Abstract

Given a nominal plant, together with a fixed neighborhood of this plant, the problem of robust stabilization is to find a controller that stabilizes all plants in that neighborhood (in an appropriate sense). If a controller achieves this design objective, we say that it robustly stabilizes the nominal plant. In this paper we formulate the robust stabilization problem in a behavioral framework, with control as interconnection. We also formulate a relevant behavioral $\mathcal{H}_\infty$ synthesis problem, which will be instrumental in solving the robust stabilization problem. We use both rational as well as polynomial representations for the behaviors under consideration. Necessary and sufficient conditions for the existence of robustly stabilizing controllers are obtained using the theory of dissipative systems. We will also find the optimal stability radius, i.e. the smallest upper bound on the radii of the neighborhoods for which there exists a robustly stabilizing controller. This smallest upper bound is expressed in terms of certain storage functions associated with nominal control system.

keywords: control, behaviors, behavioral interconnection, optimal robust stabilization, rational representations, dissipativity synthesis.

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1 Introduction

This paper deals with control in a behavioral context. We consider the problem of finding, for a given nominal plant behavior, a controller such that the interconnection of the controller with any plant in a given neighborhood of the nominal plant is stable. In other words, we consider the problem of robust stabilization in a behavioral framework.

In a behavioral framework, controlling a plant means intersecting its behavior with a controller behavior. The intersection is then called the controlled behavior, which is required to satisfy the design specifications. In terms of representations, control means that additional laws (e.g. in the form of differential equations representing the controller
behavior) are put on the plant variables. Thus, the plant and controller are interconnected by sharing their variables. In our context we do not distinguish between inputs and outputs, so the interconnection does not necessarily involve feedback.

This idea was introduced by J.C. Willems in [19] in the context of stabilization and pole placement. The problem studied in [19] was to find, for a given plant behavior, a controller behavior such that the full interconnection (i.e., the intersection of plant and controller behavior) is regular and stable, or has its characteristic exponents in prespecified positions in the complex plane. Later, in [2], this problem was extended to the partial interconnection case, where only a prespecified set of components of the plant variable (the so-called interconnection variables) can be used to interconnect with a controller behavior.

In the present paper, we will extend the stabilization problem studied in [19] to the problem of robust stabilization by behavioral interconnection. Given a nominal plant behavior and a ball around it with a given radius, we will establish necessary and sufficient conditions for the existence of a controller behavior that regularly stabilizes all plants in the given ball. We will also find the smallest upper bound on the radii of these balls, i.e., the optimal stability radius.

Of course, the problem of robust stabilization has been studied before in the literature, in an input-output framework, most prominently by McFarlane and Glover in [8] (see also [14], Chapter 15). In [8] representations based on coprime factors of the transfer matrix of the nominal plant were used to obtain conditions for the existence of robustly stabilizing controllers in terms of certain algebraic Riccati equations. Also the optimal stability radius was computed in terms of solutions of these Riccati equations.

Our work can be seen as a behavioral generalization of [8]. We use rational kernel representations of the nominal plant (see [23]) without any input-output considerations. Necessary and sufficient conditions for the existence of robustly stabilizing controllers are expressed in terms of strict dissipativity of an orthogonal behavior associated with the nominal plant behavior, and the optimal stability radius is computed in terms of the extremal storage functions associated with the orthogonal complement of the nominal plant. These can be obtained by performing suitable polynomial spectral factorizations.

As noted above, in this paper we will make use heavily of rational representations of behaviors as introduced recently in [23]. Of course, rational representations have been used before in the context of behavioral $\mathcal{H}_\infty$-control in [9].

The outline of this paper is as follows. In section 2 we discuss the basic material on linear differential behaviors, and their polynomial and rational representations. Then, in section 3, we introduce the problem of robust stabilization in the context of behaviors (which we call Problem 1), and the problem of finding the optimal stability radius (called Problem 2). Similar to the classical input-output framework, our robust stabilization problem requires the solution to a behavioral $\mathcal{H}_\infty$ synthesis problem. This problem is formulated in section 5 (called Problem 3). In section 6 we continue our review of linear differential systems, and also prove a number of new results in the context of rational representations. Section 7 gives a review of the behavioral theory of dissipative systems, quadratic differential forms, two-variable polynomial matrices and storage functions. Then, in section 7, we are in a position to resolve our $\mathcal{H}_\infty$ synthesis problem (Problem 3). Using this, in section 8 we give solutions to Problems 1 and 2. In section 9 we give a small worked-out example. Some of the proofs can be found in the Appendix, section 11.
1.1 Notation and nomenclature

A few words about the notation and nomenclature used. We use standard symbols for the fields of real and complex numbers $\mathbb{R}$ and $\mathbb{C}$. $\mathbb{C}^{-}$, $\mathbb{C}^{+}$, and $\mathbb{C}_{+}$ will denote the open left half plane, open right half plane and closed right half plane, respectively. We use $\mathbb{R}^{n}$, $\mathbb{R}^{n \times m}$, etc. for the real linear spaces of vectors and matrices with components in $\mathbb{R}$.

$\mathcal{C}^{\infty}(\mathbb{R}, \mathbb{R}^{n})$ denotes the set of infinitely often differentiable functions from $\mathbb{R}$ to $\mathbb{R}^{n}$, and its subspace consisting of functions with compact support is denoted by $\mathcal{D}(\mathbb{R}, \mathbb{R}^{n})$, or sometimes simply by $\mathcal{D}$. The space of all measurable functions $w$ from $\mathbb{R}$ to $\mathbb{R}^{n}$ such that $\int_{-\infty}^{\infty} |w(t)|^{2} dt < \infty$ is denoted by $L_{2}(\mathbb{R}, \mathbb{R}^{n})$. The $L_{2}$-norm of $w$ is $\|w\|_{2} := (\int_{-\infty}^{\infty} |w(t)|^{2} dt)^{1/2}$.

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If the domain and codomain are obvious from the context, we denote $L_{2}(\mathbb{R}, \mathbb{R}^{n})$ simply by $L_{2}$.

$\mathbb{R}[\xi]$ denotes the ring of polynomials in the indeterminate $\xi$ with real coefficients, and $\mathbb{R}(\xi)$ denotes its quotient field of real rational functions in the indeterminate $\xi$. We use $\mathbb{R}[\xi], \mathbb{R}^{n \times m}[\xi], \mathbb{R}^{n}[\xi], \mathbb{R}^{n \times m}(\xi), \mathbb{R}(\xi), \mathbb{R}[\xi]$ etc. for the spaces of vectors and matrices with components in $\mathbb{R}[\xi]$ and $\mathbb{R}(\xi)$, respectively. Elements of $\mathbb{R}^{n \times m}[\xi]$ are called real polynomial matrices, elements of $\mathbb{R}^{n \times m}(\xi)$ are called real rational matrices. Any real polynomial matrix can be written as a finite sum $X(\xi) = \sum_{k=0}^{N} X_{k} \xi^{k}$. The real matrix $(X_{0}, X_{1}, \ldots, X_{N})$ is called the coefficient matrix of $X(\xi)$, and is denoted by $X$.

We use the notation $\text{det}(A)$, to denote the determinant of a square matrix $A$. A square, nonsingular real polynomial matrix $R$ is called Hurwitz if all roots of $\text{det}(R)$ lie in the open left half complex plane $\mathbb{C}^{-}$. It is called anti-Hurwitz if all roots of $\text{det}(R)$ lie in the open right half complex plane $\mathbb{C}^{+}$. A proper real rational matrix $G$ is called stable if all its poles are in $\mathbb{C}^{-}$. We denote by $\mathbb{R}_{S}(\xi)$ the ring of all proper stable real rational functions. $\mathbb{R}^{n}_{S}(\xi)$ and $\mathbb{R}^{n \times m}_{S}(\xi)$ denote the spaces of vectors and matrices with components in $\mathbb{R}_{S}(\xi)$.

**Definition 1.1** A proper, stable real rational matrix $G$ is called left prime if it has a proper, stable right inverse, i.e. if there exists a proper, stable rational matrix $G^{\dagger}$ such that $GG^{\dagger} = I$. A proper, stable real rational matrix $G$ is called co-inner if $G(\xi)G^{\dagger}(-\xi) = I$.

Equivalent characterizations of left primeness can be found in [23]. If $G$ is a proper rational matrix and has no poles on the imaginary axis, then its $L_{\infty}$ norm is defined as $\|G\|_{\infty} := \sup_{\omega \in \mathbb{R}} \|G(i\omega)\|$. If $G$ is proper and stable, then $\|G\|_{\infty} = \sup_{\lambda \in \mathbb{C}^{+}} \|G(\lambda)\|$, the $\mathcal{H}_{\infty}$-norm of $G$.

Given a matrix $M \in \mathbb{R}^{n \times n}$, the Moore-Penrose inverse $M^{\dagger}$ of $M$ is the unique $n \times n$ matrix that satisfies the following properties: $MM^{\dagger}M = M$, $M^{\dagger}MM^{\dagger} = M^{\dagger}$, $(MM^{\dagger})^{\dagger} = MM^{\dagger}$, and $(M^{\dagger}M)^{\dagger} = M^{\dagger}M$.

Finally, we use the notation $\text{col}(w_{1}, w_{2})$ to represent the column vector formed by stacking $w_{1}$ over $w_{2}$.

2 Linear differential systems and rational representations

In this section we review the basic material on linear differential systems and their polynomial and rational representations that we need in this paper.

In the behavioral approach to linear systems, a dynamical system is given by a triple $\Sigma = (\mathbb{R}, \mathbb{R}^{w}, \mathcal{B})$, where $\mathbb{R}$ is the time axis, $\mathbb{R}^{w}$ is the signal space, and the behavior $\mathcal{B}$ is a linear subspace of $\mathcal{C}^{\infty}(\mathbb{R}, \mathbb{R}^{w})$ consisting of all solutions of a set of higher order, linear, constant coefficient differential equations. Such a triple is called a linear differential
system. The set of all linear differential systems with \( w \) variables is denoted by \( \mathcal{L}^w \).

For any linear differential system \( \Sigma = (R, R^w, \mathcal{B}) \) there exists a polynomial matrix \( R \) with \( w \) columns such that \( \mathcal{B} \) is equal to the space of solutions of

\[
R(\frac{d}{dt})w = 0.
\]

(1)

If a behavior \( \mathcal{B} \) is represented by \( R(\frac{d}{dt})w = 0 \) (or: \( \mathcal{B} = \ker(R) \)), with \( R(\xi) \) a real polynomial matrix, then we call this a polynomial kernel representation of \( \mathcal{B} \). If \( R \) has \( p \) rows, then the polynomial kernel representation is said to be minimal if every polynomial kernel representation of \( \mathcal{B} \) has at least \( p \) rows. A given polynomial kernel representation, \( \mathcal{B} = \ker(R) \), is minimal if and only if the polynomial matrix \( R \) has full row rank (see [11], Theorem 3.6.4). The number of rows in any minimal polynomial kernel representation of \( \mathcal{B} \), denoted by \( p(\mathcal{B}) \), is called the output cardinality of \( \mathcal{B} \). This number corresponds to the number of outputs in any input/output representation of \( \mathcal{B} \). For a detailed exposition of polynomial representations of behaviors, we refer to [11].

Recently, in [23], representations of linear differential systems using rational matrices instead of polynomial matrices were introduced. In [23], a meaning was given to the equation \( R(\frac{d}{dt})w = 0 \), where \( R(\xi) \) is a given real rational matrix. In order to do this, we need the concept of left coprime factorization.

**Definition 2.1** Let \( R \) be a real rational matrix. The pair of real polynomial matrices \((P,Q)\) is called a left coprime factorization of \( R \) over \( \mathbb{R}[\xi] \) if

1. \( \det(P) \neq 0 \),
2. \( R = P^{-1}Q \),
3. the matrix \((P(\lambda) Q(\lambda))\) has full row rank for all \( \lambda \in \mathbb{C} \).

A meaning to the equation

\[
R(\frac{d}{dt})w = 0,
\]

(2)

with \( R(\xi) \) a real rational matrix is then given as follows: Let \((P,Q)\) be a left coprime factorization of \( R \) over \( \mathbb{R}[\xi] \). Then we define:

\[
[w \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^w) \text{ is a solution of (2)}] \iff [Q(\frac{d}{dt})w = 0].
\]

It can be proven that the space of solutions is independent of the particular left coprime factorization. Hence (2) represents the linear differential system \( \Sigma = (\mathbb{R}, \mathbb{R}^w, \ker(Q)) \in \mathcal{L}^w \).

Since the behavior \( \mathcal{B} \) of the system \( \Sigma \) is the central item, we will mostly speak about the system \( \mathcal{B} \in \mathcal{L}^w \) (instead of \( \Sigma \in \mathcal{L}^w \)). If a behavior \( \mathcal{B} \) is represented by \( R(\frac{d}{dt})w = 0 \) (or: \( \mathcal{B} = \ker(R) \)), with \( R(\xi) \) a real rational matrix, then we call this a rational kernel representation of \( \mathcal{B} \). If \( R \) has \( p \) rows, then the rational kernel representation is called minimal if every rational kernel representation of \( \mathcal{B} \) has at least \( p \) rows. It can be shown that a given rational kernel representation \( \mathcal{B} = \ker(R) \) is minimal if and only if the rational matrix \( R \) has full row rank. As in the polynomial case, every \( \mathcal{B} \in \mathcal{L}^w \) admits a minimal rational kernel representation. The number of rows in any minimal rational kernel representation of \( \mathcal{B} \) is equal to the number of rows in any minimal polynomial kernel representation of \( \mathcal{B} \), and therefore equal to \( p(\mathcal{B}) \), the output cardinality of \( \mathcal{B} \). In general, if \( \mathcal{B} = \ker(R) \) is a rational kernel representation, then \( p(\mathcal{B}) = \text{rank}(R) \). This follows immediately from the corresponding result for polynomial kernel representations (see [11]).
Definition 2.2 A behavior $\mathcal{B} \in \mathcal{L}^{w}$ is said to be controllable, if for all $w_1, w_2 \in \mathcal{B}$, there exists $T \geq 0$, and $w \in \mathcal{B}$, such that $w(t) = w_1(t)$ for $t \leq 0$, and $w(t) = w_2(t - T)$ for $t \geq T$. It is stabilizable, if for every $w \in \mathcal{B}$, there exists $w' \in \mathcal{B}$ such that $w'(t) = w(t)$ for $t \leq 0$, and $\lim_{t \to \infty} w'(t) = 0$.

The subset of $\mathcal{L}^{w}$ of all controllable behaviors is denoted by $\mathcal{L}^{w}_{\text{cont}}$. Clearly, if $\mathcal{B} \in \mathcal{L}^{w}_{\text{cont}}$ then it is stabilizable. Recall the definition Def. 1.1 of left primeness. It was shown in [23], Theorem 5 that $\mathcal{B} \in \mathcal{L}^{w}$ is stabilizable if and only if there exists a proper, stable, left prime real rational matrix $R$ such that $\mathcal{B} = \ker(R)$. We will show now that if $\mathcal{B}$ is controllable, then $R$ can in addition be taken co-inner:

**Lemma 2.3** Let $\mathcal{B} \in \mathcal{L}^{w}_{\text{cont}}$. Then there exists a proper, stable, left prime, co-inner real rational matrix $R$ such that $\mathcal{B} = \ker(R)$.

*Proof:* See the Appendix. □

**Definition 2.4** $\mathcal{B} \in \mathcal{L}^{w}$ is called stable if for all $w \in \mathcal{B}$ we have $\lim_{t \to \infty} w(t) = 0$.

**Definition 2.5** Let $\mathcal{B} \in \mathcal{L}^{w_1+w_2}$ with system variable $w$ partitioned as $w = (w_1, w_2)$. We will call $w_2$ free in $\mathcal{B}$ if, for any $w_2 \in C^\infty(\mathbb{R}, \mathbb{R}^{q_2})$, there exists $w_1$ such that $(w_1, w_2) \in \mathcal{B}$.

**Definition 2.6** Let $\mathcal{B} \in \mathcal{L}^{w_1+w_2}$ with system variable $w$ partitioned as $w = (w_1, w_2)$. Let $\gamma > 0$. $\mathcal{B}$ is called $\gamma$-contractive if for all $(w_1, w_2) \in \mathcal{B} \cap \mathcal{L}_2$ we have $\|w_1\|_2 \leq \gamma\|w_2\|_2$. It is called strictly $\gamma$-contractive if there exists $\epsilon > 0$ such that $\mathcal{B}$ is $(\gamma - \epsilon)$-contractive.

**Remark 2.7** Of course, by a density argument, $\mathcal{B}$ is $\gamma$-contractive if and only if the contractivity condition $\|w_1\|_2 \leq \gamma\|w_2\|_2$ holds for all $(w_1, w_2) \in \mathcal{B} \cap \mathcal{D}$, i.e. for all trajectories in $\mathcal{B}$ of compact support.

## 3 Robust stabilization by interconnection

In this section we will introduce the problem of robust stabilization in a behavioral context, with control by general, regular, interconnection.

Let $\mathcal{P} \in \mathcal{L}^{w}$, to be interpreted as the nominal plant. Let $\mathcal{C} \in \mathcal{L}^{w}$, to be interpreted as a controller. Their full interconnection $\mathcal{P} \cap \mathcal{C}$ (called the controlled system) is defined as the intersection

$$\mathcal{P} \cap \mathcal{C} := \{w \in C^\infty(\mathbb{R}, \mathbb{R}^w) \mid w \in \mathcal{P} \text{ and } w \in \mathcal{C}\}.$$

The full interconnection $\mathcal{P} \cap \mathcal{C}$ is called regular (see [19]) if $p(\mathcal{P} \cap \mathcal{C}) = p(\mathcal{P}) + p(\mathcal{C})$. In that case we also call the controller $\mathcal{C}$ regular.

**Definition 3.1** A controller $\mathcal{C}$ is said to be a stabilizing controller for $\mathcal{P}$, if the full interconnection $\mathcal{P} \cap \mathcal{C}$ is stable. If $\mathcal{C}$ is a regular, stabilizing controller, then we say that $\mathcal{C}$ regularly stabilizes $\mathcal{P}$.

Recall the definition of stabilizability, Def. 2.2. The following lemma was proven in [19]:
**Lemma 3.2** Let $\mathcal{P} \in \mathcal{L}^u$. There exists a regular, stabilizing controller $\mathcal{C} \in \mathcal{L}^u$ for $\mathcal{P}$ if and only if $\mathcal{P}$ is stabilizable.

In the following, we will make the stronger assumption that our nominal plant $\mathcal{P} \in \mathcal{L}^u$ is controllable. The problem of robust stabilization is to find a controller $\mathcal{C} \in \mathcal{L}^u$ that stabilizes all plants in a given neighborhood of $\mathcal{P}$. We make the concept of neighborhood explicit as follows. Using Lemma 2.3, assume that the nominal plant $\mathcal{P}$ is represented in rational kernel representation by $R(\frac{d}{dt})w = 0$, where $R$ is a proper, stable, left prime and co-inner real rational matrix. For a given $\gamma > 0$, we now define the ball $B(\mathcal{P}, \gamma)$ with radius $\gamma$ around $\mathcal{P}$ as follows:

$$B(\mathcal{P}, \gamma) := \{ \mathcal{P}_\Delta \in \mathcal{L}^u_{\text{cont}} \mid \text{there exists a proper, stable, real rational } R_\Delta \text{ of full row rank such that } \mathcal{P}_\Delta = \ker(R_\Delta) \text{ and } \|R - R_\Delta\|_\infty \leq \gamma \} . \quad (3)$$

Of course, one should check whether this definition of ball around $\mathcal{P}$ is independent of the chosen representation. Indeed, we have the following:

**Theorem 3.3** Let $\mathcal{P} \in \mathcal{L}^u_{\text{cont}}$ and represent $\mathcal{P} = \ker(R_1) = \ker(R_2)$, with $R_1, R_2$ proper, stable, left prime and co-inner. For $i = 1, 2$, let

$$B_i(\mathcal{P}, \gamma) := \{ \mathcal{P}_\Delta \in \mathcal{L}^u_{\text{cont}} \mid \exists \text{ a proper, stable, real rational } R_\Delta \text{ of full row rank such that } \mathcal{P}_\Delta = \ker(R_\Delta) \text{ and } \|R_i - R_\Delta\|_\infty \leq \gamma \} .$$

Then we have $B_1(\mathcal{P}, \gamma) = B_2(\mathcal{P}, \gamma)$.

**Proof:** Let $\mathcal{P}_\Delta \in B_1(\mathcal{P}, \gamma)$. Then $\mathcal{P}_\Delta = \ker(R_\Delta)$ with $R_\Delta$ proper, stable, real rational and of full row rank, such that $\|R_1 - R_\Delta\|_\infty \leq \gamma$. We will show that $\mathcal{P}_\Delta \in B_2(\mathcal{P}, \gamma)$. Since $\mathcal{P} = \ker(R_1)$ with $R_1$ proper, stable and left prime, by Theorem 11.3 in the Appendix there exists a proper stable $W$, with $W^{-1}$ proper stable, such that $R_2 = WR_1$. We have $I = R_2R_2^\sim = WR_1R_1^\sim W^\sim = WW^\sim$. This yields $W^\sim W = I$ as well and hence $\|R_2 - WR_\Delta\|_\infty = \|W(R_1 - R_\Delta)\|_\infty = \|R_1 - R_\Delta\|_\infty \leq \gamma$. Since, again by Theorem 11.3, $\ker(WR_\Delta) = \ker(R_\Delta) = \mathcal{P}_\Delta$, we have $\mathcal{P}_\Delta \in B_2(\mathcal{P}, \gamma)$.

We now formulate the first main problem of this paper:

**Problem 1:** Find necessary and sufficient conditions for the existence of a controller $\mathcal{C} \in \mathcal{L}^u$ that regularly stabilizes all plants $\mathcal{P}_\Delta$ in the ball with radius $\gamma$ around $\mathcal{P}$, i.e. for all $\mathcal{P}_\Delta \in B(\mathcal{P}, \gamma)$, $\mathcal{P}_\Delta \cap \mathcal{C}$ is stabilizable and $\mathcal{P}_\Delta \cap \mathcal{C}$ is a regular interconnection.

Of course, for a given nominal plant $\mathcal{P}$, we would like to know the smallest upper bound (if it exists) of those $\gamma$’s for which there exists a controller $\mathcal{C}$ that regularly stabilizes all perturbed plants $\mathcal{P}_\Delta$ in the ball with radius $\gamma$ around $\mathcal{P}$. This is the problem of optimal robust stabilization.

**Problem 2:** Find the supremum

$$\gamma^* := \sup \{ \gamma > 0 \mid \exists \mathcal{C} \in \mathcal{L}^u \text{ that regularly stabilizes all } \mathcal{P}_\Delta \in B(\mathcal{P}, \gamma) \}. \quad (4)$$

**Remark 3.4** Of course, an urgent question is to investigate the interpretation of the ball $B(\mathcal{P}, \gamma)$ defined by (3) in terms of the gap between the behavior $\mathcal{P}$ and the behaviors $\mathcal{P}_\Delta$ in the ball. This question is currently under investigation.
4 A relevant $\mathcal{H}_\infty$ synthesis problem

Our solution of the robust stabilization problem will be based on a behavioral small gain argument. For this, we need to formulate and resolve an appropriate behavioral version of the $\mathcal{H}_\infty$ control problem, or problem of dissipative synthesis by interconnection. Such problems were studied before in [17], [22], [18] and [3], see also [9]. In particular, in the present paper we need an extension of the behavioral $\mathcal{H}_\infty$ problem that was studied and resolved in [17]. In this section, we will formulate this new problem.

We start with a system behavior $\mathcal{P}_{\text{full}} \in \mathcal{L}^{w,v,c}$, with system variable $(w, v, c)$. This behavior is called the full plant behavior. The system variable has been partitioned into $w$, $v$ and $c$. These variables represent the to-be-controlled variable, an unknown disturbance, and the interconnection variable, respectively, see also [17]. The interconnection variable $c$ is the system variable through which we are allowed to interconnect $\mathcal{P}_{\text{full}}$ with a controller $\mathcal{C} \in \mathcal{L}^c$. Interconnection leads to the interconnection of $\mathcal{P}_{\text{full}}$ and $\mathcal{C}$ through $c$:

$$\mathcal{P}_{\text{full}} \wedge_c \mathcal{C} := \{(w, v, c) | (w, v, c) \in \mathcal{P}_{\text{full}} \wedge c \in \mathcal{C}\},$$

which is called the full controlled behavior. Note that, in contrast to the full interconnection case (see section 3), the interconnection only takes place through the distinguished variable $c$, the interconnection variable of $\mathcal{P}_{\text{full}}$. The interconnection (5) is called a regular interconnection if $p(\mathcal{P}_{\text{full}} \wedge_c \mathcal{C}) = p(\mathcal{P}_{\text{full}}) + p(\mathcal{C})$. Also in this partial interconnection context the notion of regular interconnection plays an important role (see [2]). We define:

**Definition 4.1** Let $\mathcal{P}_{\text{full}} \in \mathcal{L}^{w,v,c}$ and let $\mathcal{C} \in \mathcal{L}^c$. The controller $\mathcal{C}$ is called regular if the interconnection $\mathcal{P}_{\text{full}} \wedge_c \mathcal{C}$ is a regular interconnection.

For any given controller $\mathcal{C} \in \mathcal{L}^c$, the manifest controlled behavior is defined as

$$(\mathcal{P}_{\text{full}} \wedge_c \mathcal{C})_{(w,v)} := \{(w, v) | \exists c \text{ such that } (w, v, c) \in \mathcal{P}_{\text{full}} \wedge_c \mathcal{C}\}$$

(also called the projection of the full controlled behavior onto the variable $(w, v)$, see Section 5). A given behavior $\mathcal{K} \in \mathcal{L}^{w,v}$ is called implementable (see [22]) if there exists a controller $\mathcal{C} \in \mathcal{L}^c$ such that the manifest controlled behavior is equal to $\mathcal{K}$, i.e. $\mathcal{K} = (\mathcal{P}_{\text{full}} \wedge_c \mathcal{C})_{(w,v)}$. The behavior $\mathcal{K} \in \mathcal{L}^{w,v}$ is called regularly implementable (see [2]) if there exists a regular controller $\mathcal{C} \in \mathcal{L}^c$ such that $\mathcal{K} = (\mathcal{P}_{\text{full}} \wedge_c \mathcal{C})_{(w,v)}$. The following proposition follows immediately from Theorem 4 in [2]:

**Proposition 4.2** Let $\mathcal{P}_{\text{full}} \in \mathcal{L}^{w,v,c}$ and let $\mathcal{K} \in \mathcal{L}^{w,v}$. If $\mathcal{P}_{\text{full}} \in \mathcal{L}^{w,v,c}$ is controllable then $\mathcal{K}$ is regularly implementable if and only if $\mathcal{K}$ is implementable.

In our context, the variable $v$ represents an unknown disturbance. This is formalized by assuming $v$ to be free in $\mathcal{P}_{\text{full}}$. As $v$ is interpreted as unknown disturbance, it should be free even after interconnecting the plant with a controller. In order to highlight this, we give the following definition:

**Definition 4.3** Let $\mathcal{P}_{\text{full}} \in \mathcal{L}^{w,v,c}$, with $v$ free. A controller $\mathcal{C} \in \mathcal{L}^c$ is called disturbance-free if $v$ is free in $\mathcal{P}_{\text{full}} \wedge_c \mathcal{C}$.

Following [17], in the context of $\mathcal{H}_\infty$ synthesis, a controller is called stabilizing if, whenever the disturbance $v$ is zero, the to-be-controlled variable $w$ tends to zero as time runs off to infinity:
Definition 4.4 Let $\mathcal{P}_{\text{full}} \in \mathcal{L}^{q+v+c}$, with $v$ free. A disturbance-free controller $\mathcal{C} \in \mathcal{L}^c$ is called stabilizing if $[(w,0,c) \in \mathcal{P}_{\text{full}} \wedge \mathcal{C}] \Rightarrow [\lim_{t \rightarrow \infty} w(t) = 0]$.

For the following, recall the notion of contractiveness in Def. 2.6:

Definition 4.5 Let $\mathcal{P}_{\text{full}} \in \mathcal{L}^{q+v+c}$. Let $\gamma > 0$. A controller $\mathcal{C} \in \mathcal{L}^c$ is called strictly $\gamma$-contracting if $(\mathcal{P}_{\text{full}} \wedge \mathcal{C})(w,v)$ is strictly $\gamma$-contractive.

We now formulate the $\mathcal{H}_\infty$ control problem that will be instrumental in our solution of Problems 1 and 2 as formulated in section 3.

Problem 3 : Let $\mathcal{P}_{\text{full}} \in \mathcal{L}^{q+v+c}$. Assume that $v$ is free. Let $\gamma > 0$. Find necessary and sufficient conditions for the existence of a disturbance-free, stabilizing, regular and strictly $\gamma$-contracting controller $\mathcal{C} \in \mathcal{L}^c$ for $\mathcal{P}_{\text{full}}$.

This problem was studied before in [17] without the requirement of regular interconnection. The assumptions on the full plant behavior that were used in [17] are however too restrictive for the purpose of the present paper, and we will therefore in Section 7 extend the results from [17].

5 More on linear differential systems

In this section we continue our treatment of the basic properties of linear differential systems.

Our first result gives a test in terms of the rational system representation for a part of the system variable to be free. The analogous result for polynomial kernel representations is well-known (see [2]).

Proposition 5.1 Let $\mathcal{B} \in \mathcal{L}^{q_1+q_2}$ with system variable $(w_1, w_2)$. Let $\mathcal{B}$ have a minimal rational kernel representation $\mathcal{B}$ that is given by $R_1(\frac{d}{dt})w_1 + R_2(\frac{d}{dt})w_2 = 0$. Then $w_2$ is free in $\mathcal{B}$ if and only if the rational matrix $R_1$ has full row rank.

Proof: Let $(R_1, R_2) = P^{-1}(Q_1, Q_2)$ be a left coprime factorization over $\mathbb{R}[\xi]$. Then $Q(\frac{d}{dt})w_1 + Q(\frac{d}{dt})w_2 = 0$ is a minimal polynomial kernel representation of $\mathcal{B}$. Hence $w_2$ is free if and only if $Q_1$ has full row rank, equivalently, $R_1$ has full row rank.

Next, we review the properties of observability and detectability, see also [11], Chapter 5:

Definition 5.2 Let $\mathcal{B} \in \mathcal{L}^{q_1+q_2}$ with system variable $(w_1, w_2)$. We say that $w_2$ is observable from $w_1$ in $\mathcal{B}$ if, whenever $(w_1', w_2'), (w_1, w_2') \in \mathcal{B}$, then $w_2' = w_2$. We say that $w_2$ is detectable from $w_1$ in $\mathcal{B}$ if, whenever $(w_1', w_2'), (w_1, w_2') \in \mathcal{B}$, then $\lim_{t \rightarrow \infty}(w_2' - w_2)(t) = 0$.

Often we are interested only in the behavior of one of the components, say the variable $w_1$, obtained by projecting all $(w_1, w_2) \in \mathcal{B}$ onto the first component $w_1$. This behavior $\mathcal{B}_{w_1}$ is defined by $\mathcal{B}_{w_1} := \{w_1 \mid \exists w_2 \text{ such that } (w_1, w_2) \in \mathcal{B}\}$. If $\mathcal{B} = \ker(R_1 R_2)$ is a polynomial kernel representation, then a representation for $\mathcal{B}_{w_1}$ is obtained as follows: choose a unimodular polynomial matrix $U$ such that $UR_2 = \begin{pmatrix} R_{12} \\ 0 \end{pmatrix}$, with $R_{12}$ full row rank, and conformably partition $UR_1 = \begin{pmatrix} R_{11} \\ R_{21} \end{pmatrix}$. Then $\mathcal{B}_{w_1} = \ker(R_{21})$ (see [11], section 6.2.2).
is important to note that if $\mathcal{B}$ is controllable, then the projection $\mathcal{B}_{w_1}$ is also controllable. In the context of rational kernel representations this procedure (called elimination), with $U$ unimodular replaced by: $U$ square, nonsingular, rational, only applies to controllable systems. Since we will not use this fact in this paper, we omit the details. As argued in [11], chapter 6, many models obtained from first principles modeling by interconnection include auxiliary variables, in addition to the variables the model aims at. We call the latter manifest variables, and denote them by $w$. The auxiliary variables are often called latent variables, and are denoted by $\ell$. In the context of polynomial representations, this leads to representations of the form

$$R(\frac{d}{dt})w = M(\frac{d}{dt})\ell$$

with $R, M$ real polynomial matrices. The space of all $w$ for which there exists $\ell$ such that (6) holds is called the manifest behavior. If $R = I$, the identity matrix, then (6) becomes

$$w = M(\frac{d}{dt})\ell.$$  

This is called a polynomial image representation of its manifest behavior.

**Definition 5.3** The polynomial image representation (7) is said to be observable if $\ell$ is observable from $w$ in $\mathcal{B}_f$, where $\mathcal{B}_f := \{(w, \ell) \mid (w, \ell) \text{ satisfy (7)}\}$. It is said to be detectable if $\ell$ is detectable from $w$ in $\mathcal{B}_f$.

Many properties of linear differential systems can be formulated in terms of the poles and zeros of the rational matrices appearing in their rational representations. For completeness, we first recall the notions of poles and zeros of rational matrices (see [23]):

**Proposition 5.4** Let $M \in \mathbb{R}^{n_1 \times n_2}(\xi)$. There exist $U \in \mathbb{R}^{n_1 \times n_1}[\xi], V \in \mathbb{R}^{n_2 \times n_2}[\xi]$, both unimodular, $\Pi \in \mathbb{R}^{n_1 \times n_1}[\xi]$, and $Z \in \mathbb{R}^{n_1 \times n_2}[\xi]$ such that

$$M = U\Pi^{-1}ZV, \quad \Pi = \text{diag}(\pi_1, \pi_2, \ldots, \pi_{n_1}),$$

$$Z = \begin{bmatrix}
\text{diag}(z_1, z_2, \ldots, z_r) & 0_{r \times (n_2-r)} \\
0_{(n_1-r) \times r} & 0_{(n_1-r) \times (n_2-r)}
\end{bmatrix}$$

with $z_1, z_2, \ldots, z_r, \pi_1, \pi_2, \ldots, \pi_{n_1}$ non-zero monic elements of $\mathbb{R}[\xi]$, the pairs $z_k, \pi_k$ coprime for $k = 1, 2, \ldots, r$, $\pi_k = 1$ for $k = r+1, r+2, \ldots, n_1$, and where $z_{k-1}$ is a factor of $z_k$ and $\pi_k$ is a factor $\pi_{k-1}$, for $k = 2, \ldots, r$. Of course, $r = \text{rank}(M)$. The roots of $\pi_k$'s (hence of $\pi_1$ disregarding the multiplicity issue) are called the poles of $M$, and the roots of the $z_k$'s (hence of $z_r$, disregarding the multiplicity issue) the zeros of $M$. If $M$ is a polynomial matrix, the $\pi_k$'s are absent (they are equal to 1). We then speak of the Smith form.

It is well-known (see [11]) that if $\mathcal{B} = \ker(R)$ is a minimal polynomial kernel representation, then $\mathcal{B}$ is stable if and only if $R$ is Hurwitz. Likewise, for rational representations we have:

**Lemma 5.5** Let $\mathcal{B} \in \mathcal{L}^w$ and let $\mathcal{B} = \ker(R)$ be a minimal rational kernel representation. Then $\mathcal{B}$ is stable if and only if $R$ is square, nonsingular, and has no zeros in $\mathbb{C}^+$. 

**Proof:** Let $R = P^{-1}Q$ be a left coprime factorization over $\mathbb{R}[\xi]$. Note that the zeros of $R$ coincide with the roots of $\det(Q)$. Since $\mathcal{B} = \ker(Q)$ is a minimal polynomial kernel
representation, \( \mathfrak{B} \) is stable if and only if \( Q \) is square and nonsingular, and \( \det(Q) \) has no roots in \( \bar{\mathbb{C}}^+ \), equivalently, \( R \) is square, nonsingular, and has no zeros in \( \bar{\mathbb{C}}^+ \).

The following can be found in [23]:

**Proposition 5.6** Let \( R \) be a real rational matrix and let \( M \) be real polynomial matrix. Then we have:

1. The rational kernel representation \( R(d/dt)w = 0 \) represents a controllable system if and only if \( R \) has no zeros.
2. The rational kernel representation \( R(d/dt)w = 0 \) defines a stabilizable system if and only if \( R \) has no zeros in \( \bar{\mathbb{C}}^+ \).
3. \( \mathfrak{B} \in \mathcal{L}^w \) is controllable if and only if \( \mathfrak{B} \) admits an observable polynomial image representation \( w = M(d/dt)\ell \).
4. The polynomial image representation \( w = M(d/dt)\ell \) is observable if and only if \( M \) has full column rank and has no zeros. It is detectable if and only if \( M \) has full column rank, and has no zeros in \( \bar{\mathbb{C}}^+ \).

Our next proposition states under what conditions a controller in rational kernel representation regularly stabilizes a given plant:

**Proposition 5.7** Let \( P, C \in \mathcal{L}^w \). Let \( R(d/dt)w = 0 \) and \( C(d/dt)v = 0 \) be minimal rational kernel representations of \( P \) and \( C \), respectively. Then \( C \) is a regular, stabilizing controller for \( P \) if and only if \( \begin{pmatrix} R \\ C \end{pmatrix} \) is square and nonsingular, and has no zeros in \( \bar{\mathbb{C}}^+ \).

**Proof:** \((\Rightarrow)\) \( C \) is regular, so \( p(\mathcal{P} \cap C) = p(\mathcal{P}) + p(C) = \text{rowdim}(R) + \text{rowdim}(C) \). This implies that \( \mathcal{P} \cap C = \ker \begin{pmatrix} R \\ C \end{pmatrix} \) is a minimal rational kernel representation. Since \( \mathcal{P} \cap C \) is also stable, it follows from Lemma 5.5 that \( \begin{pmatrix} R \\ C \end{pmatrix} \) is square and nonsingular, and has no zeros in \( \bar{\mathbb{C}}^+ \).

\((\Leftarrow)\) The converse implication is proven along the same lines. \( \Box \)

The following result characterizes the property that a controller is disturbance-free, stabilizing and regular in terms of the rational matrices in the given kernel representations:

**Proposition 5.8** Let \( \mathcal{P}_{\text{full}} \in \mathcal{L}^{w+v+c} \) and \( C \in \mathcal{L}^c \). Let \( R_1(d/dt)w + R_2(d/dt)v + R_3(d/dt)c = 0 \) and \( C(d/dt)c = 0 \) be minimal rational kernel representations of \( \mathcal{P}_{\text{full}} \) and \( C \), respectively. Assume that in \( \mathcal{P}_{\text{full}} \) \( c \) is observable from \( (w,v) \). Then the following are equivalent:

1. \( C \) is a disturbance-free, stabilizing, regular controller for \( \mathcal{P}_{\text{full}} \),
2. \( \begin{pmatrix} R_1 & R_3 \\ 0 & C \end{pmatrix} \) is square, nonsingular and has no zeros in \( \bar{\mathbb{C}}^+ \).

**Proof:** \((1) \Rightarrow (2)\) Since \( R_1, R_2, R_3 \) and \( C \) have full row rank, \( C \) is regular if and only if \( \begin{pmatrix} R_1 & R_2 & R_3 \\ 0 & 0 & C \end{pmatrix} \) has full row rank. Thus, by Prop. 5.1, \( v \) is free in \( \mathcal{P}_{\text{full}} \land_c C \) if and only if \( \begin{pmatrix} R_1 & R_3 \\ 0 & C \end{pmatrix} \) has full row rank. Consider now the behavior \( N \in \mathcal{L}^{w+c} \) represented
by \( \begin{pmatrix} R_1 & R_3 \\ 0 & C \end{pmatrix} \begin{pmatrix} w \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \). We will show that \( N \) is stable. Since \( C \) is disturbance-free stabilizing, \((w,c) \in N\) implies \( w(t) \to 0 (t \to \infty)\). This implies that the projection \( N_w \) of \( N \) onto the variable \( w \) is stable. It is easily seen that in \( N \), \( c \) is observable from \( w \), and that therefore also \( c(t) \to 0 (t \to \infty)\). Hence \( N \) is stable so \( \begin{pmatrix} R_1 & R_3 \\ 0 & C \end{pmatrix} \) is square, nonsingular and has no zeros in \( \mathbb{C}^+ \) (see Lemma 5.5). The converse implication \( (2) \Rightarrow (1) \) is proven in a similar way. \( \square \)

Finally, we characterize the property of strict contractiveness in terms of rational representations:

**Proposition 5.9** Let \( \mathcal{B} \in \mathcal{L}^{w_1+w_2} \) with system variable \((w_1,w_2)\). Let \( \gamma > 0 \). Let a minimal rational kernel representation of \( \mathcal{B} \) be given by \( R_1 \frac{d}{dt} w_1 + R_2 \frac{d}{dt} w_2 = 0 \). Assume that \( R_1 \) is square and nonsingular. Then \( \mathcal{B} \) is strictly \( \gamma \)-contractive if and only if \( R^{-1}_1 R_2 \) is proper, has no poles on the imaginary axis, and \( \| R^{-1}_1 R_2 \|_\infty < \gamma \).

**Proof:** See the Appendix \( \square \)

To conclude this section, we review the notion of orthogonal complement of a behavior (see [20]). Let \( \mathcal{B} \in \mathcal{L}^\gamma \) be a controllable behavior. Then we define its orthogonal complement \( \mathcal{B}^\perp \) by

\[
\mathcal{B}^\perp := \{ w \in \mathcal{C}^\infty(\mathbb{R},\mathbb{R}^\gamma) \mid \int_{-\infty}^{\infty} w^T w' dt = 0 \text{ for all } w' \in \mathcal{B} \cap \mathcal{D} \}.
\]

\( \mathcal{B}^\perp \) is again controllable. If \( R(\frac{d}{dt})w = 0 \) is a minimal polynomial kernel representation of \( \mathcal{B} \), then \( \tilde{w} = R^T(\frac{d}{dt}) \ell \) is an observable polynomial image representation of \( \mathcal{B}^\perp \) (see [20], Section 10).

### 6 Two-variable polynomial matrices, QDF’s and dissipative systems

In this paper, our aim is to establish conditions on a given plant \( P \) for the existence of robustly stabilizing controllers, and determine the optimal stability radius. A major role in our development will be played by the notions of dissipativeness, strict dissipativeness, and storage function in a behavioral context. These notions have been studied before in [21], [6], [15]. We also refer to [10]. In the present section we review these notions. An important role is played by two-variable polynomial matrices and quadratic differential forms. An extensive treatment was given in [20]. We will first give a brief review.

An \( l \times l \) two-variable polynomial matrix in the indeterminates \( \zeta \) and \( \eta \) is an expression of the form \( \Phi(\zeta, \eta) = \sum_{h=0}^{N} \Phi_{h,k} \zeta^h \eta^k \) where \( \Phi_{h,k} \) are real \( l \times l \) matrices, and where \( N \geq 0 \) is an integer. With any such two-variable polynomial matrix we can associate a bilinear functional \( L_\Phi : \mathcal{C}^\infty(\mathbb{R},\mathbb{R}^{l^2}) \times \mathcal{C}^\infty(\mathbb{R},\mathbb{R}^{l^2}) \to \mathcal{C}^\infty(\mathbb{R},\mathbb{R}) \) by defining \( L_\Phi(\ell_1, \ell_2) = \sum_{h,k=0}^{N} \Phi_{h,k} \frac{d^h \ell_1}{dt^h} \frac{d^k \ell_2}{dt^k} \). The two-variable polynomial matrix \( \Phi(\zeta, \eta) \) is called symmetric if \( \Phi_{h,k} = \Phi^T_{k,h} \) for all \( h, k \). In that case we also associate with \( \Phi(\zeta, \eta) \) the quadratic differential form (QDF) \( Q_\Phi(\ell) := L_\Phi(\ell, \ell) \).

The properties of the two-variable polynomial matrix \( \Phi(\zeta, \eta) \) are completely determined by the real constant \( (N+1)l \times (N+1)l \) matrix \( \check{\Phi} \) whose \((h,k)\)th block is equal to \( \Phi_{h,k} \). This
matrix will be called the coefficient matrix associated with $\Phi(\zeta, \eta)$. Factorizations of the coefficient matrix immediately give rise to corresponding factorizations of the associated two-variable polynomial matrix and quadratic differential form. The QDF $Q_\Phi$ is called non-negative if $Q_\Phi(\ell) \geq 0$, in the sense that $Q_\Phi(\ell)(t) \geq 0$ for all $t \in \mathbb{R}$. It is easily seen that $Q_\Phi$ is non-negative if and only if the coefficient matrix $\hat{\Phi}$ satisfies $\hat{\Phi} \geq 0$.

Consider, in general, a controllable linear differential system $\mathcal{B} \in \mathbb{L}^w$, represented by the observable polynomial image representation

$$ w = W\left(\frac{d}{dt}\right)\ell $$

with $W \in \mathbb{R}^{w \times 1}[\zeta]$. In addition, let $Q_\Phi$ be the QDF associated with the symmetric two-variable polynomial matrix $\Phi \in \mathbb{R}^{w \times w}[\zeta, \eta]$. $Q_\Phi$ will be called the supply rate. The system $\mathcal{B}$ will be called dissipative with respect to the supply rate $Q_\Phi$ if for all $w \in \mathcal{B} \cap \mathcal{D}(\mathbb{R}, \mathbb{R}^w)$ we have

$$ \int_{-\infty}^{\infty} Q_\Phi(w) dt \geq 0. \quad (9) $$

$\mathcal{B}$ is called strictly dissipative with respect to the supply rate $Q_\Phi$ if there exists $\epsilon > 0$ such that for all $w \in \mathcal{B} \cap \mathcal{D}(\mathbb{R}, \mathbb{R}^w)$

$$ \int_{-\infty}^{\infty} Q_\Phi(w) dt \geq \epsilon^2 \int_{-\infty}^{\infty} ||w(t)||^2 dt. \quad (10) $$

In this paper the supply rate will often be given by a constant real symmetric matrix, say $\Sigma$. In that case we have $Q_\Sigma(w) = w^T \Sigma w$. We say that the system $\mathcal{B}$ is (strictly) $\Sigma$-dissipative if it is (strictly) dissipative with respect to the supply rate $Q_\Sigma$.

Given the polynomial image representation (8) and the polynomial matrix $\Phi(\zeta, \eta)$, define $\Phi' \in \mathbb{R}^{1 \times 1}[\zeta, \eta]$ by $\Phi'(\zeta, \eta) := W^T(\zeta)\Phi(\zeta, \eta)W(\eta)$. It is easily verified that, if $w$ and $\ell$ are related by (8), then $Q_\Phi(w) = Q_{\Phi'}(\ell)$. Therefore, the system is dissipative if and only if for all $\ell \in \mathcal{D}(\mathbb{R}, \mathbb{R}^1)$ we have $\int_{-\infty}^{\infty} Q_{\Phi'}(\ell) dt \geq 0$, and strictly dissipative if and only if there exists $\epsilon > 0$ such that, for all $\ell \in \mathcal{D}(\mathbb{R}, \mathbb{R}^1)$ we have

$$ \int_{-\infty}^{\infty} Q_{\Phi'}(\ell) dt \geq \epsilon^2 \int_{-\infty}^{\infty} ||W\left(\frac{d}{dt}\right)\ell||^2 dt. $$

These conditions are equivalent to

$$ \Phi'(-i\omega, i\omega) \geq 0 \quad \text{for all } \omega \in \mathbb{R} \quad (11) $$

and

$$ \Phi'(-i\omega, i\omega) \geq \epsilon^2 W^T(-i\omega)W(i\omega) \quad \text{for all } \omega \in \mathbb{R} \quad (12) $$

respectively (see [20]). It is well known (see [4], [5], [13], [7]) that, if (11) holds then we can factorize $\partial \Phi'(\xi) := \Phi'(-\xi, \xi) = F^T(-\xi)F(\xi)$, with $F \in \mathbb{R}^{1 \times 1}[\xi]$. If (12) holds, then $F$ can be chosen Hurwitz, and also anti-Hurwitz. Introduce now the two-variable polynomial $\Delta$, defined by $\Delta(\zeta, \eta) := \Phi'(\zeta, \eta) - F^T(\xi)F(\eta)$. Since $\Delta(-\xi, \xi) = 0$, the two-variable polynomial $\Delta$ must contain a factor $\zeta + \eta$ (see [20], theorem 3.1), and therefore we can define the new two-variable polynomial $\Psi$ by

$$ \Psi(\zeta, \eta) := (\zeta + \eta)^{-1} \Delta(\zeta, \eta). \quad (13) $$
Consider now the QDF’s $Q_\Psi$ and $Q_\Delta$ associated with $\Psi$ and $\Delta$, respectively. We have $Q_\Delta(\ell) = Q_\Psi(\ell) - \|F(\ell, \ell)\|^2$. Furthermore, (13) is equivalent to: $\frac{dQ_\Psi(\ell)}{dt} = Q_\Delta(\ell)$ for all $\ell \in C^\infty(\mathbb{R}, \mathbb{R}^1)$. Thus we obtain
\[
\frac{dQ_\Psi(\ell)}{dt}(t) \leq Q_\Psi(\ell)(t),
\]
for all $\ell \in C^\infty(\mathbb{R}, \mathbb{R}^1)$, for all $t \in \mathbb{R}$. If we interpret $Q_\Psi(\ell)(t)$ as the amount of supply (e.g., energy) stored inside the system at time $t$, then (14) expresses the fact that the rate at which the internal storage increases does not exceed the rate at which supply flows into the system. The inequality (14) is called the dissipation inequality. Any quadratic differential form $Q_\Psi(\ell)$ that satisfies this inequality is called a storage function for $\mathcal{B}$. It can be shown that, $\mathcal{B}$ is dissipative if and only if there exists a symmetric two-variable polynomial matrix $\Psi(\zeta, \eta)$ such that, the corresponding QDF $Q_\Psi$ satisfies (14). In general, storage functions are not unique. In fact, we quote [20], Theorem 5.7:

**Proposition 6.1** Let $\mathcal{B}$ be represented by the observable image representation (8). Assume $\mathcal{B}$ is dissipative with respect to $Q_\Psi$. Then there exist storage functions $Q_{\Psi-}$ and $Q_{\Psi+}$ such that any other storage function $Q_\Psi$ satisfies $Q_{\Psi-} \leq Q_\Psi \leq Q_{\Psi+}$. If $\mathcal{B}$ is strictly dissipative then $\Psi_-$ and $\Psi_+$ may be constructed as follows. Let $H$ and $A$ be respectively Hurwitz and anti-Hurwitz factorizations of $\partial \Psi'$. Then
\[
\Psi_+(\zeta, \eta) = \frac{\Phi'(\zeta, \eta) - A^\top(\zeta)A(\eta)}{\zeta + \eta}
\]
and
\[
\Psi_-(\zeta, \eta) = \frac{\Phi'(\zeta, \eta) - H^\top(\zeta)H(\eta)}{\zeta + \eta}.
\]

Finally, we spend some words on state maps, and on positive and negative definiteness of storage functions.

Again, let $\mathcal{B}$ be given in image representation $w = W(\frac{d}{dt})\ell$, with $W$ a polynomial matrix. An $n \times 1$ polynomial matrix, $X$, is said to define a state map for $\mathcal{B}$ if, $x := X(\frac{d}{dt})\ell$ is a state variable for $\mathcal{B}$ (see [12]). The dimension of the state space of a state-minimal representation of $\mathcal{B} \in \mathcal{L}^n$ is called the McMillan degree of $\mathcal{B}$ and is denoted by $n(\mathcal{B})$. Often, $n(\mathcal{B})$ is denoted by $n$. A state map $X$ for $\mathcal{B}$ is called a minimal state map if its number of rows is equal to $n(\mathcal{B})$. It was shown in [12] how to construct minimal state maps $X$ for a given $\mathcal{B}$. In this paper, we need state maps with the additional property that their coefficient matrix (see subsection 1.1) is part of an orthogonal matrix:

**Lemma 6.2** Let $\mathcal{B} \in \mathcal{L}^n$. There exists a polynomial matrix $X(\xi)$ defining a state map $X(\frac{d}{dt})$ for $\mathcal{B}$, such that the coefficient matrix $X$ of $X(\xi)$ satisfies $X^\top X = I$.

**Proof:** Let $X(\xi)$ yield a minimal state map. Then its rows are linearly independent over $\mathbb{R}$ (see [12]), and hence its coefficient matrix $X$ has full row rank. Thus, there exists a square nonsingular real $n(\mathcal{B}) \times n(\mathcal{B})$ matrix $S$ such that $X^\top X = SS^\top$. Define now a new polynomial matrix $X'(\xi)$ by $X'(\xi) := S^{-1}X(\xi)$. Clearly $X'^\top X' = I$ and $X'(\frac{d}{dt})$ also defines a minimal state map for $\mathcal{B}$, since $S$ merely represents a nonsingular state transformation. \qed

The following proposition obtained in [20] (also see [16]) will also play an important role in the sequel.
Proposition 6.3 Let $\mathcal{B}$ be represented by the observable image representation (8). Assume $\mathcal{B}$ is dissipative with respect to $Q_\Sigma(w) = w^\top \Sigma w$, where $\Sigma \in \mathbb{R}^{n \times n}$, and let $Q_\Psi(\ell)$ be a storage function. Let $X \in \mathbb{R}^{n \times |\ell|}$ define a minimal state map of $\mathcal{B}$. Then there exists a real symmetric matrix $K \in \mathbb{R}^{n \times n}$ such that $Q_\Psi(\ell) = (X(d\ell/dt))^\top KX(d\ell/dt)$ for all $\ell \in C^{\infty}(\mathbb{R}, \mathbb{R}^1)$.

Definition 6.4 A storage function $Q_\Psi$ for $\mathcal{B}$ is called positive (negative) definite if there exists a minimal state map $X$ for $\mathcal{B}$ and a real symmetric matrix $K > 0$ ($K < 0$) such that $Q_\Psi(\ell) = (X(d\ell/dt))^\top KX(d\ell/dt)$ for all $\ell \in C^{\infty}(\mathbb{R}, \mathbb{R}^1)$.

7 A solution to the $H_\infty$ synthesis problem

In this section we will study the $H_\infty$ synthesis problem stated as Problem 3 in section 4. This problem was already studied extensively in [17]. However, in order to be able to apply it to our robust stabilization problem, we need to extend the main result of [17]. This will be done in this section.

Let $P_{\text{full}} \in \mathcal{L}^{w+v+c}$. Let $\gamma > 0$. It is well known that strict contractiveness and strict dissipativeness are equivalent, in the sense that a controller $C \in \mathcal{L}^c$ is strictly $\frac{1}{\gamma}$-contracting if and only if $(P_{\text{full}} \land_c C)_{(w,v)}$ is strictly $\Sigma_\gamma$-dissipative, where

$$
\Sigma_\gamma := \begin{pmatrix} -I_v & 0 \\ 0 & -\frac{1}{\gamma^2}I_v \end{pmatrix}.
$$

Note that

$$
-\Sigma_\gamma^{-1} = \begin{pmatrix} I_v & 0 \\ 0 & -\gamma^2I_v \end{pmatrix}.
$$

In [17], necessary and sufficient conditions for the existence of a disturbance-free, stabilizing, strictly $\frac{1}{\gamma}$-contracting controller (however, without regularity condition) for $P_{\text{full}}$ were established in terms of $-\Sigma_\gamma^{-1}$-dissipativeness of an orthogonal behavior associated with $P_{\text{full}}$. We summarize the relevant results here as propositions. Recall from section 5 that $(P_{\text{full}})_{(w,v)}$ denotes the projection of $P_{\text{full}}$ onto the variable $(w,v)$, while $(P_{\text{full}})_{(w,v)}^\perp$ denotes its orthogonal complement. Our first proposition is a restatement of Lemma 9.2 from [17]:

Proposition 7.1 Let $P_{\text{full}} \in \mathcal{L}^{q+v+c}_{\text{cont}}$. Assume $v$ is free. Let $\gamma > 0$. If there exists a disturbance-free, stabilizing, strictly $\frac{1}{\gamma}$-contracting controller for $P_{\text{full}}$ then $(P_{\text{full}})_{(w,v)}^\perp$ is strictly $-\Sigma_\gamma^{-1}$-dissipative and has a negative definite storage function.

The next proposition can be found as Theorem 9.1 in [17]. It states that if our synthesis problem is a full information problem (i.e. $(w,v)$ observable from $c$), then the conditions in Proposition 7.1 are also sufficient:

Proposition 7.2 Let $P_{\text{full}} \in \mathcal{L}^{q+v+c}_{\text{cont}}$. Assume $v$ is free, and

1. $(w,v)$ is observable from $c$,
2. $c$ is observable from $(w,v)$. 

Let \( \gamma > 0 \). Then there exists a disturbance-free, stabilizing, strictly \( \frac{1}{\gamma} \)-contracting controller for \( P_{\text{full}} \) if and only if \( (P_{\text{full}})_{(w,v)}^{+} \) is \( -\Sigma_{\gamma}^{-1} \)-dissipative and has a negative definite storage function.

For the purpose of this paper we need to extend the above proposition in two directions. In the first place, we need to relax condition 1) of the proposition to the condition that \( (w,v) \) is only detectable from \( c \). Secondly, we need to know under what conditions the controller in the statement of the proposition can, in addition, be taken regular. The following theorem is the main result of this section. It states that the necessary and sufficient conditions of Proposition 7.2 remain valid:

**Theorem 7.3** Let \( P_{\text{full}} \in \mathcal{L}^{w+v+c}_{\text{cont}} \). Assume \( v \) is free, and

1. \( (w,v) \) is detectable from \( c \),
2. \( c \) is observable from \( (w,v) \).

Let \( \gamma > 0 \). Then there exists a disturbance-free, stabilizing, regular, and strictly \( \frac{1}{\gamma} \)-contracting controller for \( P_{\text{full}} \) if and only if \( (P_{\text{full}})_{(w,v)}^{+} \) is \( -\Sigma_{\gamma}^{-1} \)-dissipative and has a negative definite storage function.

In the remainder of this section we will give a proof of Theorem 7.3. The idea is, starting from \( P_{\text{full}} \), to construct a new full plant behavior \( P'_{\text{full}} \) that satisfies the conditions of Proposition 7.2. We then apply this proposition to \( P'_{\text{full}} \), and finally translate back to \( P_{\text{full}} \) to obtain a proof of Theorem 7.3.

Since \( P_{\text{full}} \) is controllable it admits an observable polynomial image representation (see Proposition 5.6):

\[
P_{\text{full}} = \{ \begin{pmatrix} w \\ v \\ c \end{pmatrix} | \exists \ell \in \mathcal{C}^{\infty}(\mathbb{R}, \mathbb{R}^n) \text{ such that } \begin{pmatrix} w \\ v \\ c \end{pmatrix} = \begin{pmatrix} W(\frac{d}{dt}) \\ V(\frac{d}{dt}) \\ C(\frac{d}{dt}) \end{pmatrix} \ell \}. \tag{17}
\]

It is easily verified that \( c \) is observable from \( \text{col}(w,v) \) if and only if the matrix \( \begin{pmatrix} W(\lambda) \\ V(\lambda) \end{pmatrix} \) has full column rank for all \( \lambda \in \mathbb{C} \), and \( \text{col}(w,v) \) is detectable from \( c \) if and only if the matrix \( C(\lambda) \) has full column rank for all \( \lambda \in \mathbb{C}^{+} \). Therefore we can factorize \( C \) as \( C = C'L \), with \( L \) and \( C' \) polynomial matrices such that \( L \) is Hurwitz and \( C'(\lambda) \) has full column rank for all \( \lambda \in \mathbb{C} \). Define now a new behavior \( P'_{\text{full}} \in \mathcal{L}^{w+v+c}_{\text{cont}} \) as follows:

\[
P'_{\text{full}} := \{ \begin{pmatrix} w \\ v \\ c' \end{pmatrix} | \exists \ell \in \mathcal{C}^{\infty}(\mathbb{R}, \mathbb{R}^n) \text{ such that } \begin{pmatrix} w \\ v \\ c' \end{pmatrix} = \begin{pmatrix} W(\frac{d}{dt}) \\ V(\frac{d}{dt}) \\ C'(\frac{d}{dt}) \end{pmatrix} \ell \}. \]

Clearly \( (P_{\text{full}})_{(w,v)} = (P'_{\text{full}})_{(w,v)} \), and, most important, in \( P'_{\text{full}} \), \( (w,v) \) is observable from \( c' \) (use the fact that \( C'(\lambda) \) has full column rank for all \( \lambda \in \mathbb{C} \)). We now first prove the following lemma which states that, due to the fact that in \( P'_{\text{full}} \) \( (w,v) \) is observable from \( c' \), the full controlled behavior corresponding to a given controller can also be implemented by a controller of the form \( c' = C'(\frac{d}{dt})\ell', K(\frac{d}{dt})\ell' = 0 \):

**Lemma 7.4** Consider the system \( P'_{\text{full}} \). Let \( c' \in \mathcal{L}^{c} \). There exists a full row rank polynomial matrix \( K \) such that \( \mathcal{C}' := \{ c' | \exists \ell' \in \mathcal{C}^{\infty}(\mathbb{R}, \mathbb{R}^n) \text{ such that } c' = C'(\frac{d}{dt})\ell', K(\frac{d}{dt})\ell' = 0 \} \) satisfies \( P_{\text{full}} \wedge c \mathcal{C}' = P'_{\text{full}} \wedge c \mathcal{C}' \).
Proof: Let \( \mathcal{C}' \) be represented by, say, \( S(\frac{d}{dt})c' = 0 \). Let \( K \) be a full row rank polynomial matrix such that \( \ker(SC') = \ker(K) \). We claim that, with such \( K \), the statement of the lemma holds. Indeed, let \( \col(w, v, c') \in \mathcal{P}_1 \wedge_c \mathcal{C}' \). Then there exists \( \ell \) such that \( w = W(\frac{d}{dt})\ell, v = V(\frac{d}{dt})\ell, c' = C'(\frac{d}{dt})\ell, \) and \( S(\frac{d}{dt})c' = 0 \). Since \( S(\frac{d}{dt})C'(\frac{d}{dt})\ell = 0 \), we get \( K(\frac{d}{dt})\ell = 0 \). Thus \( c' \in \mathcal{C}' \), so \( \col(w, v, c') \in \mathcal{P}_1 \wedge_c \mathcal{C}' \).

Conversely, let \( \col(w, v, c') \in \mathcal{P}_1 \wedge_c \mathcal{C}' \). Then there exist \( \ell \) and \( \ell' \) such that \( w = W(\frac{d}{dt})\ell, v = V(\frac{d}{dt})\ell', c' = C'(\frac{d}{dt})\ell', \) and \( K(\frac{d}{dt})\ell' = 0 \). Since \( C'(\lambda) \) has full column rank for all \( \lambda \), this implies that \( \ell = \ell' \). Thus, \( S(\frac{d}{dt})c' = S(\frac{d}{dt})C'(\frac{d}{dt})\ell' = 0 \) since \( K(\frac{d}{dt})\ell' = 0 \). We conclude that \( \col(w, v, c') \in \mathcal{P}_1 \wedge_c \mathcal{C}' \). □

In order to proceed, we formulate and prove the following theorem:

**Theorem 7.5** Let \( \mathcal{P}_{\text{full}} \) and \( \mathcal{P}_1 \) be as above. Let \( \gamma > 0 \). If there exists a disturbance-free, stabilizing, strictly \( \frac{1}{\gamma} \)-contracting controller for \( \mathcal{P}_{\text{full}} \) then there exists a disturbance-free, stabilizing, strictly \( \frac{1}{\gamma} \)-contracting controller for \( \mathcal{P}_1 \).

Proof: Using Lemma 7.4, let

\[
\mathcal{C}' = \{ c' \mid c' = C'(\frac{d}{dt})\ell', K(\frac{d}{dt})\ell' = 0 \} \tag{18}
\]

be a disturbance-free stabilizing, strictly \( \frac{1}{\gamma} \)-contracting controller for \( \mathcal{P}_1 \). Define \( \mathcal{K}_1 \in \mathcal{S}_{\omega^+\gamma} \) by

\[
\mathcal{K}_1 := \{ \begin{pmatrix} w \\ v \end{pmatrix} \mid \exists \ell' \text{ such that } \begin{pmatrix} w \\ v \end{pmatrix} = \begin{pmatrix} W(\frac{d}{dt}) \\ V(\frac{d}{dt}) \end{pmatrix} \ell', K(\frac{d}{dt})\ell' = 0 \}. \tag{19}
\]

From observability of \( \begin{pmatrix} w \\ v \end{pmatrix} \) from \( c' \) in \( \mathcal{P}_{\text{full}} \) we have \( \mathcal{K}_1 = \mathcal{P}_{\text{full}} \wedge_c \mathcal{C}'(w, v) \).

Let \( L \) be the Hurwitz polynomial matrix obtained from the factorization \( C = C'L \) above. As \( L \) is Hurwitz, the matrix \( KL^{-1} \) is a stable rational matrix. Factorize \( KL^{-1} = P_1^{-1}Q_1 \), where \( P_1 \) is Hurwitz. Then we have

\[
P_1K = Q_1L. \tag{20}
\]

Define

\[
\mathcal{C} := \{ c \mid \exists \ell \text{ such that } c = C(\frac{d}{dt})\ell, P_1(\frac{d}{dt})K(\frac{d}{dt})\ell = 0 \}, \tag{21}
\]

and \( \mathcal{K}_2 \in \mathcal{S}_{\omega^+\gamma} \) by

\[
\mathcal{K}_2 := \{ \begin{pmatrix} w \\ v \end{pmatrix} \mid \exists \ell \text{ such that } \begin{pmatrix} w \\ v \end{pmatrix} = \begin{pmatrix} W(\frac{d}{dt}) \\ V(\frac{d}{dt}) \end{pmatrix} \ell, P_1(\frac{d}{dt})K(\frac{d}{dt})\ell = 0 \}. \tag{22}
\]

**Lemma 7.6** \( (\mathcal{P}_{\text{full}} \wedge_c \mathcal{C})(w, v) = \mathcal{K}_2 \).

Proof: The implication \( \mathcal{K}_2 \subseteq (\mathcal{P}_{\text{full}} \wedge_c \mathcal{C})(w, v) \) is trivial. The converse implication is proven as follows.

Let \( \begin{pmatrix} w \\ v \end{pmatrix} \in (\mathcal{P}_{\text{full}} \wedge_c \mathcal{C})(w, v) \). Then there exists a \( c \) such that \( \begin{pmatrix} w \\ v \\ c \end{pmatrix} \in \mathcal{P}_{\text{full}} \) and \( c \in \mathcal{C} \). From the representations of \( \mathcal{P}_{\text{full}} \) and \( \mathcal{C} \) it is evident that there exists an \( \ell \) such
that \( \begin{pmatrix} w \\ v \end{pmatrix} = \begin{pmatrix} W(\frac{d}{dt}) \\ V(\frac{d}{dt}) \end{pmatrix} \ell \), \( c = C'(\frac{d}{dt})L(\frac{d}{dt})\ell \), and there exists an \( \hat{\ell} \) such that \( c = C'(\frac{d}{dt})L(\frac{d}{dt})\hat{\ell} \). Let the controllers \( C \) that stabilize \( P \) is a disturbance-free, stabilizing controller for \( P \). Let \( \text{ker}(P_1) = \text{ker}(K_1) = 0 \). As \( C'(\lambda) \) has full column rank for all \( \lambda \in \mathbb{C} \) we get
\[
L(\frac{d}{dt})\ell = L(\frac{d}{dt})\hat{\ell}.
\]
Now use (20) and (23) to obtain \( P_1(\frac{d}{dt})K(\frac{d}{dt})\ell = 0 \). As \( \begin{pmatrix} w \\ v \end{pmatrix} = \begin{pmatrix} W(\frac{d}{dt}) \\ V(\frac{d}{dt}) \end{pmatrix} \ell \) and \( P_1(\frac{d}{dt})K(\frac{d}{dt})\ell = 0 \) we conclude that \( \begin{pmatrix} w \\ v \end{pmatrix} \in \mathcal{K}_2 \). \( \square \)

**Lemma 7.7** Let the controllers \( \mathcal{C}' \) and \( \mathcal{C} \) be given by (18) and (21), respectively. Then \( \mathcal{C} \) is a disturbance-free, stabilizing controller for \( \mathcal{P}_\text{full} \) if and only if \( \mathcal{C}' \) is a disturbance-free stabilizing controller for \( \mathcal{P}'_\text{full} \).

**Proof:** By Lemma 7.6, \( \mathcal{C} \) is a disturbance-free, stabilizing controller for \( \mathcal{P}_\text{full} \) if and only if in \( \mathcal{K}_2 \) we have: \( v \) is free, and \( v = 0 \) implies \( w(t) \rightarrow 0 (t \rightarrow \infty) \). From the representation (22) of \( \mathcal{K}_2 \) it is easily seen that \( v \) is free if and only if \( \begin{pmatrix} V \\ P_1K \end{pmatrix} \) has full row rank. Moreover, \( w(t) \rightarrow 0 (t \rightarrow \infty) \) if and only if \( \begin{pmatrix} V(\lambda) \\ P_1(\lambda)K(\lambda) \end{pmatrix} \) has full column rank for all \( \lambda \in \mathbb{C}^+ \) (use the fact that \( \begin{pmatrix} W(\lambda) \\ V(\lambda) \end{pmatrix} \) has full column rank for all \( \lambda \), so \( v = 0 \) and \( w(t) \rightarrow 0 (t \rightarrow \infty) \) imply \( \ell(t) \rightarrow 0 (t \rightarrow \infty) \)). Thus, \( \mathcal{C} \) is disturbance-free and stabilizing if and only if \( \begin{pmatrix} V \\ P_1K \end{pmatrix} \) is Hurwitz. In the same way, using the representation (19) of \( \mathcal{K}_1 \) we can show that \( \mathcal{C}' \) is disturbance-free and stabilizing for \( \mathcal{P}'_\text{full} \) if and only if \( \begin{pmatrix} V \\ K \end{pmatrix} \) is Hurwitz. The proof is then completed by noting that \( \begin{pmatrix} V \\ P_1K \end{pmatrix} \) is Hurwitz if and only if \( \begin{pmatrix} V \\ K \end{pmatrix} \) is Hurwitz (use the fact that \( P_1 \) is Hurwitz). \( \square \)

In the following, recall that \( \mathcal{D} \) denotes the space of compact support functions

**Lemma 7.8** Let \( \mathcal{K}_1 \) and \( \mathcal{K}_2 \) be given by (19) and (22), respectively. Then \( \mathcal{K}_1 \cap \mathcal{D} = \mathcal{K}_2 \cap \mathcal{D} \). Consequently, for any \( \gamma > 0 \), \( \mathcal{K}_1 \) is strictly \( \frac{1}{\gamma} \)-contractive if and only if \( \mathcal{K}_2 \) is strictly \( \frac{1}{\gamma} \)-contractive.

**Proof:** In the following, we suppress the symbol \( \frac{d}{dt} \) in the polynomial differential operators. We first prove that \( \mathcal{D} \cap \ker(K) = \mathcal{D} \cap \ker(P_1K) \). The implication \( \mathcal{D} \cap \ker(M) \subseteq \mathcal{D} \cap \ker(P_1M) \) is obvious. To show the converse implication, assume that \( \ell \in \mathcal{D} \cap \ker(P_1K) \). Define \( y := K\ell \). Then \( y \in \mathcal{D} \cap \ker(P_1) \). As \( \ker(P_1) \cap \mathcal{D} = 0 \) (since \( P_1 \) is nonsingular) we have \( y = 0 \). Hence \( \ell \in \mathcal{D} \cap \ker(K) \).

Since \( \begin{pmatrix} w \\ v \end{pmatrix} = \begin{pmatrix} W(\frac{d}{dt}) \\ V(\frac{d}{dt}) \end{pmatrix} \ell \) is an observable representation, \( \ell \in \mathcal{D} \) if and only if \( \begin{pmatrix} w \\ v \end{pmatrix} \in \mathcal{D} \). Then, from the definitions of \( \mathcal{K}_1 \) and \( \mathcal{K}_2 \) we have the equality \( \mathcal{K}_1 \cap \mathcal{D} = \mathcal{K}_2 \cap \mathcal{D} \). Therefore, immediately from Definition 2.6, \( \mathcal{K}_1 \) is strictly \( \frac{1}{\gamma} \)-contractive if and only if \( \mathcal{K}_2 \) is strictly \( \frac{1}{\gamma} \)-contractive. \( \square \)
Applying the previous lemmas, we can now complete the proof of Theorem 7.5: from lemmas 7.6 to 7.8 we conclude that, starting with the disturbance-free, stabilizing strictly $\frac{1}{\ell}$-contracting controller $\mathcal{C}'$ for $\mathcal{P}'_{\text{full}}$, the controller $\mathcal{C}$ is a disturbance-free, stabilizing strictly $\frac{1}{\ell}$-contracting controller $\mathcal{P}_{\text{full}}$. \[\square\]

We are now in a position to give a proof of Theorem 7.3:

**Proof of Theorem 7.3:** Starting with $\mathcal{P}_{\text{full}}$, introduce the new behavior $\mathcal{P}'_{\text{full}}$ as above. We have $(\mathcal{P}_{\text{full}})_{(w,v)} = (\mathcal{P}'_{\text{full}})_{(w,v)}$. Thus, if $(\mathcal{P}_{\text{full}})_{(w,v)}$ is strictly $-\Sigma_\gamma^{-1}$-dissipative and has a negative definite storage function, then the same holds for $(\mathcal{P}'_{\text{full}})_{(w,v)}$. By Proposition 7.2 there exists a disturbance-free, stabilizing strictly $\frac{1}{\ell}$-contracting controller for $\mathcal{P}_{\text{full}}$. Then there also exists such controller for the original $\mathcal{P}_{\text{full}}$. Finally, we should prove that also a regular controller for $\mathcal{P}_{\text{full}}$ exists with these properties. Again, note that $(\mathcal{P}_{\text{full}} \land_c \mathcal{C})_{(w,v)} = \mathcal{K}_2$ (see Lemma 7.6). Now, $\mathcal{K}_2$ is obviously implementable with respect to $\mathcal{P}_{\text{full}}$. Since $\mathcal{P}_{\text{full}}$ is assumed to be controllable, Proposition 4.2 then asserts that $\mathcal{K}_2$ is also regularly implementable. Any regular controller that implements $\mathcal{K}_2$ is then of course a disturbance-free, stabilizing strictly $\frac{1}{\ell}$-contracting controller for $\mathcal{P}_{\text{full}}$.

The converse implication follows immediately from Proposition 7.1 above. \[\square\]

**Remark 7.9** Without going into details, in this remark we will outline how to actually compute a disturbance-free, stabilizing strictly $\frac{1}{\ell}$-contracting, regular controller for $\mathcal{P}_{\text{full}}$ from the polynomial matrices $W$, $V$ and $C$ appearing in its image representation (17) (see also [17]). In the following, let $\ell$ denote the number of columns of $W$ (i.e. the dimension of the latent variable $\ell$). Let $\Sigma_\gamma$ be given by (15). Denote $R^\sim(\xi) := R^\top(-\xi)$.

1. Factorize: $-(\begin{bmatrix} W \\ V \end{bmatrix})^\sim \Sigma_\gamma (\begin{bmatrix} W \\ V \end{bmatrix}) = (\begin{bmatrix} F_+ \\ F_- \end{bmatrix})^\sim (\begin{bmatrix} I_{1-\ell} & 0 \\ 0 & -I_{\ell} \end{bmatrix}) (\begin{bmatrix} F_+ \\ F_- \end{bmatrix})$ such that
   
   (a) $(\begin{bmatrix} F_+ \\ F_- \end{bmatrix})$ is a Hurwitz polynomial matrix,
   
   (b) $(\begin{bmatrix} W \\ V \end{bmatrix})(\begin{bmatrix} F_+ \\ F_- \end{bmatrix})^{-1}$ is proper,
   
   (c) $(\begin{bmatrix} V \\ F_+ \end{bmatrix})$ is Hurwitz.

2. Factorize: $C = C'L$ with $C'$ and $L$ polynomial matrices such that $C'(\lambda)$ has full column rank for all $\lambda \in \mathbb{C}$, and $L$ Hurwitz.

3. Factorize: $F_+L^{-1} = P_1^{-1}Q_1$ with $P_1, Q_1$ polynomial matrices, $P_1$ Hurwitz.

Define then a controller $\mathcal{C}$ for $\mathcal{P}_{\text{full}}$ by:

$$\mathcal{C} := \{c \mid \exists \ell \in C^\infty(\mathbb{R}, \mathbb{R}^1) \text{ such that } c = C(\frac{d}{dt})\ell, P_1(\frac{d}{dt})F_+(\frac{d}{dt})\ell = 0\},$$

(24)

The controller $\mathcal{C}$ is then disturbance-free, stabilizing and strictly $\frac{1}{\ell}$-contracting. It can be shown that if $c$ is free in $\mathcal{P}_{\text{full}}$ then the above controller $\mathcal{C}$ is also regular. If $c$ is not free in $\mathcal{P}_{\text{full}}$ then, starting with $\mathcal{C}$ given by (24), a regular, disturbance-free, stabilizing and strictly $\frac{1}{\ell}$-contracting controller can be constructed using ideas from [1]. Since in the context of robust stabilization $c$ will turn out to be free, we do not explicitly give this construction here.
8 A solution to the robust stabilization problem

In this section we study and resolve Problem 1 and Problem 2 as introduced in section 3.

8.1 Solution to Problem 1

Let \( P \in L^w \) be controllable, and let it be represented in rational kernel representation by \( R(\frac{d}{dt})w = 0 \), where \( R \) is proper, stable, real rational, left prime and co-inner (see Lemma 2.3). Clearly, \( R \) has full row rank, and its number of rows is equal to \( p := p(P) \). Recall (see (3)) that for given \( \gamma > 0 \) we have defined the ball \( B(P, \gamma) \) with radius \( \gamma \) around \( P \) as:

\[
B(P, \gamma) := \{ P_{\Delta} \in L^w_{\text{cont}} \mid \text{there exists a proper, stable, real rational } \quad R_{\Delta} \quad \text{of full row rank such that } \quad P_{\Delta} = \ker(R_{\Delta}) \\
\text{and } \quad \| R - R_{\Delta} \|_{\infty} \leq \gamma \}.
\]

Define the auxiliary system \( P_{\text{aux}} \in L^{w+v+w} \) as

\[
P_{\text{aux}} := \{ (w, v, c) \mid R(\frac{d}{dt})w + v = 0, \ c = w \}.
\]

Let \( R(\xi) = P^{-1}(\xi)Q(\xi) \) be a left coprime factorization over \( \mathbb{R}[\xi] \), with \( P \) Hurwitz. Then by definition

\[
P_{\text{aux}} = \{ (w, v, c) \mid Q(\frac{d}{dt})w + P(\frac{d}{dt})v = 0, \ c = w \}.
\]

(26)

It is important to note that the projection \( (P_{\text{aux}})_{(w,v)} \) of \( P_{\text{aux}} \) onto the variable \( (w,v) \) is represented by

\[
(P_{\text{aux}})_{(w,v)} = \{ (w, v) \mid Q(\frac{d}{dt})w + P(\frac{d}{dt})v = 0 \},
\]

and that \( (P_{\text{aux}})_{(w,v)} \) is controllable (see Proposition 5.6, item 1.). Thus, the orthogonal complement of this projection, \( (P_{\text{aux}})^\perp_{(w,v)} \), is well-defined, and given in polynomial image representation by

\[
\left( \begin{array}{c}
\tilde{w} \\
\tilde{v}
\end{array} \right) = \left( \begin{array}{c}
Q^T(\frac{d}{dt}) \\
P^T(\frac{d}{dt})
\end{array} \right) \ell,
\]

(28)

(see the last paragraph of section 5). Note that this image representation is observable. The following theorem provides a solution to Problem 1:

**Theorem 8.1** Let \( \gamma > 0 \). There exists a controller \( \mathcal{C} \in L^v \) such that \( P_{\Delta} \cap \mathcal{C} \) is stable and \( P_{\Delta} \cap \mathcal{C} \) is a regular interconnection for all \( P_{\Delta} \in B(P, \gamma) \) if and only if \( (P_{\text{aux}})^\perp_{(w,v)} \) is strictly \( -\Sigma_\gamma^{-1} \)-dissipative and has a negative definite storage function.

This theorem follows immediately from the equivalence of items (1.) and (4.) in the following lemma, Lemma 8.2, after applying Theorem 7.3 to the system \( P_{\text{aux}} \). Indeed, Theorem 7.3 applies to \( P_{\text{aux}} \) since the following conditions are satisfied:

1. \( v \) is free in \( P_{\text{aux}} \) since \( R \) is a full row rank polynomial matrix (see Prop. 5.1),
2. \( c \) is observable from \( (w, v) \) in \( P_{\text{aux}} \) since, trivially, \( (w, v) = 0 \) implies \( c = 0 \),
3. \( (w, v) \) is detectable from \( c \) in \( P_{\text{aux}} \): if \( c = 0 \) then \( w = 0 \) and \( \lim_{t \to \infty} v(t) = 0 \) (use the fact that in (26) \( P \) is Hurwitz).
The lemma formulates a behavioral version of the ‘small gain theorem’:

**Lemma 8.2** Let $P_{aux}$ be the auxiliary system represented by (25). Let $C \in L^w$ be represented in minimal rational kernel representation by $C(\frac{d}{dt})c = 0$. Let $\gamma > 0$. Then the following statements are equivalent:

1. $C$ regularly stabilizes $P_{\Delta}$ for all $P_{\Delta} \in B(P, \gamma)$, i.e., $P_{\Delta} \cap C$ is stable and $P_{\Delta} \cap C$ is a regular interconnection for all $P_{\Delta} \in B(P, \gamma)$.

2. $\left( \begin{array}{c} R_{\Delta} \\ C \end{array} \right)$ is square, nonsingular and has no zeros in $\bar{C}^+$ for all stable, proper real rational $R_{\Delta}$ of full row rank such that $\|R - R_{\Delta}\|_\infty \leq \gamma$.

3. $\left( \begin{array}{c} R \\ C \end{array} \right)^{-1} \left( \begin{array}{c} I \\ 0 \end{array} \right)$ is proper and satisfies $\|G\|_\infty < \frac{1}{\gamma}$.

4. $C$ is a disturbance-free, stabilizing, regular and strictly $\frac{1}{\gamma}$-contracting controller for $P_{aux}$.

In the remainder of this section we will establish a proof of Lemma 8.2

**Proof:** (1.) $\iff$ (2.) Since $P_{\Delta}$ is represented minimally by $R_{\Delta}(\frac{d}{dt})w = 0$ and $C$ by $C(\frac{d}{dt})w = 0$, the equivalence between statements (1.) and (2.) immediately follows from Proposition 5.7.

(2.) $\iff$ (3.) Our proof of the equivalence of statements (2.) and (3.) hinges on the following lemma, that has appeared in the literature in various forms:

**Lemma 8.3** Let $N$ be a real rational matrix, and let $E$ and $F$ be stable, proper real rational matrices. Then the following statements are equivalent:

1. $N + E\Delta F$ is square, nonsingular and has no zeros in $\bar{C}^+$ for all stable, proper real rational $\Delta$ such that $\|\Delta\|_\infty \leq \gamma$.

2. $N$ is square, nonsingular and has no zeros in $\bar{C}^+$, $FN^{-1}E$ is proper, and $\|FN^{-1}E\|_\infty < \frac{1}{\gamma}$.

**Proof:** See the Appendix. □

Denote $\Delta := R_{\Delta} - R$. Then $\left( \begin{array}{c} R_{\Delta} \\ C \end{array} \right) = \left( \begin{array}{c} R \\ C \end{array} \right) + \left( \begin{array}{c} I \\ 0 \end{array} \right) \Delta$. Obviously, $\left( \begin{array}{c} R_{\Delta} \\ C \end{array} \right)$ is square, nonsingular and has no zeros in $\bar{C}^+$ for all $R_{\Delta}$ such that $\|R - R_{\Delta}\|_\infty \leq \gamma$ if and only if $\left( \begin{array}{c} R \\ C \end{array} \right) + \left( \begin{array}{c} I \\ 0 \end{array} \right) \Delta$ is square, nonsingular and has no zeros in $\bar{C}^+$ for all $\Delta$ such that $\|\Delta\|_\infty \leq \gamma$. The equivalence of (2.) and (3.) then follows from Lemma 8.3 by taking $E = \left( \begin{array}{c} I \\ 0 \end{array} \right)$ and $F = I$.

(3.) $\iff$ (4.) Finally we will prove the equivalence of (3.) and (4.) By Proposition 5.8, $C$ is a disturbance-free, stabilizing, regular controller for $P_{aux}$ if and only if $\left( \begin{array}{ccc} R & 0 \\ I & -I \end{array} \right)$ is
square, nonsingular and has no zeros in $\tilde{C}^+$, which in turn is equivalent with: $\begin{pmatrix} R \\ C \end{pmatrix}$ is square, nonsingular and has no zeros in $\tilde{C}^+$.

We also have that $\begin{pmatrix} R(d) \\ C(d) \end{pmatrix} w + \begin{pmatrix} I \\ 0 \end{pmatrix} v = 0$ is a minimal rational kernel representation of $(\mathcal{P}_{aux} \wedge \mathfrak{C})(w,v)$, which then, by Proposition 5.9 is strictly $\gamma$-contractive if and only if $\| \begin{pmatrix} R \\ C \end{pmatrix}^{-1} \begin{pmatrix} I \\ 0 \end{pmatrix} \|_\infty < \gamma$. This completes the proof of Lemma 8.2. □

**Remark 8.4** Given the nominal plant $\mathcal{P} \in \mathcal{L}_{cont}$ with minimal kernel representation $R(d)w = 0$, with $R$ proper, stable, real rational, left prime and co-inner, and given $\gamma > 0$, a robustly stabilizing controller is computed as follows:

1. Factorize: let $R = -VW^{-1}$ be a right coprime factorization over $\mathbb{R}[\xi]$. Then $W$ is Hurwitz, and it can be shown that an observable image representation of $\mathcal{P}_{aux}$ is given by $\begin{pmatrix} w \\ v \\ c \end{pmatrix} = \begin{pmatrix} W(d) \\ -V(d) \\ W(d) \end{pmatrix} \ell$. Note that $c$ is free in $\mathcal{P}_{aux}$ (use the fact that $W$ has full row rank).

By carefully following the steps in Remark 7.9, we see that the next steps are:

2. Factorize $F_+W^{-1} = P_1^{-1}Q_1$, with $P_1$ Hurwitz.

3. Define $\mathfrak{C} := \ker(Q_1)$.

Then $\mathfrak{C}$ is a controller that regularly stabilizes all $\mathcal{P}_\Delta$ in the ball $B(\mathcal{P}, \gamma)$. Indeed, from Remark 7.9, a required controller is given by

$$\mathfrak{C} := \{ c \mid \exists \ell \in \mathcal{C}\infty(\mathbb{R}, \mathbb{R}^1) \text{ such that } c = W(d)\ell, P_1(d)F_+(d)\ell = 0 \},$$

but, by noting that $P_1F_+ = Q_1W$, we see that, in fact, $\mathfrak{C} = \ker(Q_1)$.

### 8.2 Solution to Problem 2

Again, consider a nominal plant $\mathcal{P} \in \mathcal{L}_{cont}$. Let $R(d)w = 0$ be a minimal rational kernel representation of $\mathcal{P}$, with $R$ proper, stable, left prime and co-inner. Let $R = P^{-1}Q$ be a left coprime factorization over $\mathbb{R}[\xi]$. Then $\tilde{w} = Q^T(-d)\ell$ is an observable polynomial image representation of the orthogonal behavior $\mathcal{P}_{\perp}$ (note that by controllability of $\mathcal{P}$, $Q(\lambda)$ must have full row rank for all $\lambda$). Consider $\mathcal{P}_{\perp}$, together with the supply rate $\|\tilde{w}\|^2$. Clearly, by the form of this supply rate, $\mathcal{P}_{\perp}$ is strictly dissipative. We denote by $Q_{\Psi_-}(\ell)$ and $Q_{\Psi_+}(\ell)$ its smallest and largest storage function, respectively. We have $Q_{\Psi_-} \leq 0$ and $Q_{\Psi_+} \geq 0$ (see [20]). We compute the underlying two-variable polynomials $\Psi_-$ and $\Psi_+$ as follows. Since the rational matrix $R$ is co-inner, $R(\xi)R^T(-\xi) = I$, we have

$$Q(\xi)Q^T(-\xi) = P(\xi)P^T(-\xi). \quad (29)$$

Note that $P^T(-\xi)$ is anti-Hurwitz. Thus, (29) displays an anti-Hurwitz polynomial spectral factorization of $Q(\xi)Q^T(-\xi)$. Consequently, by Proposition 6.1,

$$\Psi_+(\zeta, \eta) = \frac{Q(-\zeta)Q^T(-\eta) - P(-\zeta)P^T(-\eta)}{\zeta + \eta} \quad (30)$$
yields the largest storage function of $\mathcal{P}^\perp$ with respect to the supply rate $\|\dot{w}\|^2$. Next, we compute $\Psi_-(\zeta, \eta)$. Let

$$Q(\xi)Q^\top(-\xi) = H^\top(-\xi)H(\xi),$$

be a Hurwitz polynomial spectral factorization. Then we have

$$\Psi_-(\zeta, \eta) = \frac{Q(-\zeta)Q^\top(-\eta) - H(\zeta)H^\top(\eta)}{\zeta + \eta},$$

for the smallest storage function of $\mathcal{P}^\perp$ with respect to the supply rate $\|\dot{w}\|^2$.

We will now formulate the main theorem of this section, which yields a solution to Problem 2, the problem of optimal robust stabilization. Recall (4):

$$\gamma^* = \sup\{\gamma > 0 \mid \exists \mathcal{C} \in \mathcal{L}^w \text{ that regularly stabilizes all } \mathcal{P}_\Delta \in B(\mathcal{P}, \gamma)\}. \quad (33)$$

In fact, our theorem gives the optimum $\gamma^*$ in terms of the coefficient matrices of $\Psi_-$ and $\Psi_+$:

**Theorem 8.5** Let $\tilde{\Psi}_-$ and $\tilde{\Psi}_+$ be the coefficient matrices of $\Psi_-$ and $\Psi_+$ given by (32) and (30), respectively. Let $\tilde{\Psi}_+^\dagger$ be the Moore-Penrose inverse of $\tilde{\Psi}_+$. Then we have $\lambda_{\text{max}}(\tilde{\Psi}_-\tilde{\Psi}_+^\dagger) < 0$ and

$$\gamma^* = \frac{\sqrt{|\lambda_{\text{max}}(\tilde{\Psi}_-\tilde{\Psi}_+^\dagger)|}}{1 + |\lambda_{\text{max}}(\tilde{\Psi}_-\tilde{\Psi}_+^\dagger)|}. \quad (34)$$

In particular, for $\gamma > 0$ the following holds: there exists $\mathcal{C} \in \mathcal{L}^w$ such that $\mathcal{P}_\Delta \cap \mathcal{C}$ is stable, and $\mathcal{P}_\Delta \cap \mathcal{C}$ is a regular interconnection for all $P_\Delta \in B(\mathcal{P}, \gamma)$ if and only if $\gamma < \gamma^*$.

In the remainder of this section we will establish a proof of Theorem 8.5.

**Proof:** Consider $(\mathcal{P}_{\text{aux}})^\perp_{(w,v)}$, which, as we already know, has an observable polynomial image representation given by (28), together with the supply rate $\|\dot{w}\|^2 - \gamma^2\|\dot{v}\|^2$, where $\gamma > 0$. Recall that this supply rate is associated with the matrix $-\Sigma^{-1}_\gamma$ given by (16). We now investigate strict $-\Sigma^{-1}_\gamma$-dissipativity of $(\mathcal{P}_{\text{aux}})^\perp_{(w,v)}$.

It turns out that the smallest storage function of $(\mathcal{P}_{\text{aux}})^\perp_{(w,v)}$ as a $-\Sigma^{-1}_\gamma$-dissipative system can be expressed in terms of the smallest and largest storage functions $\Psi_-$ and $\Psi_+$ of $\mathcal{P}^\perp$ with respect to the supply rate $\|\dot{w}\|^2$:

**Lemma 8.6** Let $\gamma > 0$. Then $(\mathcal{P}_{\text{aux}})^\perp_{(w,v)}$ is strictly $-\Sigma^{-1}_\gamma$-dissipative if and only if $0 < \gamma < 1$. The smallest storage function of $(\mathcal{P}_{\text{aux}})^\perp_{(w,v)}$ as a $-\Sigma^{-1}_\gamma$-dissipative system is given by the two-variable polynomial matrix

$$\Psi_\gamma = (1 - \gamma^2)\Psi_- + \gamma^2\Psi_+,$$

where $\Psi_-$ and $\Psi_+$ are given by (32) and (30), respectively.

**Proof:** Let $\gamma \in (0, 1)$ and $\delta := 1 - \gamma^2$. Since $Q(i\omega)Q^\top(-i\omega) = P(i\omega)P^\top(-i\omega)$, we have $Q(i\omega)Q^\top(-i\omega) - \gamma^2P(i\omega)P^\top(-i\omega) = \delta P(i\omega)P^\top(-i\omega) = \epsilon (P(i\omega)P^\top(-i\omega) + Q(i\omega)Q^\top(-i\omega))$ for all $\omega \in \mathbb{R}$, with $\epsilon := \delta/2 > 0$. This shows strict $-\Sigma^{-1}_\gamma$-dissipativeness of $(\mathcal{P}_{\text{aux}})^\perp_{(w,v)}$ (see section 6, condition (12)). In a similar way, it follows from strict $-\Sigma^{-1}_\gamma$-dissipativeness of $(\mathcal{P}_{\text{aux}})^\perp_{(w,v)}$ that $\gamma < 1$. 

22
For all $\gamma \in (0, 1)$ it follows from (29) and (31) that
\[
Q(\xi)Q^T(-\xi) - \gamma^2 P(\xi)P^T(-\xi) = (1 - \gamma^2)H^T(-\xi)H(\xi).
\] (36)

Define $H'(\xi) := \sqrt{1 - \gamma^2}H(\xi)$. Then (36) displays a Hurwitz polynomial spectral factorization, with Hurwitz spectral factor $H'(\xi)$. The smallest storage function $\Psi^-_\gamma$ of $(P_{aux})_{(w,v)}^\perp$ must therefore be given by
\[
\Psi^-_\gamma = \frac{Q(-\xi)Q^T(-\eta) - \gamma^2 P(-\xi)P^T(-\eta) - (1 - \gamma^2)H^T(\xi)H(\eta)}{\zeta + \eta}
\]
\[
= (1 - \gamma^2) \frac{Q(-\xi)Q^T(-\eta) - H^T(\xi)H(\eta)}{\zeta + \eta}
\]
\[
+ \gamma^2 \frac{Q(-\xi)Q^T(-\eta) - P(-\xi)P^T(-\eta)}{\zeta + \eta}
\]
\[
= (1 - \gamma^2)\Psi_-(\xi, \eta) + \gamma^2\Psi_+(\xi, \eta).
\]

This completes the proof of the lemma. □

Now, according to Theorem 8.1, there exists $C \in \mathcal{C}$ such that $P_{\Delta} \cap C$ is stable for all $P_{\Delta} \in B(P, \gamma)$ if and only if $(P_{aux})_{(w,v)}^\perp$ is strictly $-\Sigma^{-1}_\gamma$-dissipative and its smallest storage function $\Psi^-_\gamma$ is negative definite. Using Lemma 8.6, we will now establish necessary and sufficient conditions for the smallest storage function $\Psi^-_\gamma$ of $(P_{aux})_{(w,v)}^\perp$ to be negative definite.

Let $X(\xi)$ be a polynomial matrix such that, $X(\frac{d}{dt})$ is a minimal state map for $P_{\perp}$ and such that its coefficient matrix $\dot{X}$ satisfies $\dot{X}X^T = I$ (see Lemma 6.2). It is easily seen that $X(\frac{d}{dt})$ is also a minimal state map for $(P_{aux})_{(w,v)}^\perp$. Let $n$ be the number of rows of $X$. Now, there exist real symmetric $n \times n$ matrices $K_-$ and $K_+$ such that $\Psi_-(\xi, \eta) = X^T(\xi)K_-X(\eta)$ and $\Psi_+(\xi, \eta) = X^T(\xi)K_+X(\eta)$. Since, by inspection, $P_{\perp}$ is strictly dissipative both on $\mathbb{R}_-$ and on $\mathbb{R}_+$ with respect to the supply rate $\|\tilde{w}\|_2^2$, it follows from Lemma 6 of [3], that $K_- < 0$ and $K_+ > 0$. As a consequence, the smallest storage function $\Psi^-_\gamma(\xi, \eta)$ of $(P_{aux})_{(w,v)}^\perp$ is induced by the two-variable polynomial matrix
\[
\Psi^-_\gamma(\xi, \eta) = X^T(\xi) \left( (1 - \gamma^2)K_- + \gamma^2K_+ \right) X(\eta).
\]

Therefore, $\Psi^-_\gamma$ yields a negative definite storage function for $(P_{aux})_{(w,v)}^\perp$ if and only if $(1 - \gamma^2)K_- + \gamma^2K_+ < 0$. The latter can be expressed equivalently in terms of the largest eigenvalue of $K_-K_+^{-1}$:

**Lemma 8.7** Let $0 < \gamma < 1$. Then we have: $(1 - \gamma^2)K_- + \gamma^2K_+ < 0$ if and only if $\gamma^2 < \frac{\lambda_{\text{max}}(K_-K_+^{-1})}{1 + \lambda_{\text{max}}(K_-K_+^{-1})}$.

**Proof:** First note that $K_- < 0 < K_+$. Thus all eigenvalues of $K_-K_+^{-1}$ are real and $\lambda_{\text{max}}(K_-K_+^{-1}) < 0$. The following equivalent statements hold:
\[
(1 - \gamma^2)K_- + \gamma^2K_+ < 0 \iff (1 - \gamma^2)\lambda_{\text{max}}(K_-K_+^{-1}) < -\gamma^2 \iff \lambda_{\text{max}}(K_-K_+^{-1}) < -[1 - \lambda_{\text{max}}(K_-K_+^{-1})]\gamma^2 \iff \gamma^2 < \frac{\lambda_{\text{max}}(K_-K_+^{-1})}{1 + \lambda_{\text{max}}(K_-K_+^{-1})}.
\]
\[
\gamma^2 < \frac{\lambda_{\text{max}}(K_- K_{+1}^{-1})}{\lambda_{\text{max}}(K_- K_{+1}^{-1}) - 1} \quad \iff \quad \\
\gamma^2 < \frac{|\lambda_{\text{max}}(K_- K_{+1}^{-1})|}{1 + |\lambda_{\text{max}}(K_- K_{+1}^{-1})|},
\]

Finally, we will show that the nonzero eigenvalues of \( K_- K_{+1}^{-1} \) and \( \tilde{\Psi}_- \tilde{\Psi}_+^\dagger \) coincide. Again let \( \tilde{X} \) be the coefficient matrix of the polynomial matrix \( X(\xi) \) such that \( \tilde{X} \tilde{X}^\top = I \). Choose \( \tilde{Y} \) such that \( \begin{pmatrix} \tilde{X} \\ \tilde{Y} \end{pmatrix} \) is orthogonal (such a \( \tilde{Y} \) exists since \( \tilde{X} \tilde{X}^\top = I \)). Let \( \tilde{\Psi}_- \) and \( \tilde{\Psi}_+ \) be the coefficient matrices of \( \Psi_- \) and \( \Psi_+ \) respectively. Then we have \( \tilde{\Psi}_- = \tilde{X}^\top K_- \tilde{X} = \begin{pmatrix} \tilde{X} \\ \tilde{Y} \end{pmatrix}^\top \begin{pmatrix} K_- & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \tilde{X} \\ \tilde{Y} \end{pmatrix} \), and \( \tilde{\Psi}_+ = \tilde{X}^\top K_+ \tilde{X} = \begin{pmatrix} \tilde{X} \\ \tilde{Y} \end{pmatrix}^\top \begin{pmatrix} K_+ & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \tilde{X} \\ \tilde{Y} \end{pmatrix} \). It can be easily verified that \( \tilde{\Psi}_+^\top = (\begin{pmatrix} \tilde{X} \\ \tilde{Y} \end{pmatrix}^\top \begin{pmatrix} K_+^{-1} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \tilde{X} \\ \tilde{Y} \end{pmatrix} \) is the Moore-Penrose inverse of \( \tilde{\Psi}_+ \). Also, we can compute \( \tilde{\Psi}_- \tilde{\Psi}_+^\dagger = \begin{pmatrix} \tilde{X} \\ \tilde{Y} \end{pmatrix}^\top \begin{pmatrix} K_- K_+^{-1} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \tilde{X} \\ \tilde{Y} \end{pmatrix} \). Thus, the nonzero eigenvalues of \( K_- K_{+1}^{-1} \) and \( \tilde{\Psi}_- \tilde{\Psi}_+^\dagger \) coincide. In particular this implies

\[
\lambda_{\text{max}}(\tilde{\Psi}_- \tilde{\Psi}_+^\dagger) = \lambda_{\text{max}}(K_- K_{+1}^{-1}). \tag{37}
\]

Thus, \( \Psi_-^\gamma \) yields a negative definite storage function for \((P_{\text{aux}})^\perp_{(w,v)}\) if and only if

\[
\gamma < \sqrt{\frac{|\lambda_{\text{max}}(\tilde{\Psi}_- \tilde{\Psi}_+^\dagger)|}{1 + |\lambda_{\text{max}}(\tilde{\Psi}_- \tilde{\Psi}_+^\dagger)|}}. \tag{38}
\]

This completes the proof of Theorem 8.5.

\[\square\]

9 Example

In order to illustrate the result of Theorem 8.5, we now present a simple worked-out example.

Example 9.1 Let \( P \in \mathcal{L}^2 \), the nominal plant, be given by \( P = \{ w \mid R(\frac{d}{dt})w = 0 \} \), where \( R(\xi) = \begin{pmatrix} 1 & \xi \\ \xi + 1 & \xi + 1 \end{pmatrix} \). A left coprime factorization of \( R(\xi) \) is given by \( R(\xi) = P^{-1}(\xi)Q(\xi) \), where

\[
P(\xi) = \xi + 1 \tag{39}
\]

and

\[
Q(\xi) = \begin{pmatrix} 1 & \xi \end{pmatrix}. \tag{40}
\]

Then the system \( P^\perp \) is given by the rational image representation \( \tilde{w} = R^\top(-\frac{d}{dt})\ell \) and the polynomial image representation \( \tilde{w} = Q^\top(-\frac{d}{dt})\ell \). As argued in subsection 8.2,
$P^\perp$ is strictly dissipative with respect to supply rate $\|\tilde{w}\|^2$. We have, $Q(\xi)Q^T(-\xi) = P(\xi)P^T(-\xi)$, and, $Q(\xi)Q^T(-\xi) = H^T(-\xi)H(\xi)$ as anti-Hurwitz and Hurwitz polynomial spectral factorization respectively, where $H(\xi) = \xi + 1$. The largest and smallest storage functions of $P^\perp$, as $\|\tilde{w}\|^2$ dissipative system, are obtained by using (30), (32), (39) and (40), as follows:

$$\Psi_+(\zeta, \eta) = \frac{Q(-\zeta)Q^T(-\eta) - P(-\zeta)P^T(-\eta)}{\zeta + \eta} = 1,$$

$$\Psi_-(\zeta, \eta) = \frac{Q(-\zeta)Q^T(-\eta) - H(\zeta)H^T(\eta)}{\zeta + \eta} = -1.$$  

From this we get $\tilde{\Psi}_+ = 1$, $\tilde{\Psi}_- = -1$, and, $\tilde{\Psi}_+^\dagger = 1$. Thus, by Theorem 8.5, there exists $C \in L^2$ such that $P \Delta \cap C$ is stable, and $P \Delta \cap C$ is a regular interconnection for all $P \Delta \in B(P, \gamma)$ if and only if $\gamma < \gamma^*$, where $\gamma^* = \sqrt{\frac{1}{1 + |\lambda_{\max}(\Psi_+\Psi_-^\dagger)|}} = \sqrt{\frac{1}{2}}$.

10 Conclusions

In this paper we have formulated and resolved the problems of robust stabilization and of finding the optimal stability radius in the context of behavioral control. In this context, controllers act on the plant using general interconnection, without a priori input-output considerations. We have restricted ourselves to the full interconnection case, where all variables can be used for interconnection. We have shown that a controller robustly stabilizes a given nominal plant if and only if it solves a particular $H_\infty$ synthesis problem. This generalizes the well-known input-output small gain argument to the behavioral context. We have shown that a robustly stabilizing controller exists if and only if a given orthogonal behavior associated with the nominal plant is strictly dissipative and has a negative definite storage function. Finally, we have expressed the optimal stability radius in terms of eigenvalues associated with the extremal storage functions of the orthogonal complement of the nominal plant, and have shown that this optimal radius can be computed using polynomial spectral factorization.

A very urgent problem for future research is to interpret the concept of ball around a nominal plant introduced in this paper in terms of the gap between the nominal plant and the behaviors contained in the ball. Future research will also involve the extension of the theory presented to the case that the nominal plant behavior is given in rational image representation, or in output nulling of driving variable representation. A very challenging problem is also the extension of the results in this paper to the case of partial interconnection, where only part of the nominal plant system variable can be used for interconnection, thus establishing an extension to the robust stabilization context of the results in [2].

11 Appendix: Proofs

In this section we provide proofs of some of the lemmas in this paper.

Proof of Lemma 5.9: ($\Rightarrow$) Let $(R_1, R_2) = P^{-1}(Q_1, Q_2)$ be a left coprime factorization over $\mathbb{R}[\xi]$. Then $Q_1(\frac{d}{dt})w_1 + Q_2(\frac{d}{dt})w_2 = 0$ is a minimal polynomial kernel representation,
and $Q_1$ is square, nonsingular. Clearly, $G := R_1^{-1}R_2$ is equal to $Q_1^{-1}Q_2$. Let $-ND^{-1}$ be a right coprime factorization of $G$ over $\mathbb{R}[\xi]$. Since $Q_1N + Q_2D = 0$, we have $\text{col}(w_1, w_2) = \text{col}(N(\frac{d}{dt})\ell, D(\frac{d}{dt})\ell) \in \mathcal{B} \cap \mathcal{D}$ for all $\ell \in \mathcal{D}$. Thus, by assumption, there exists $\epsilon > 0$ such that $\|N(\frac{d}{dt})\ell\|_2 \leq (\gamma - \epsilon)\|D(\frac{d}{dt})\ell\|_2$ for all $\ell \in \mathcal{D}$. Taking Fourier transforms it follows from Parseval’s theorem that $N^\top(-i\omega)N(i\omega) \leq (\gamma - \epsilon)D^\top(-i\omega)D(i\omega)$ for all $\omega \in \mathbb{R}$. Using that $ND^{-1}$ is a right coprime factorization, this implies that $D(i\omega)$ is nonsingular for all $\omega \in \mathbb{R}$. Thus $G$ has no poles on the imaginary axis and $G^\top(-i\omega)G(i\omega) \leq (\gamma - \epsilon)I$ for all $\omega$. This implies that $G$ is proper and $\|G\|_\infty < \gamma$.

(\Leftarrow) Conversely, in $\mathcal{B}$ $w_1$ is output and $w_2$ is input, and the transfer matrix from $w_2$ to $w_1$ is equal to $G = R_1^{-1}R_2$. Since $G$ is proper and has no poles on the imaginary axis, the system $\mathcal{B}$ induces a bounded operator that maps $w_2 \in \mathcal{L}_2$ to $w_1 \in \mathcal{L}_2$. The norm of this operator is equal to $\|G\|_\infty < \gamma$, and therefore there exists $\epsilon > 0$ such that $\|w_1\|_2 \leq (\gamma - \epsilon)\|w_2\|_2$ for all $(w_1, w_2) \in \mathcal{B} \cap \mathcal{L}_2$.

\[ \text{Proof of Lemma } 8.3: (1.) \Rightarrow (2.) \text{ The claims about } N \text{ are immediate (consider the case } \Delta = 0). \text{ Note that } FN^{-1}E \text{ is stable. To show } \|FN^{-1}E\|_\infty < \frac{1}{\gamma}, \text{ on the contrary assume that there exists a } \lambda \text{ such that } \text{Re}(\lambda) \geq 0, \text{ and a complex vector } v, \|v\| = 1 \text{ such that} \]

\[ \|F(\lambda)N^{-1}(\lambda)E(\lambda)v\| = \tilde{\gamma} > \frac{1}{\gamma}. \] \hfill (41)

Define $w := F(\lambda)N^{-1}(\lambda)E(\lambda)v$. Then (41) implies $w^*w - \tilde{\gamma}^2 = 0$ which, in turn, implies $\det(I - w\frac{1}{\tilde{\gamma}}w^*) = 0$. Define a constant complex matrix $W := -\frac{1}{\tilde{\gamma}}vw^*$. Then we have $\det(I + F(\lambda)N^{-1}(\lambda)E(\lambda)W) = 0$. Thus we obtain

\[ \det(I + E(\lambda)WF(\lambda)N^{-1}(\lambda)) = \det(I + F(\lambda)N^{-1}(\lambda)E(\lambda)W) = 0, \]

and therefore $I + E(\lambda)WF(\lambda)N^{-1}(\lambda)$ is singular. Postmultiplying with $N(\lambda)$ results in $N(\lambda) + E(\lambda)WF(\lambda)$ being singular. Using an idea similar as in [24], Section 8.2, we can now construct a stable, proper real rational matrix $\Delta$ such that $\|\Delta\|_\infty > \gamma$, and $\Delta(\lambda) = W$. We omit the details here. For this $\Delta$, $N + E\Delta F$ has a zero in $\mathbb{C}^+$, which is a contradiction.

(2.) $\Rightarrow$ (1.) On the contrary, assume that there exists a stable, proper $\Delta$ such that $\|\Delta\|_\infty \leq \gamma$, and $\lambda$ such that $\text{Re}(\lambda) \geq 0$ such that

\[ \det(N(\lambda) + E(\lambda)\Delta(\lambda)F(\lambda)) = 0. \] \hfill (42)

Since $N(\lambda)$ is nonsingular, equation (42) implies that

\[ \det(I + F(\lambda)N^{-1}(\lambda)E(\lambda)\Delta(\lambda)) = \det(I + E(\lambda)\Delta(\lambda)F(\lambda)N^{-1}(\lambda)) = 0. \]

Therefore, there exists a complex vector $w \neq 0$, such that

\[ (I + F(\lambda)N^{-1}(\lambda)E(\lambda)\Delta(\lambda))w = 0. \]

Define $v := \Delta(\lambda)w$. Then we have $w = -F(\lambda)N^{-1}(\lambda)E(\lambda)v$. As $\|FN^{-1}E\|_\infty < \frac{1}{\gamma}$, we have $\|w\| < \frac{1}{\tilde{\gamma}}\|v\|$. This contradicts the assumption $\|\Delta\|_\infty \leq \gamma$ which requires $\|w\| \geq \frac{1}{\tilde{\gamma}}\|v\|$.

\[ \Box \]

\[ \text{Proof of Lemma } 2.3: \text{ To prove this lemma, we need to prove some preliminary results on rational representations of behaviors. First, from } [23], \text{ Section 7, we recall the concepts of polynomial and rational annihilators of a given behavior. Here, we introduce proper stable rational annihilators:} \]
Definition 11.1 Let $\mathcal{B} \in \mathcal{L}^u$.

1. $n \in \mathbb{R}^{1 \times \lambda}(\xi)$ is called a polynomial annihilator of $\mathcal{B}$ if $n(\frac{d}{dt})w = 0$ for all $w \in \mathcal{B}$.

2. $n \in \mathbb{R}^{1 \times \lambda}(\xi)$ is called a proper stable rational annihilator of $\mathcal{B}$ if $n(\frac{d}{dt})w = 0$ for all $w \in \mathcal{B}$.

We denote the sets of polynomial and proper stable rational annihilators of $\mathcal{B} \in \mathcal{L}^u$ by $\mathcal{B}^{\perp R[\xi]}$ and $\mathcal{B}^{\perp R_S(\xi)}$ respectively. It is a well-known result that for $\mathcal{B} \in \mathcal{L}^u$, $\mathcal{B}^{\perp R[\xi]}$ is a finitely generated $\mathbb{R}[\xi]$-submodule of $\mathbb{R}^{1 \times \lambda}[\xi]$. Moreover, if $\mathcal{B} = \ker(R)$ is a minimal polynomial kernel representation, then this submodule is generated by the rows of $R$.

In the context of proper stable left prime rational representations we need to impose controllability:

Lemma 11.2 Let $\mathcal{B} \in \mathcal{L}^u_{\text{cont}}$ be represented by $R(\frac{d}{dt})w = 0$, where $R$ is proper, stable real rational and left prime. Then $\mathcal{B}^{\perp R_S(\xi)}$ is an $\mathbb{R}_S(\xi)$-submodule of $\mathbb{R}_S^{1 \times \lambda}(\xi)$, and the rows of $R$ form a basis of $\mathcal{B}^{\perp R_S(\xi)}$.

Proof: If $\mathcal{B}$ is controllable, then $\mathcal{B}^{\perp R_S(\xi)}$ forms a $\mathbb{R}_S(\xi)$-submodule of $\mathbb{R}_S^{1 \times \lambda}(\xi)$. This can be proven along the same lines as the proof of Theorem 11 in [23].

Let $R = P^{-1}Q$ be a left coprime factorization over $\mathbb{R}[\xi]$ of $R$. Then $\mathcal{B} = \ker(Q)$ is a minimal polynomial kernel representation. Let $n \in \mathcal{B}^{\perp R_S(\xi)}$. Then by Def. 11.1 $n(\frac{d}{dt})w = 0$ for all $w \in \mathcal{B}$. Let $n = u^{-1}v$ be a left coprime factorization of $n$ over $\mathbb{R}[\xi]$. Note that $u$ is Hurwitz. Then by definition we have $n(\frac{d}{dt})w = 0$ for all $w \in \mathcal{B}$ if and only if $v(\frac{d}{dt})w = 0$ for all $w \in \mathcal{B}$. Thus, by Def. 11.1, $v \in \mathcal{B}^{\perp R[\xi]}$. Consequently, there exists a $l \in \mathbb{R}^{1 \times \lambda}[\xi]$ such that $v = lQ$. Hence $n = u^{-1}v = u^{-1}lQ = (u^{-1}lP)(P^{-1}Q) = (u^{-1}lP)R$. Define $m := u^{-1}lP$. Then we have

$$n = mR.$$  

As $R$ is left prime, there exists a proper stable rational matrix $M$ such that $RM = I$. Multiplying (43) on both sides with $M$ we obtain $nM = mRM = m$. As $n$ and $M$ are proper and stable, we conclude that $m$ is proper and stable. Hence the rows of $R$ span the $\mathbb{R}_S(\xi)$-module $\mathcal{B}^{\perp R_S(\xi)}$. Finally, as $\mathcal{B} = \ker(R)$ is a minimal proper stable rational kernel representation, the rows of $R$ are linearly independent over $\mathbb{R}_S(\xi)$. We conclude then that these rows form a basis of $\mathcal{B}^{\perp R_S(\xi)}$.

The following lemma addresses the question under what conditions two proper, stable, left prime rational kernel representations represent the same controllable behavior:

Theorem 11.3 Let $\mathcal{B}_1, \mathcal{B}_2 \in \mathcal{L}^u_{\text{cont}}$. Let $\mathcal{B}_1 = \ker(R_1)$ and $\mathcal{B}_2 = \ker(R_2)$ be minimal rational kernel representations, where $R_1$ and $R_2$ are proper, stable real rational and left prime. Then $\mathcal{B}_1 = \mathcal{B}_2$ if and only if there exists a square, nonsingular, proper, stable real rational matrix $W$, with $W^{-1}$ proper and stable, such that $R_1 = WR_2$.

Proof: As $\mathcal{B}_1 = \mathcal{B}_2$ we have $\mathcal{B}_1^{\perp R_S(\xi)} = \mathcal{B}_2^{\perp R_S(\xi)} =: \mathcal{M}$. From Lemma 11.2, the rows of $R_1$ and $R_2$ both form a basis for the module $\mathcal{M}$. Then from the theory of modules we conclude that there exists a square, nonsingular, proper stable real rational matrix $W$ with $W^{-1}$ proper and stable such that $R_1 = WR_2$. 

27
Conversely, let \( R_1 = P_1^{-1}Q_1 \), \( R_2 = P_2^{-1}Q_2 \) be left coprime factorizations over \( \mathbb{R}[\xi] \) of \( R_1 \) and \( R_2 \). Let \( W = LM^{-1} \) be a right coprime factorization over \( \mathbb{R}[\xi] \) of \( W \). Then both \( L \) and \( M \) are nonsingular. By definition we have \( \mathfrak{B}_1 = \ker(Q_1) \) and \( \mathfrak{B}_2 = \ker(Q_2) \). Then,

\[
R_1 = WR_2 \iff P_1^{-1}Q_1 = LM^{-1}P_2^{-1}Q_2 \\
\iff L^{-1}P_1^{-1}Q_1 = M^{-1}P_2^{-1}Q_2 \\
\iff (P_1L)^{-1}Q_1 = (P_2M)^{-1}Q_2.
\]

Since \( \mathfrak{B}_1 \) and \( \mathfrak{B}_2 \) are controllable behaviors both \( Q_1(\lambda) \) and \( Q_2(\lambda) \) have full row rank for all \( \lambda \in \mathbb{C} \). This implies that \( (P_1L)Q_1(\lambda) \) and \( (P_2M)Q_2(\lambda) \) have full row rank for all \( \lambda \in \mathbb{C} \). Define \( \hat{R} := (P_1L)^{-1}Q_1 = (P_2M)^{-1}Q_2. \) This displays two left coprime factorizations of \( \hat{R} \), so \( \mathfrak{B}_1 = \ker(Q_1) = \ker(R) = \ker(Q_2) = \mathfrak{B}_2. \)

We are in a position to prove Lemma 2.3:

**Proof of Lemma 2.3:** Since \( \mathfrak{B} \) is controllable, by [23], Theorem 5, it admits a representation \( \mathfrak{B} = \ker(R) \) such that \( R \) is proper, stable, real rational and left prime. Clearly, \( R \) then has no zeros. Define \( Z(\xi) := R(\xi)R^\top(-\xi). \) Obviously \( Z(\xi) = Z^\top(-\xi) \) and \( Z \) has no poles and zeros on the imaginary axis, so \( Z(i\omega) > 0 \) for all \( \omega \in \mathbb{R} \). Thus there exists a square, nonsingular, proper stable real rational matrix \( W \) with \( W^{-1} \) proper and stable such that \( R(\xi)R^\top(-\xi) = W(\xi)W^{-\top}(\xi). \) Define \( R' := W^{-1}R. \) Clearly \( R' \) is co-inner. As \( R \) is left prime there exists a proper stable rational matrix \( M \) such that \( RM = I. \) We have \( R'MW = W^{-1}RMW = I. \) Hence from Def. 1.1 we conclude that \( R' \) is left prime. Finally, by Theorem 11.3, \( \mathfrak{B} = \ker(R'). \)

\[
\square
\]

**References**


