as in the proof of lemma 14, there exists 0 $\neq d \in \mathcal{D}$ such that for all $n \in N$ we have $dn \in M$. Hence for all $n' \in N_S$ we have $dn' \in M_S$, so $n' = dn'/d \in M_S$ This implies $N_S \subseteq M_S$. The converse inclusion $M_S \subseteq N_S$ is obvious.

(2) : Let $\{e_1, e_2, \ldots, e_m\}$ be a basis for $N$ and let $\bar{e}_1, \bar{e}_2, \ldots, \bar{e}_m$ be their equivalence classes in $N/M$. For $i = 1, 2, \ldots, m$, consider the sequence $\{\bar{x}_i, \bar{x}_i^2, \bar{x}_i^3, \ldots\}$ in $N/M$. By finite dimensionality, there exists a positive integer $s$ and $\alpha_0, \alpha_1, \ldots, \alpha_s \in k$ such that $(\alpha_0 + \alpha_1 x_1 + \ldots + \alpha_s x_s^s)\bar{e}_i = 0$. Define a polynomial $p_i(x_1) := \alpha_0 + \alpha_1 x_1 + \ldots + \alpha_s x_s^s$. Then $p_i(x_1)\bar{e}_i \in M$. Likewise, one can find polynomials $q_i(x_2)$ such that $q_i(x_2)\bar{e}_i \in M$. Now, let $I_i$ be the ideal generated by $p_i(x_1)$ and $q_i(x_2)$. Then $I_i \subseteq \mathcal{D}$ and $I_iN \subseteq M$.

Define $\tilde{I}$ to be the product of the ideals $I_i$. Then still $\tilde{I} \subseteq \mathcal{D}$, and $\tilde{I}N \subseteq M$. Finally, let $\xi \in N$. Then the ideal $I = \{ p \in \mathcal{D} \mid p\xi \in M \}$ satisfies $\tilde{I} \subseteq I$. Thus $I \subseteq \mathcal{D}$.

(3) : Let $\{e_1, \ldots, e_m\}$ be a basis for $N$ and $\xi = \xi_1 e_1 + \cdots + \xi_m e_m$ with all $\xi_i \in D$. Using that $\mathcal{D}$ is a unique factorization domain one can write each $\xi_i$ as $f_i/g_i$. $f_i, g_i \in \mathcal{D}$ with $g_i \cdot \text{c.d.}(f_i, g_i) = 1$. Let $g$ be the smallest common multiple of $g_1, \ldots, g_m$. Thus $I = g\mathcal{D}$, which does not have finite codimension in $\mathcal{D}$. $\square$

**Corollary 23.** Let $M$ be a finitely generated torsion free $\mathcal{D}$-module and $S = \mathcal{D} \setminus \{0\}$. Define

$$K(M) := \{ \xi \in M_S \mid p \in \mathcal{D}, p\xi \in M \} \text{ has finite codimension } \} \quad (3.5)$$

Then $K(M)$ is the unique free $\mathcal{D}$-module $K$ such that $M \subseteq K$. Denote this unique free module by $M^+$. Furthermore, if $N$ and $M$ are finitely generated torsion free $\mathcal{D}$-modules then the following hold:

1. If $M$ is free then $M^+ = M$,
2. \((M^+)^+ = M^+\),
3. if \(N \subset M\), then \(N^+ \subset M^+\),
4. if \(N \subset M\), then \(N^+ = M^+\),
5. if \(N^+ = M^+\), then \(N \cap M \subset M\).

**Proof.** According to lemma 22 there exists a free \(\mathcal{D}\)-module \(N\) such that \(M \subset N\).

We will prove that \(N = K\). Let \(\xi \in N\). By lemma 22, part 2, \(I = \{p \in \mathcal{D} \mid p\xi \in M\} \subset \mathcal{D}\). Since \(N \subset N_S \subset M_S\), we have \(\xi \in K\). Conversely, suppose \(\xi \in K\) but \(\xi \notin N\). According to lemma 22, part 3, \(I = \{p \in \mathcal{D} \mid p\xi \in M\} \subset \mathcal{D}\) does not have finite codimension, which contradicts \(\xi \in K\).

Statement 1 immediately follows from the uniqueness, whereas statement 2 follows immediately from statement 1. A proof of statement 3 follows immediately from the characterization (3.5) \(M^+ = K(M)\) and \(N^+ = K(N)\).

Statement 4 follows from the fact that \(N \subset M \subset M^+\) so \(N \subset M^+\). Since \(M^+\) is free, this implies \(N^+ = M^+\).

Finally, we prove statement 5. In fact, we prove that \((N \cap M)^+ = M^+\). Obviously, this implies \(N \cap M \subset M^+\), and since \(N \cap M \subset M\) this also yields \(N \cap M \subset M\).

Let \(\xi \in M^+ = N^+\). Then \(\xi \in M_S\) and the ideal \(I_1 := \{p \in \mathcal{D} \mid p\xi \in M\}\) has finite codimension in \(\mathcal{D}\). Likewise, \(\xi \in N_S\) and the ideal \(I_2 := \{p \in \mathcal{D} \mid p\xi \in N\}\) has finite codimension in \(\mathcal{D}\). Thus, \(\xi \in M_S \cap N_S = (M \cap N)_S\). Finally, the ideal \(I_3 := \{p \in \mathcal{D} \mid p\xi \in M \cap N\}\) has finite codimension in \(\mathcal{D}\) since for the product of \(I_1\) and \(I_2\) we have \(I_1I_2 \subset \mathcal{D}\), and \(I_1I_2 \subset I_3\). We conclude that \(\xi \in (M \cap N)^+\). 

The actual computation of \(M^+\) will be essential for solving the problems we consider in this paper. In the following, for a given \(\mathcal{D}\)-module \(N\), let \(N^+ := \text{Hom}(N, \mathcal{D})\). Also, \(N^{**} = \text{Hom}(N^*, \mathcal{D})\). There is a canonical homomorphism

\[
c_N : N \rightarrow N^{**}, \quad n \in N \mapsto \begin{cases} \varphi_n : N^* \rightarrow \mathcal{D} \\ g \mapsto \varphi_n(g) = g(n) \end{cases}.
\]

If \(N\) is finitely generated and free, then \(c_N\) is an isomorphism. Finally, for any homomorphism \(f : A \rightarrow B\) between modules, there is an induced homomorphism \(f^* : B^* \rightarrow A^*\) defined by \(g \mapsto g \circ f\). Next, there is also an induced homomorphism \(f^{**} : A^{**} \rightarrow B^{**}\).

We now provide a theorem which allows us to compute \(M^+\).

**Theorem 24.** Let \(M\) be a finitely generated torsion free \(\mathcal{D}\)-module, and let \(M^+\) be the unique free \(\mathcal{D}\)-module such that \(M \subset M^+\). Then \(M^+\) is isomorphic to \(M^{**}\), with isomorphism \(\psi : M^+ \rightarrow M^{**}\) given by \(\psi = (i^*)^{-1} \circ c_{M^+}\). Here, \(i : M \rightarrow M^+\) is the inclusion map, and \(c_{M^+} : M^+ \rightarrow M^{**}\) the canonical homomorphism.

**Proof.** Let \(i : M \rightarrow M^+\) be the inclusion. We claim that \(i^* : (M^+)^* \rightarrow M^*\) is bijective. From this the claims of the theorem will follow. Indeed, if \(i^*\) is bijective, then also \(i^{**}\) is bijective. Since also \(c_{M^+}\) is an isomorphisms (note that \(M^+\) is free), the claims follows by inspecting the diagram

\[
M^\ast \xrightarrow{c_{M^+}} M^{**} \xrightarrow{i^{**}} (M^+)^* \xrightarrow{i^*} M^+.
\]

We will now prove that \(i^*\) is bijective. First note that there exists an ideal \(I \subset \mathcal{D}\) such that \(I \cdot M^+/M = 0\) or, equivalently, \(I \cdot M^+ \subset M\). We also know that \(I\) contains
nonzero elements \( p = p(x_1), q = q(x_2) \). The ideal \((p, q)\) has also finite codimension and we may for notational convenience assume that \( I = (p, q) \).

We now first show that the map \( i^* : (M^+)^* \to M^* \) is injective. Suppose \( 0 \neq \ell_1 \in (M^+)^* \) such that \( i^*(\ell_1) = \ell_1(i) = 0 \). If \( n \in M^+ \setminus M \), since \( M^+/M \) is a torsion module there exists \( d \) such that \( dn \in M \) and therefore \( 0 = \ell_1(0) = \ell_1(dn) = d\ell(n) \) and because \((M^+)^* \) is torsion free one has that \( \ell(n) = 0 \) for all \( n \in M^+ \).

The induced map \( (M^+_\ast)^* \to M^*_\ast \) is a bijection because \( M^+/M \) is a torsion module. Consider an element \( \ell \in M^*_\ast \), then \( p \cdot \ell \) lies in the image of \( i^* \). Indeed, define \( \ell_1 \in (M^+)\ast \) by \( \ell_1(f) = \ell(pf) \) (note that \( pM^+ \subset M \)). By construction \( i^*(\ell_1) = p \cdot \ell \). Also \( q \cdot \ell \) lies in the image of \( i^* \). Let \( a_1, \ldots, a_r \) be a free basis of \((M^+)^* \). Further \( \ell \) is the image under (the extension of) \( i^* \) of some element \( \xi \) of \((M^+_\ast)^* \) which can be written as \( q_1a_1 + \cdots + q_ra_r \) with \( q_1, \ldots, q_r \in \mathbb{Q}t \) where \( \mathbb{Q}t \) is the field of fractions. Let \( d \) be the common denominator of \( q_1, \ldots, q_r \). Then the ideal \( J := \{ r \in \mathbb{D} \mid r\xi \in (M^+)^* \} = d\mathbb{D} \).

This ideal contains \( p \) and \( q \). Since \( \text{g.c.d.}(p, q) = 1 \) one has \( d = 1 \) and thus \( \xi \in (M^+)^* \).

The bi-dualising functor \((-)^{**} = \text{Hom}_\mathbb{D}(\text{Hom}_\mathbb{D}(-, \mathbb{D}), \mathbb{D}) \) is implemented in the MAPLE package \texttt{homalg} (see [2]). \texttt{homalg} expects the arguments of the functors to be \( \mathbb{D} \)-modules given by a finite free presentation, i.e. by specifying a relation matrix \( R \in \mathbb{D}^{\times t} \), where \( M = \text{coker}(R) \). For a torsion free \( \mathbb{D} \)-module \( M \), given by a finite free presentation, one can use the \texttt{homalg} command \texttt{HomHomR} to effectively computed the \( \mathbb{D} \)-module \( M^+ \cong M^{**} \), also given by a finite presentation. But since we want to consider the torsion free \( M \) as a submodule of the free module \( F := \mathbb{D}^q \) we also want to realize \( M^+ \) and hence \( M^{**} \) as a submodule of \( F \): \( M \leq M^+ \leq \mathbb{D}^q \). To achieve this in \texttt{homalg} we consider the embedding \( M \hookrightarrow F \), where the rows of \( \iota \) are simply the generators of \( M \leq F \). The diagram

\[
\begin{array}{ccc}
M & \xrightarrow{\iota} & F \\
\downarrow{\varepsilon_M} & & \downarrow{\iota^*} \\
M^{**} & \xrightarrow{i^{**}} & F^{**}
\end{array}
\]

enables one to also embed \( M^{**} \) in \( F = F^{**} \), where \( \varepsilon_M \) is the evaluation map. To compute \( i^{**} \) using \texttt{homalg} we apply morphism part of the functor \texttt{HomHomR} which is called \texttt{HomHomMap.R}. Again, we interpret the rows of \( i^{**} \) as the generators of \( M^+ \leq F \).

For more details consult the library of examples on the web page of \texttt{homalg} [2].

We will now give some examples.

**Example 25.** Let \( \mathbb{D} = \mathbb{R}[x_1, x_2] \). Let \( M \subset \mathbb{D} \) be the \( \mathbb{D} \)-module generated by \( x_1 \) and \( x_2 \). Clearly \( M \) is torsion free. Since \( \mathbb{D}/(x_1, x_2) \) is generated by 1, the codimension of \( M \) in \( \mathbb{D} \) equals 1, so \( M^+ \) must be equal to \( \mathbb{D} \).

**Example 26.** Let \( \mathbb{D} = \mathbb{R}[x_1, x_2] \). Consider the matrix

\[
R(x_1, x_2) := \begin{pmatrix}
0 & (x_1 + x_2)^2 & 0 \\
0 & x_2 & x_1 \\
x_1(x_2 + 1) & 0 & 0 \\
x_1^2 - x_1x_2 & 0 & 0
\end{pmatrix}.
\]

Let \( M \subset \mathbb{D}^q \) be the \( \mathbb{D} \)-module generated by the rows of \( R \). Obviously it is a torsion
Implementability problem is formulated as follows: given a plant behavior obtained in section 3 to an alternative version of the problem of regular implementability. It is easily verified that \( \text{Hom}(M) \) is finitely generated torsion free \( \mathcal{D} \)-module since it is contained in the free module \( \mathcal{D}^3 \). Define

\[
R_1(x_1, x_2) := \begin{pmatrix}
0 & (x_2 + x_1)^2 & 0 \\
x_2 & x_1 x_2 & x_1 \\
x_1 & 0 & 0
\end{pmatrix}.
\]

It can be computed that \( M^+ \) is the (free) \( \mathcal{D} \)-module generated by the rows of \( R_1 \). The codimension of \( M \) in \( M^+ \) is equal to 1.

Example 27. For notational convenience, in this example we use as indeterminates \( x \) and \( y \). Let \( \mathcal{D} = \mathbb{R}[x, y] \). The rows of the following matrix are the generators of the module \( M \) as a submodule of the free module \( \mathcal{D}^2 \):

\[
\begin{pmatrix}
-y(x-y) & y x^2 - x - 2 y + 2 \\
x^2 y^2 + x^3 + 2 x y + x^2 + 2 x y - 2 y^2 & x y^3 - x y^3 - y^3 - 2 x^2 - x y + 4 y^2 - 2 y \\
x^3 y - x^3 + 2 x y^2 - y^3 - x^2 - 4 x y + 4 y^2 & 3 y^3 + 2 x^2 - x y - 10 y^2 + 6 y \\
x^4 + 2 x^3 + 2 x y^2 - 2 x y - x^2 + 5 y - 3 y^2 & 3 x y^2 - y^3 - 4 x^2 - 6 x y + 6 y^2 + 4 x - 4 y
\end{pmatrix}.
\]

It can be computed that \( M^+ \) is the (free) \( \mathcal{D} \)-module generated by the rows of the following matrix:

\[
\begin{pmatrix}
x + 1 & y - 2 \\
2 x - y + 1 & 3 y - 4 - x^2 + x
\end{pmatrix}.
\]

We conclude this section with a lemma that we will need in the following section.

Lemma 28. Let \( N_1 \) and \( N_2 \) be finitely generated torsion free \( \mathcal{D} \)-modules. Then we have: \( (N_1 \oplus N_2)^* = N_1^* \oplus N_2^* \).

Proof. Clearly, \( N_1 \oplus N_2 \) is finitely generated and torsion free, so \( (N_1 \oplus N_2)^* \) is defined. It is easily verified that \( \text{Hom}(N_1 \oplus N_2, \mathcal{D}) = \text{Hom}(N_1, \mathcal{D}) \oplus \text{Hom}(N_2, \mathcal{D}) \), so \( (N_1 \oplus N_2)^* = N_1^* \oplus N_2^* \). Applying this argument twice yields \( (N_1 \oplus N_2)^{**} = N_1^{**} \oplus N_2^{**} \). The proof can then be completed by inspecting the isomorphisms established in theorem 24.

4. Regular “almost” implementability. In this section we apply the result obtained in section 3 to an alternative version of the problem of regular implementability, called the problem of regular almost implementability. The original regular implementability problem is formulated as follows: given a plant behavior \( \mathcal{B} \) and a desired behavior \( \mathcal{B}_d \), find a controller behavior \( \mathcal{B}_c \) such that \( \mathcal{B}_d = \mathcal{B} \cap \mathcal{B}_c \), and the interconnection is regular. This problem was studied for 1D behaviors in [23], and for general \( n \)D behaviors in [14, 15, 28] and [18]. A necessary and sufficient condition for the existence of a required controller is that the \( \mathcal{D} \)-module \( \mathcal{B}_d^\perp \) is a direct summand of the \( \mathcal{D} \)-module \( \mathcal{B}_c^\perp \). For general \( \mathcal{D} \)-modules, checking this condition is a hard problem. On the other hand, checking the direct summand condition for free \( \mathcal{D} \)-modules turns out to be feasible, and has in fact been implemented in MAPLE. In this section it will turn out that relaxation of the original regular implementability problem to its almost version yields solvability conditions involving a direct summand condition for free \( \mathcal{D} \)-modules. Our result will use the existence of a unique free module containing a given one, and is only valid for 2D systems.

We consider the following problem:

Problem 29. Given an \( n \)D behavior \( \mathcal{B} \) (the plant) and a desired \( n \)D behavior \( \mathcal{B}_d \subset \mathcal{B} \), find an \( n \)D behavior \( \mathcal{B}_c \) (a controller) such that
1. \( \mathfrak{B} \cap \mathfrak{B}_C \subseteq \mathfrak{B}_d \),

2. \( \mathfrak{B}^+ \cap \mathfrak{B}^+_C = 0 \), i.e., the interconnection is regular.

If such \( \mathfrak{B}_c \) exists, then we say that \( \mathfrak{B}_d \) is regularly almost implementable with respect to \( \mathfrak{B} \). The problem of finding such a \( \mathfrak{B}_c \) is called the problem of regular almost implementability of \( \mathfrak{B}_d \) w.r.t. \( \mathfrak{B} \).

The following theorem is valid for the case that \( n = 2 \). It reduces the problem to a problem of checking whether a given free \( \mathfrak{D} \)-module is a direct summand of a larger free \( \mathfrak{D} \)-module.

**Theorem 30.** Given 2D behaviors \( \mathfrak{B}_d \subset \mathfrak{B} \), denote \( N = \mathfrak{B}^+ \), and \( N_d = \mathfrak{B}^+_d \). If \( \mathfrak{B}_d \) is almost regularly implementable by regular interconnection w.r.t. \( \mathfrak{B} \) then \( N^+ \) is a direct summand of \( N_d^+ \). Furthermore, if \( N^+ \) is direct summand of \( N_d^+ \), and \( \mathfrak{B} \) is a regular behavior then \( \mathfrak{B}_d \) is almost regularly implementable w.r.t. \( \mathfrak{B} \).

**Proof.** If \( \mathfrak{B}_d \) is regularly almost implementable w.r.t. \( \mathfrak{B} \) then there exists \( \mathfrak{B}_c \) such that conditions 1 and 2 of problem 29 hold. By lemma 14, for \( N_c := \mathfrak{B}^+_C \) we then have \( N_d \subseteq N \oplus N_c \). By corollary 23 we obtain \( N_d^f = (N \oplus N_c)^f \), which by lemma 28 is equal to \( N^+ \oplus N_c^+ \).

Conversely, suppose there exists \( N_c \) such that \( N^+ \oplus N_c = N_d^+ \). Now, \( N \) is free (since \( \mathfrak{B} \) is a regular behavior), so \( N^+ = N \). This implies \( N_d \subseteq N \oplus N_c \) so, again by lemma 14, \( \mathfrak{B}_c := N_1^+ \) regularly almost implements \( \mathfrak{B}_d \) w.r.t. \( \mathfrak{B} \). \( \square \)

Theorem 30 provides a necessary and a sufficient condition for solving the problem of regular almost implementability of \( \mathfrak{B}_d \) from \( \mathfrak{B} \). Indeed, such a necessary condition (and also sufficient if \( N \) is free) can be computed by checking whether \( N^+ \) is direct summand of \( N_d^+ \). This is computationally very effective since both \( N^+ \) and \( N_d^+ \) are free modules.

One way to do it is as follows: choose a basis \( e_1, e_2, \ldots, e_n \) of \( N^+ \) and a basis \( f_1, f_2, \ldots, f_m \) of \( N_d^+ \) \( (n \leq m) \). Next, choose \( r_{ij} \in \mathfrak{D} \) such that \( e_i = r_{i,1}f_1 + r_{i,2}f_2 + \ldots + r_{i,m}f_m \) \( (i = 1, 2, \ldots, n) \). Define an \( m \times n \) matrix \( R \) by \( R = (r_{ij}) \). Then obviously \( N^+ \) is a direct summand of \( N_d^+ \) if and only if the \( n \times n \) minors of \( R \) generate the unit ideal \( \mathfrak{D} \). Applying theorem 30, we thus obtain: if the \( n \times n \) minors of \( R \) do not generate \( \mathfrak{D} \), then \( \mathfrak{B}_d \) is not almost implementable by regular interconnection w.r.t. \( \mathfrak{B} \). On the other hand, if the \( n \times n \) minors of \( R \) do generate the unit ideal \( \mathfrak{D} \) and if \( N \) is free, then \( N \) is direct summand of \( N_d^+ \). A direct summand can then be computed by extending the matrix \( R \) to a unimodular \( m \times m \) matrix, and defining \( e_{n+1}, \ldots, e_m \in N_d^+ \) by \( e_i = r_{i,1}f_1 + r_{i,2}f_2 + \ldots + r_{i,m}f_m \) \( (i = n + 1, \ldots, m) \). Then define \( N_c \) as the module generated by \( \{e_{n+1}, \ldots, e_m\} \). The controller \( \mathfrak{B}_c := N_c^+ \) then regularly almost implements \( \mathfrak{B}_d \) w.r.t. \( \mathfrak{B} \).

The package homalg (see [2]) provides the commands \texttt{HomHom_ R} and \texttt{HomHomMap_ R} to perform the operation \((−)^f \), and a command \texttt{LeftinverseC} to check whether a given free submodule of a free module is a direct summand. If so, it provides the computation of a direct summand.

We conclude this section with an example.

**Example 31.** Let \( \mathfrak{D} = \mathbb{R}[x_1, x_2] \). Define

\[
R(x_1, x_2) := \begin{pmatrix}
0 & x_2 x_1^2 & 0 \\
x_1 & 0 & 0 \\
x_2 & 0 & 0
\end{pmatrix},
\]
and

\[ R_d(x_1, x_2) = \begin{pmatrix} 0 & x_1^2 & 0 \\ 0 & 0 & 1 \\ x_1 & 0 & 0 \\ x_2 & 0 & 0 \end{pmatrix}. \]

Let \( \mathfrak{B} = \ker(R) \) and \( \mathfrak{B}_d = \ker(R_d) \). \( \mathfrak{B}^\perp = N \subseteq \mathcal{D}^3 \) is the module generated by the rows of \( R \), \( \mathfrak{B}_d^\perp = N_d \) is the module generated by the rows of \( R_d \). We compute that \( N^+ = \langle (0, x_2 x_1^2, 0), (1, 0, 0) \rangle \) and \( N_d^+ = \langle (0, x_1^2, 0), (0, 0, 1), (1, 0, 0) \rangle \). It can be checked that \( N^+ \) is not a direct summand of \( N_d^+ \), and therefore \( \mathfrak{B}_d \) is not is regularly almost implementable with respect to \( \mathfrak{B} \).

5. Autonomous-controllable decomposition with finite dimensional intersection. In the following two sections we address the problem of decomposing a behavior into smaller components. We do this in two steps. First, in this section we look at the autonomous-controllable decomposition, and in the next section we treat further decompositions of the controllable part. The autonomous-controllable decomposition has played an important role in the theory of linear systems. It has been studied extensively in the context of 1D behaviors [21], in the context of 2D behaviors in [6, 20], and for higher dimensional systems in [25, 26]. In our study to decompose a given behavior into smaller subbehaviors, it is natural to study first whether it is possible to have an autonomous-controllable decomposition with finite dimensional intersection. In this section we show that an autonomous-controllable decomposition with finite dimensional intersection is always possible for 2D behaviors. We also show that for \( n = 3 \) it is not always possible to write a behavior as the sum of its controllable part and an autonomous part with finite dimensional intersection. This will be done by giving a counterexample.

The following lemma is still valid for any \( n \):

**Lemma 32.** Let \( \mathfrak{B} \subseteq A^q \) be an \( n \mathcal{D} \) behavior, and let \( \mathfrak{B}_{\text{cont}} \) be its controllable part. Denote \( M := \mathcal{D}^q/\mathfrak{B}^\perp \) and let \( M_t \) be the torsion submodule of \( M \). Consider the exact sequence \( 0 \rightarrow M_t \rightarrow M \xrightarrow{\beta} N \approx M/M_t \rightarrow 0 \). The following statements are equivalent:

1. there exists an autonomous behavior \( \mathfrak{B}_{\text{aut}} \subseteq \mathfrak{B} \) such that
   \[ \mathfrak{B} = \mathfrak{B}_{\text{cont}} + \mathfrak{B}_{\text{aut}} \text{ and } \mathfrak{B}_{\text{cont}} \cap \mathfrak{B}_{\text{aut}} \text{ has finite dimension}, \]

2. there exists an autonomous behavior \( \mathfrak{B}_{\text{aut}} \subseteq \mathfrak{B} \) such that
   \[ \mathfrak{B}^\perp = (\mathfrak{B}_{\text{cont}})^\perp \cap (\mathfrak{B}_{\text{aut}})^\perp \text{ and } (\mathfrak{B}_{\text{cont}})^\perp + (\mathfrak{B}_{\text{aut}})^\perp \subseteq \mathfrak{D}^q, \]

3. there exists a \( \mathcal{D} \)-module \( A \subseteq M \) such that \( D(M) = D(M/M_t) + D(M/A) \) and \( D(M/M_t) \cap D(M/A) \) has finite dimension,

4. there exists a \( \mathcal{D} \)-module \( A \subseteq M \) such that \( A \cap M_t = 0 \) and \( A + M_t \subseteq M \),

5. there exists a \( \mathcal{D} \)-module \( N' \subset N \) such that \( 0 \rightarrow M_t \rightarrow M' \xrightarrow{\beta} N' \rightarrow 0 \) splits, i.e. \( M_t \) is direct summand of \( M' \).

The following remark and lemma will be used in the proof of lemma 32.

**Remark 33.** Consider the following exact sequence:

\[ 0 \rightarrow M_t \rightarrow M \xrightarrow{\beta} N \rightarrow 0. \]
If $N'$ is a submodule of $N$, and if we define $M' = \beta^{-1}(N')$, then the sequence

$$0 \longrightarrow M_t \longrightarrow M' \xrightarrow{\beta} N' \longrightarrow 0$$

(5.4)
is also exact. Consequently, if $N'$ has the property that $\dim(N/N') < \infty$, then also $\dim(M/M') < \infty$ since $\frac{N}{N'} \cong \frac{M/M_t}{M/M'} \cong \frac{M}{M'}$.

**Lemma 34.** Let $A \subset M$ be two $\mathcal{D}$-modules, and let $M_t$ be the torsion part of $M$.

Then the following hold:

1. $D(M/(M_t + A)) = D(M/M_t) \cap D(M/A)$,
2. $M_t \cap A = 0 \iff D(M) = D(M/M_t) + D(M/A)$.

**Proof.** This follows immediately from [24], corollary 3. □

**Proof of lemma 32:** (1) $\iff$ (2): Clearly, $\mathfrak{B} = \mathfrak{B}_{\text{cont}} + \mathfrak{B}_{\text{aut}}$ if and only if $\mathfrak{B}^\perp = (\mathfrak{B}_{\text{cont}})^\perp \cap (\mathfrak{B}_{\text{aut}})^\perp$. Also, the intersection $\mathfrak{B}_{\text{cont}} \cap \mathfrak{B}_{\text{aut}}$ has finite dimension if and only if $D(\mathfrak{D}/(\mathfrak{B}_{\text{cont}} \cap \mathfrak{B}_{\text{aut}})^\perp)$ has finite dimension. By lemma 14, this holds if and only if $\mathfrak{D}/(\mathfrak{B}_{\text{cont}} \cap \mathfrak{B}_{\text{aut}}) = \mathfrak{D}/(\mathfrak{B}_{\text{cont}}^\perp + \mathfrak{B}_{\text{aut}})$ has finite dimension.

(1) $\Rightarrow$ (3): Take for $A$ any submodule of $M$ such that $\mathfrak{B}_{\text{aut}} = D(\mathfrak{D}/A)$. Since $\mathfrak{B}_{\text{cont}} = D(M/M_t)$, this proves (3).

(3) $\Rightarrow$ (4): By part (2) of lemma 34, $M_t \cap A = 0$ if and only if $D(M) = D(M/M_t) + D(M/A)$. By part (1) of lemma 34, $D(M/M_t) \cap D(M/A) = D(M/(M_t + A))$, which has finite dimension if and only if $M/(M_t + A)$ has finite dimension (see lemma 14), equivalently, $M_t + A \ll M$.

(4) $\Rightarrow$ (5): Since the sequence splits there exists a $\mathcal{D}$-module $A$ such that $M_t \oplus A = M$’ so by remark 33 one has that $M_t \ll M$.

(4) $\Rightarrow$ (5): Define a submodule of $N$ by $N' := (M_t \oplus A)/M_t$. Then the sequence

$$0 \longrightarrow M_t \longrightarrow M_t \oplus A \xrightarrow{\beta} (M_t \oplus A)/M_t \longrightarrow 0$$
is exact, splits and we have $(M_t \oplus A)/M_t \subset M_t \subset M/M_t$, since $M_t \oplus A \subset M$. □

The following theorem states that for 2D systems an autonomous-controllable decomposition with finite dimensional intersection always exists. Using a completely different approach, an equivalent result for discrete behaviors ($A = k^{2^\infty}$) is given in [20] (theorem 3.1).

**Theorem 35.** Let $\mathcal{D} = k[x_1, x_2]$. Let $M$ be a finitely generated $\mathcal{D}$-module and let $M_t$ be the torsion submodule of $M$. Then there exists a submodule $A \subset M$ such that $A \cap M_t = 0$ and $A + M_t \ll M$.

**Proof:** Using remark 33, it is enough to check that there exists $N' \subset N$ of finite dimension such that the sequence (5.4) splits. There exists $d \in R, d \neq 0$ with $dM_t = 0$. Since $N \cong M/M_t$, $N$ is torsion free and applying theorem 18 there is a free module $F = Re_1 + \cdots + Re_m \supset N$ such that $F/N$ has finite dimension. By lemma 14 there is an ideal $J \subset \mathcal{D}$ such that $N$ contains the submodule $Je_1 + \cdots + Je_m$ of the free module $\mathcal{D}e_1 + \cdots + \mathcal{D}e_m$ i.e. $J \cdot F/N = 0$. 


Further $J$ contains a non zero multiple $p$ of $d$ since $J \subseteq \mathcal{D}$. We want to show that there exists an element $q \in J$ such that the ideal $(p, q) \subset \mathcal{D}$ (or equivalently $\text{g.c.d.}(p, q) = 1$).

The radical $\sqrt{J}$ corresponds to a finite set $S$ of points in the plane $\mathbb{C}^2$. For any non zero element $f \in J$, the radical $\sqrt{(f)}$ corresponds to a curve $\Gamma$ passing through $S$. This curve $\Gamma$ is a finite union of irreducible curves. The converse is valid: let $\Gamma$ be a curve, passing through $S$, then the radical ideal corresponding to $\Gamma$ has the form $Rg$ for some element $g \in \sqrt{J}$ and thus, for some integer $N \geq 1$ one has $g^N \in J$.

Now $\sqrt{J}p$ defines a curve $\Gamma$ passing through $S$, which is a finite union of irreducible curves $\Gamma_1, \ldots, \Gamma_r$. It is clear that there exists a curve $\Gamma'$ passing through $S$, corresponding to some radical ideal $Rg$, such that none of the irreducible components of $\Gamma'$ coincides with a $\Gamma_i$. Let $q := g^N \in J$. The radical ideal $\sqrt{(p, q)}$ corresponds to the intersection $\Gamma \cap \Gamma'$. This is a finite set and thus $\sqrt{(p, q)}$ and also $I := (p, q)$ are ideals in $R$ with finite codimension. In particular $\text{g.c.d.}(p, q) = 1$.

We may replace $N$ by $Ie_1 + \cdots + Ie_m$ and $M$ accordingly. Choose, for $i = 1, \ldots, m$, elements $u_i, v_i \in M$ with images $pe_i, qe_i$ in $N$.

Consider an expression $\sum_{i=1}^{m} (a_i u_i + b_i v_i)$ (all $a_i, b_i \in R$) having image 0 in $N$. Then $\sum_{i=1}^{m} (a_i p + b_i q)v_i = 0$ and it follows that for suitable $c_i \in R$ we have $a_i = c_i q, b_i = -c_i p$. Note that $pu_i - pv_i \in M_i$ for all $i$ and thus $p(\sum_{i=1}^{m} (a_i u_i + b_i v_i)) = 0$.

Next, consider the submodule $A$ of $M$ generated by the elements $pu_i, qv_i$ for all $i$. Then $\beta(A) = (p^2, q^2)e_1 + \cdots + (p^2, q^2)e_m$ has finite codimension in $N$. We verify now that $A \cap M_i = 0$.

Suppose that $\xi := \sum_{i=1}^{m} (a_i pu_i + b_i qv_i)$ lies in $M_i$. Then there are $c_i$ such that $a_i p = c_i q, b_i q = -c_i p$ for all $i$. Then $c_i$ is divisible by $pq$ since $\text{g.c.d.}(p, q) = 1$. Write $c_i = pqd_i$. Then $\xi = \sum_{i=1}^{m} pqd_i(au_i - pv_i)$. This expression is zero. \(\Box\)

The proof of theorem 35 can be converted into an algorithm e.g. using the package homalg ([2]) in MAPLE. By using the appropriate commands, the following example was computed. The details of the computations can be found following the link in [2].

**Example 36.** For notational convenience, in this example we use as indeterminates $x$ and $y$. Let $\mathcal{D} = \mathbb{R}[x, y]$. Define

\[
R(x, y) = \begin{pmatrix} 0 & 0 & 0 & x & y - 1 \\
0 & x^4 - x^2 - y^2x & yx^3 - yx - y^3 & 0 & 0 \\
x^4 - x^2 - y^2x & yx^3 - yx - y^3 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}.
\]

We consider $\mathcal{B} = \ker(R) \subset \mathcal{A}^4$. We have $\mathcal{B} = D(M)$ where $M = \mathcal{D}^4/\mathcal{B}^4$. The torsion part of $M$ is computed as $M_t = (((x, y, 0, 0)) + \mathcal{B}^4)/\mathcal{B}^4 \subset M$.

We can compute a $\mathcal{D}$-module $A$ such that $M/A$ is torsion and $A \cap M_t = 0$:

\[
A = \langle (0, 0, y^3 - y^2, 0), (y^3 - 2y^2 + y, 0, 0), (0, 0, x^7 - 2x^5 - 2y^2 x^4 + x^3 + 2y^2 x^2 + y^4 x), (0, x^7 - 2x^5 - 2y^2 x^4 + 2y^2 x^3 + 2y^2 x^2 + y^4 x, 0, 0)) + \mathcal{B}^4 \rangle / \mathcal{B}^4.
\]

Since $\mathcal{B}_{\text{cont}} = D(M/M_t)$ we can compute that $\mathcal{B}_{\text{cont}} = \ker(R_1)$ with

\[
R_1(x, y) = \begin{pmatrix} 0 & 0 & x & y - 1 \\
x & y & 0 & 0 \\
\end{pmatrix}.
\]

An autonomous subbehavior of $\mathcal{B}$ such that $\mathcal{B} = \mathcal{B}_{\text{cont}} + \mathcal{B}_{\text{aut}}$ and $\mathcal{B}_{\text{cont}} \cap \mathcal{B}_{\text{aut}}$ is finite-dimensional is then given by $\mathcal{B}_{\text{aut}} = D(M/A) = T^2$ where $T \subset \mathcal{D}^4$ is the module given by:

\[
T = \langle (0, 0, x, y - 1), (0, 0, y^3 - y^2, 0), (y^3 - 2y^2 + y, 0, 0, 0), (x^4 - x^2 - y^2x, yx^3 - yx - y^3, 0, 0), (0, 0, x^7 - 2x^5 - 2y^2 x^4 - 2x^5 + 2y^2 x^2 + 2y^3 x + x^3 - y^2 x), (0, x^7 - 2x^5 - 2y^2 x^4 + x^3 + 2y^2 x^2 + y^4 x, 0, 0) \rangle.
\]
We conclude this section with a counterexample that shows that theorem 35 does not hold for $n \geq 3$. A 3D behavior will be constructed which can not be written as the sum of its controllable part and an autonomous subbehavior, while their intersection is finite dimensional.

**Example 37.** Let $\mathfrak{D} = k[x_1, x_2, x_3]$. Consider the submodule $\langle (x_1x_2, -x_1^2) \rangle \subset \mathfrak{D}^2$, i.e. the submodule generated by the row-vector $(x_1x_2, -x_1^2) \in \mathfrak{D}^2$. Define

$$M := \mathfrak{D}^2/\langle (x_1x_2, -x_1^2) \rangle.$$  

It can be verified that its torsion submodule is given by

$$M_t = \langle (x_2, -x_1) \rangle/\langle (x_1x_2, -x_1^2) \rangle.$$  

Let $I \subset \mathfrak{D}$ be the ideal generated by $x_1$ and $x_2$ in $\mathfrak{D}$, i.e. $I = \langle x_1, x_2 \rangle$. Finally, define a homomorphism $\beta : \mathfrak{D}^2 \to I$ by $\beta(a, b) = x_1a + x_2b$. Then $\ker(\beta) = \langle (x_2, -x_1) \rangle$. Let $\beta : M \to I$ be defined by $\beta \sigma = \beta$, where $\pi : \mathfrak{D}^2 \to M$ is the canonical projection. It is easily verified that the sequence

$$0 \to M_t \to M \to I \to 0$$

is exact. We will now prove that in this example, condition (5) of lemma 32 is violated: there does not exist an ideal $J \subset I$ such that $\dim(I/J) < \infty$ and such that the sequence

$$0 \to M_t \to \beta^{-1}(J) \to J \to 0$$

splits. From $J \subset I$, there exists a nonzero polynomial $p_3 = p_3(x_3)$ such that $p_3I/J = 0$. This implies $x_1p_3$ and $x_2p_3 \in J$. Now consider $M_t$. For the generator of this module we write $(x_2, -x_1)$. Every element in $M_t$ can be uniquely written as $f(x_2, x_3)(x_2, -x_1)$, with $f(x_2, x_3) \in k[x_2, x_3]$. Suppose now there exists $B \subset \beta^{-1}(J)$ which maps onto $J$ and $B \cap M_t = 0$ (this will lead to a contradiction). Now, $B$ contains an element $\xi_1$ which is mapped to $x_1p_3$ and an element $\xi_2$ which is mapped to $x_2p_3$. Thus

\[
\xi_1 = (p_3, 0) + f_1(x_2, x_3)(x_2, -x_1),
\]

\[
\xi_2 = (0, p_3) + f_2(x_2, x_3)(x_2, -x_1).
\]

Then $x_2\xi_1 - x_1\xi_2 = p_3(x_2, -x_1) + x_2f_1(x_2, x_3)(x_2, -x_1) - x_1f_2(x_2, x_3)(x_2, -x_1) = p_3(x_3) + x_2f_1(x_2, x_3)(x_2, -x_1)$. This is a nonzero element of $M_t \cap B$ because $p_3(x_3) + x_2f_1(x_2, x_3)$ is a nonzero element of $k[x_2, x_3]$.

The example shows that for the 3D behavior $\mathfrak{B} \subset \mathcal{A}^2$, given by $\mathfrak{B} = D(M)$, i.e.,

\[
\mathfrak{B} = \{(w_1, w_2) \in \mathcal{A}^2 \mid \frac{\partial^2}{\partial x_1 \partial x_2} w_1 = \frac{\partial^2}{\partial x_1^2} w_2\}
\]

no autonomous subbehavior $\mathfrak{B}_{\text{aut}}$ exists such that $\mathfrak{B} = \mathfrak{B}_{\text{cont}} + \mathfrak{B}_{\text{aut}}$, with $\mathfrak{B}_{\text{cont}} \cap \mathfrak{B}_{\text{aut}}$ finite dimensional.
6. Decomposition of the controllable part. Once we have decomposed the behavior into its controllable part and an autonomous subbehavior, one may look for a finer decomposition. The autonomous part of discrete 2D behaviors has been studied in detail in [20, 19, 3, 6]. In this section we study further decompositions of the controllable part of a given behavior. In fact, we treat the following problem:

**Problem 38.** Given a controllable nD behavior \( \mathfrak{B} \subset \mathfrak{A}^q \) and a subbehavior \( \mathfrak{B}_1 \subset \mathfrak{B} \), find a subbehavior \( \mathfrak{B}_2 \subset \mathfrak{B} \) such that

1. \( \mathfrak{B}_1 + \mathfrak{B}_2 = \mathfrak{B} \),
2. \( \mathfrak{B}_1 \cap \mathfrak{B}_2 \) has finite dimension.

**Theorem 39.** Let \( \mathfrak{B} \subset \mathfrak{A}^q \) be a controllable nD behavior and let \( \mathfrak{B}_1 \subset \mathfrak{B} \) be a subbehavior. Let \( M = \mathfrak{D}^q / \mathfrak{B}^\perp \) and \( N_1 = \mathfrak{B}_1^+ / \mathfrak{B}^\perp \). Then the following statements are equivalent:

1. there exists an nD behavior \( \mathfrak{B}_2 \subset \mathfrak{B} \) such that \( \mathfrak{B}_1 + \mathfrak{B}_2 = \mathfrak{B} \) and \( \mathfrak{B}_1 \cap \mathfrak{B}_2 \) has finite dimension,
2. there exists \( \mathfrak{D} \)-module \( N_2 \subset M \) such that \( N_1 \cap N_2 = 0 \) and \( N_1 + N_2 \subset \mathfrak{B}^\perp \).

In the case that \( n = 2 \), both of these conditions are equivalent with

3. \( N_1^+ \) is a direct summand of \( M^+ \).

**Proof.** (1) \( \Rightarrow \) (2): Passing to the modules of \( \mathfrak{B}, \mathfrak{B}_1 \) and \( \mathfrak{B}_2 \) we get \( \mathfrak{B}_1^+ \cap \mathfrak{B}_2^+ = \mathfrak{B}^\perp \). Also, \( 0 \subset \mathfrak{B}_1 \cap \mathfrak{B}_2 \), so by lemma 14, \( \mathfrak{B}_1^+ + \mathfrak{B}_2^+ \subset \mathfrak{D}^q \). Taking quotients with \( \mathfrak{B}^\perp \) and defining \( N_2 = \mathfrak{B}_2^+ / \mathfrak{B}^\perp \), we obtain (2).

(2) \( \Rightarrow \) (3) If \( N_1 \cap N_2 \subset \mathfrak{B}^\perp \), then by corollary 23, (4), \( (N_1 \cap N_2)^+ = M^+ \). By lemma 28 we then have \( N_1^+ \cap N_2^+ = M^+ \).

(3) \( \Leftarrow \) (2) Let \( N_2 \) be a \( \mathfrak{D} \)-module such that \( N_1^+ \cap N_2^+ = M^+ \). Since \( M^+ \) is free, the same holds for \( N_2 \) by the theorem of Quillen and Suslin, so \( N_2 = N_2^+ \). Also, \( N_1 \cap N_2 = 0 \). Since \( (N_1 \cap N_2)^+ = N_1^+ \cap N_2^+ = M^+ \), by corollary 23, (5), we obtain \( N_1 \cap (N_2 \cap M) = (N_1 \cap N_2) \cap M \subset N_2 \). This proves (2).

From this theorem, the problem is to compute, for given \( N_1 \subset M \), the free extensions \( N_1^+ \) and \( M^+ \), and to check whether \( N_1^+ \) is a direct summand of \( M^+ \). In the comments following theorem 30 it was already explained how this can be done, using the property that the modules are free. Finally, if \( N_1^+ \cap N_2 = M^+ \), then \( \mathfrak{B}_2 = \text{Hom}(M \cap N_2, \mathfrak{A}) \) satisfies the conditions of problem 38.

We will now provide an example.

**Example 40.** Consider the 2D behavior \( \mathfrak{B} \) given as \( \mathfrak{B} = \text{Hom}(M, \mathfrak{A}) \) with \( M \) the torsion free \( \mathfrak{D} \)-module generated given by the rows of the matrix

\[
K(x_1, x_2) = \begin{pmatrix}
x_2 & x_1(x_2^2 - 1) & 1 - x_1 \\
x_1 x_2 & x_1 x_2 & x_1 x_2 + 2 x_2^2 \\
x_1^2 & x_1^2 & x_1^2 + 2 x_2 x_1 \\
0 & 0 & (x_1 + x_2)^2 \\
0 & 0 & x_2^2 + x_1 x_2
\end{pmatrix}.
\]

Let \( \mathfrak{B}_1 \subset \mathfrak{B} \) be the subbehavior given as \( \mathfrak{B}_1 = \text{Hom}(N_1, \mathfrak{A}) \), with \( N_1 \) the submodule of \( M \) generated by the first three rows of \( K(x_1, x_2) \).

We compute: \( M^+ = \langle x_1 + x_2, x_1 x_2, 2 x_2 + 1 \rangle \), \( (x_1, x_1, x_1 + x_2) \), and \( N_1^+ = \langle (x_1 + x_2, x_1 x_2, 2 x_2 + 1), (x_1, x_1, x_1 + 2 x_2) \rangle \). Using "homalg" we can check that \( N_1^+ \) is a direct summand of \( M^+ \). In fact, a direct summand is given by \( N_2 = \langle (0,0,x_1+ \ldots, x_2) \rangle \).
finally, the intersection of $N_2 \cap M$ is equal to $\langle (0,0,(x_1 + x_2)^2), (0,0,x_1^2 + x_1x_2) \rangle$. The subbehavior $\mathfrak{B}_c = \text{Hom}(N_2 \cap M, A)$ satisfies the conditions of problem 38. We conclude this section by noting that condition 2 of theorem 39 can also be given a meaning in terms of the following alternative version of the regular almost implementability problem as treated in section 4 of this paper (see also [28]): given an $n$D plant behavior $\mathfrak{B}$ and a desired behavior $\mathfrak{B}_d$, find a controller behavior $\mathfrak{B}_c$ such that
1. $\mathfrak{B}_d \subseteq \mathfrak{B} \cap \mathfrak{B}_c$
2. $\mathfrak{B}_c^\perp \cap \mathfrak{B}_c = 0$ (i.e. the interconnection is regular).

Defining $N := \mathfrak{B}^\perp$ and $N_d := \mathfrak{B}_d^\perp$ it is easily seen that $\mathfrak{B}_c$ exists if and only if there exists a $\mathfrak{D}$-module $N_c$ such that $N \cap N_c = 0$ and $N + N_c \subseteq N_d$. By theorem 39, for $n = 2$ this is equivalent to the condition that $N^+$ is a direct summand of $N_d^+$.

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