Tracking and regulation in the behavioral framework

Shaik Fiaz, K. Takaba, H.L. Trentelman

Johann Bernoulli Institute for Mathematics and Computer Science, University of Groningen, P.O Box 407, 9700 AK Groningen, The Netherlands

Department of Applied Mathematics and Physics, Kyoto University, Kyoto 606-8501, Japan

Abstract

Given a plant together with an exosystem generating the disturbances and the reference signals, the problem of asymptotic tracking and regulation is to find a controller such that the plant variable tracks the reference signal regardless of the disturbance acting on the system. If a controller achieves this design objective, we call it a regulator for the plant with respect to the given exosystem. In this paper, we formulate the asymptotic tracking and regulation problem in the behavioral framework, with control by interconnection.

1. Introduction

This paper deals with control in a behavioral context. We consider the problem of finding an admissible, stabilizing controller that regulates the tracking error to zero in the presence of a class of exogenous inputs. In other words, we consider the problem of asymptotic tracking and regulation in the behavioral framework.

In the behavioral framework, controlling a plant means restricting its behavior to a desired subset. This restriction is brought about by interconnecting the plant with a controller that we design. The restricted behavior is then called the controlled behavior, which is required to satisfy the design specifications. In terms of representations, control means that additional laws (e.g., in the form of differential equations representing the controller) are imposed on some of the plant variables. Thus, the plant and controller are interconnected through some of their variables. In our context, we do not distinguish between inputs and outputs and we do not restrict ourselves to feedback control. This idea was introduced by Willems (1997) in the context of stabilization and pole placement. In this paper, we use these ideas to solve the problem of asymptotic tracking and regulation.

The problem of asymptotic tracking and regulation has been studied before in the literature, in an input–output framework. See for instance Davison (1975), Davison and Goldenberg (1975), Francis (1977) and Francis and Wonham (1975). The theory has also been extended to nonlinear systems by Isidori and Byrnes (1990). Many results have been collected by Saberi et al. in the book Saberi, Stoorvogel, and Sannuti (2000) (see also Trentelman, Stoorvogel, and Hautus (2001)). In these, the concept of internal model principle plays a pivotal role in obtaining a solution to the asymptotic tracking and regulation problem. According to the internal model principle, in order to achieve regulation the controlled system must contain the dynamics of the exosystem.

Our work can be seen as the behavioral generalization of Davison and Goldenberg (1975), Francis (1977) and Francis and Wonham (1975). We use polynomial kernel representations of the plant (see Polderman & Willems, 1997) without input–output considerations. This problem was initially studied by Takaba (2009). In the work of Takaba, only necessary conditions were obtained for the existence of a regulator. In Fiaz, Takaba, and Trentelman (2010) necessary and sufficient conditions were obtained. It was assumed that the underlying exosystem is anti-stable and that the underlying plant does not annihilate any signal generated by the exosystem. In this paper, we generalize these results to the case when the underlying exosystem can be any autonomous system (not necessarily anti-stable) and the underlying plant might annihilate signals generated by the exosystem. Necessary and sufficient conditions for the existence of suitable controllers are expressed in terms of the plant and the exosystem. Also, a procedure to construct such controllers is given using the polynomial matrices appearing in the kernel representations of the plant and the exosystem.
A few words about the notation and nomenclature used. We use standard symbols for the fields of real and complex numbers \( \mathbb{R} \) and \( \mathbb{C} \), \( \mathbb{C}^- \) and \( \mathbb{C}_+ \) will denote the open left half plane and closed right half plane, respectively. We use \( \mathbb{R}^n \), \( \mathbb{R}^{m \times n} \), etc., for the real linear space of vectors and matrices. \( w(t) \) with components in \( \mathbb{R} \), \( \mathbb{R}^n \) denotes the set of infinitely often differentiable functions from \( \mathbb{R} \) to \( \mathbb{R}^n \). \( \mathbb{R}[t] \) denotes the ring of polynomials in the indeterminate \( t \) with real coefficients. We use \( \mathbb{R}[t]^n, \mathbb{R}[t]^m \), for the spaces of vectors and matrices with components in \( \mathbb{R}[t] \). Elements of \( \mathbb{R}[t]^n \) are called real polynomial matrices.

We use the notation \( \det(A) \) to denote the determinant of a square matrix \( A \). A square, nonsingular real polynomial matrix \( R \) is called unimodular if \( \det(R) \) is a non-zero constant. It is called Hurwitz if all roots of \( \det(R) \) lie in the open left half plane \( \mathbb{C}^- \). It is called anti-Hurwitz if all roots of \( \det(R) \) lie in the closed right half plane \( \mathbb{C}^+ \).

2. Linear differential systems and polynomial kernel representations

2.1. A function of the form

\[
    h(t) = \sum_{i=1}^{N} p_i(t)e^{\alpha_i t} \cos(b_i t) + q_i(t)e^{\beta_i t} \sin(b_i t),
\]

with \( p_i, q_i \) real valued polynomials in the indeterminate \( t \), and \( \alpha_i, \beta_i \in \mathbb{R} \), called a Bohl function. A Bohl function \( h(t) \) is called stable Bohl if in addition \( \lim_{t \to \infty} h(t) = 0 \). A nonzero Bohl function \( h(t) \) is called anti-stable Bohl if we have either \( \lim_{t \to \infty} h(t) \neq 0 \) or \( \lim_{t \to -\infty} h(t) \) does not exist.

It follows immediately from Polderman and Willems (1997, Theorem 3.2.16) that \( \mathbb{B} \in \mathbb{L}^2 \) is autonomous if and only if every \( w \in \mathbb{B} \) is a Bohl function, and that \( \mathbb{B} \) is stable if and only if every \( w \in \mathbb{B} \) is a stable Bohl function. Also, \( \mathbb{B} \) is anti-stable if and only if every nonzero \( w \in \mathbb{B} \) is an anti-stable Bohl function.

The next proposition which states that every autonomous behavior can be written as a direct sum of a stable and an anti-stable behavior follows immediately from results in Bisio and Valcher (2001a,b) (also see Proposition 2.6.8 in Fiazi, 2010).

Proposition 2.2. Let \( \mathbb{B} \in \mathbb{L}^2_{\text{aut}} \) then there exists a stable \( \mathbb{B}_1 \in \mathbb{L}^2_{\text{aut}} \) and an anti-stable \( \mathbb{B}_2 \in \mathbb{L}^2_{\text{aut}} \) such that \( \mathbb{B} = \mathbb{B}_1 \oplus \mathbb{B}_2 \).

Let \( \mathbb{B} \in \mathbb{L}^{1+2} \) with system variable \( w \) partitioned as \( w = (w_1, w_2) \). Assume that the first component \( w_1 \) is viewed as an observed variable, and the second component \( w_2 \) as a to-be-deduced variable. In such systems we can talk about observability. We say that \( w_2 \) is observable from \( w_1 \) in \( \mathbb{B} \) if, whenever \( (w_1, w_2) \in \mathbb{B} \), then \( w_2 = w_2(t) \). The weaker notion of detectability is defined along similar lines. We say that \( w_2 \) is detectable from \( w_1 \) in \( \mathbb{B} \) if, whenever \( (w_1, w_2) \in \mathbb{B} \), then \( \lim_{t \to \infty} (w_2 - w_2(t)) = 0 \). If \( R_1(\frac{d}{dt}) w_1 + R_2(\frac{d}{dt}) w_2 = 0 \) is a minimal representation of \( \mathbb{B} \), then \( w_2 \) is observable from \( w_1 \) in \( \mathbb{B} \) if and only if \( R_2(\lambda) \) has full column rank for all \( \lambda \in \mathbb{C} \) and \( w_2 \) is detectable from \( w_1 \) in \( \mathbb{B} \) if and only if \( R_2(\lambda) \) has full column rank for all \( \lambda \in \mathbb{C}^\ast \) (see Polderman & Willems, 1997).

Let \( \mathbb{B} \in \mathbb{L}^{1+2} \) with system variable \( (w_1, w_2) \). Often we are interested only in the behavior of one of the components, say the variable \( w_1 \), obtained by projecting \( \mathbb{B} \) onto this component. This behavior \( \langle \mathbb{B} \rangle_{w_1} \) is defined by \( \langle \mathbb{B} \rangle_{w_1} := \{(w_1, w_2) \in \mathbb{B} \} \). Starting with a polynomial kernel representation of \( \mathbb{B} \), in the following proposition we give a procedure for obtaining a polynomial kernel representation for \( \langle \mathbb{B} \rangle_{w_1} \) (see Polderman & Willems, 1997).

Proposition 2.3. Let \( \mathbb{B} \in \mathbb{L}^{1+2} \) with system variable \( (w_1, w_2) \) be represented by \( R_1(\frac{d}{dt}) w_1 + R_2(\frac{d}{dt}) w_2 = 0 \). Let \( U \) be a unimodular matrix such that \( UR_2 \in \mathbb{L}^2_{\text{full row rank}} \). Then \( \langle \mathbb{B} \rangle_{w_1} \) is partitioned into \( UR_1 \in \mathbb{L}^2 \) full rank row. Let \( UR_1 \) be a kernel representation of \( \langle \mathbb{B} \rangle_{w_1} \) is given by \( R_1(\frac{d}{dt}) w_1 = 0 \).

3. Review of stabilization by interconnection

In this section we will briefly recall the notion of stabilization by interconnection. We will first look at the full interconnection case, i.e. the case when all the plant variables are available for interconnection.

Definition 3.1. Let \( \mathcal{P} \in \mathbb{L}^2 \) be a plant behavior. A controller for \( \mathcal{P} \) is a system behavior \( \mathcal{E} \in \mathbb{L}^2 \). The full interconnection of \( \mathcal{P} \) and \( \mathcal{E} \), shown schematically in Fig. 1, is defined as the system with behavior \( \mathcal{P} \cap \mathcal{E} \). This behavior is referred to as the controlled behavior, and is also an element of \( \mathbb{L}^2 \). The full interconnection is called regular if \( \mathcal{P} \cap \mathcal{E} = \mathcal{P} + \mathcal{E} \). In that case we call \( \mathcal{E} \) a regular controller.

In full interconnection, the regularity condition is equivalent to: \( \mathcal{E} \) does not re-impose restrictions on the plant variable \( w \) that are already present in the laws of \( \mathcal{P} \) (see Willems, 1997).

A behavior \( \mathbb{B} \in \mathbb{L}^2 \) is said to be stabilizable if for every \( w \in \mathbb{B} \) there exists \( w' \in \mathbb{B} \) such that \( w'(t) = w(t) \) for \( t \leq 0 \),
4. Asymptotic tracking and regulation

For a given plant behavior with its to-be-controlled variable \( w \) and reference signal \( r \), an important synthesis problem in control is to design a controller such that the plant variable \( w \) follows the reference signal \( r \) in the resulting system after interconnecting the plant and the controller. This is called the asymptotic tracking problem. A classical approach to this problem is to let the reference signal be generated by an autonomous system called the exosystem. One then incorporates the dynamics of the exosystem into the dynamics of the plant and defines a new variable \( e \) as the difference between the reference signal \( r \) and \( w \). The asymptotic tracking problem is then reformulated as: design a controller that, after interconnection with the plant, drives the signal \( e \) to zero.

A second important synthesis problem is the problem of regulation. For a given plant with to-be-controlled variable \( w \), and external disturbance acting on the plant (which is assumed to be free in the plant), the problem here is to design a controller such that in the resulting system after interconnection of the plant and the controller, the disturbance remains free and the plant variable \( w \) converges to zero as time tends to infinity, regardless of the disturbance acting on the plant. A controller such that after interconnection with the plant, the disturbance remains free is called an admissible controller. In line with the approach to the regulation problem in Francis (1977) and Francis and Wonham (1975) and similarly to the asymptotic tracking problem given above, we approach this problem by assuming the disturbance to be generated by some linear time invariant autonomous system, again called the exosystem. Then one incorporates the dynamics of the exosystem into the dynamics of the plant, and requires the variable \( w \) in this interconnected system to converge to zero as time tends to infinity.

Combining these two synthesis problems we can formulate a single new synthesis problem by requiring the design of a controller such that the interconnected system variable tracks a given reference signal, regardless of the disturbance. This is done by combining the two exosystems into a single one and requires regulation of the tracking error.

In addition to the requirements of asymptotic tracking and regulation, a realistic design requires the system to go to rest in the absence of disturbances (i.e., if the disturbance signal is identically equal to zero). An admissible controller that takes the system to rest in the absence of disturbances is called a stabilizing controller. An admissible controller which achieves all three requirements, i.e. asymptotic tracking, regulation and stabilization, is called a regulator.

4.1. Problem formulation

In this subsection we will introduce the problem of asymptotic tracking and regulation in a behavioral context, with control by regular, partial interconnection. We start with a plant behavior \( P \in \mathbb{L}^\alpha \) with plant variables \( (w, c, v) \), shown schematically in Fig. 3(b). The system variable has been partitioned into \( w, c \) and \( v \). These variables represent the to-be-controlled variable (including tracking error), the interconnection variable (such as sensor measurements and actuator inputs), and the external disturbances and reference signals, respectively. The interconnection variable \( c \) is the system variable through which we are allowed to interconnect \( P \) with the controller \( C \in \mathbb{L}^\alpha \). As the components of the variable \( v \) represent reference signals and external disturbances, we assume \( v \) to be free in \( P \). In addition to the plant \( P \), let an exosystem \( E \in \mathbb{L}^\alpha \) which generates the disturbance and the reference signal be given, as shown schematically in Fig. 3(a).

Let \( E \in \mathbb{L}^\alpha \), shown schematically in Fig. 3(c). Then the interconnection of \( P \) with \( C \) (shown schematically in Fig. 4) is given by

\[
P \wedge_C E = \{(w, c, v) \mid (w, c, v) \in P \text{ and } c \in E\}.
\]
As \( v \) is interpreted as unknown disturbance, it should remain free after interconnecting the plant with a controller. In order to highlight this, we give the following definition:

**Definition 4.1.** Let \( \mathcal{P} \in \mathbb{L}^{w+c+\tau} \). Assume \( v \) is free in \( \mathcal{P} \). Then \( \mathcal{E} \in \mathbb{L}^c \) is called an admissible controller for \( \mathcal{P} \) if \( v \) is free in \( \mathcal{P} \land \mathcal{E} \).

In the context of asymptotic tracking and regulation a controller is called stabilizing if, whenever the disturbance \( v \) is zero, the to-be-regulated variable \( w \) and interconnection variable \( c \) tend to zero as time runs off to infinity:

**Definition 4.2.** Let \( \mathcal{P} \in \mathbb{L}^{w+c+\tau} \), with \( v \) free. An admissible controller \( \mathcal{E} \in \mathbb{L}^c \) is called stabilizing if \( \lim_{t \to \infty} \left( c(t), P(t) \right) = (0, 0) \) for all \( (w, c, 0) \in \mathcal{P} \land \mathcal{E} \) (equivalently, \( N_{(w,c)}(\mathcal{P}) \land \mathcal{E} \) is stable).

The following theorem establishes necessary and sufficient conditions on the plant for the existence of a regular, admissible, stabilizing controller:

**Theorem 4.3.** Let \( \mathcal{P} \in \mathbb{L}^{w+c+\tau} \). Assume \( v \) is free in \( \mathcal{P} \). Then there exists a regular, admissible, stabilizing controller for \( \mathcal{P} \) if and only if

\[ N_{(w,c)}(\mathcal{P}) \] is stabilizable, and

\[ (w, c, 0) \in \mathcal{P} \land \mathcal{E} \] is detectable from \((c, v)\) in \( \mathcal{P} \).

**Proof.** Let \( R_1 (\frac{d}{dt}) w + R_2 (\frac{d}{dt}) c + R_3 (\frac{d}{dt}) v \) = 0 be a minimal representation of \( \mathcal{P} \). Then there exists a unimodular matrix \( U \) such that

\[
U \begin{bmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix},
\]

where \( R_{11} \) has full row rank. Then we have

\[ \mathcal{P} = \ker \begin{bmatrix} R_{11} (\frac{d}{dt}) & R_{12} (\frac{d}{dt}) & R_{13} (\frac{d}{dt}) \\ 1 & 0 & 0 \end{bmatrix}, \]

(2)

\[ N_{(w,c)}(\mathcal{P}) = \ker \begin{bmatrix} R_{11} (\frac{d}{dt}) & R_{12} (\frac{d}{dt}) \\ 0 & R_{22} (\frac{d}{dt}) \end{bmatrix}. \]

Since \( v \) is free in \( \mathcal{P} \land \mathcal{E} \) and \( N_{(w,c)}(\mathcal{P}) \land \mathcal{E} \) is stable, \[ \begin{bmatrix} R_{11} & R_{12} \\ 0 & 0 \end{bmatrix} \] is square, nonsingular and Hurwitz, which in turn implies that \( R_{11} \) is square, nonsingular and Hurwitz and \[ \begin{bmatrix} R_{11} & R_{12} \\ 0 & 0 \end{bmatrix} \] has full row rank for all \( \lambda \in \tilde{\mathbb{C}}^+ \). From Eq. (2) \( w \) is detectable from \((c, v)\) in \( \mathcal{P} \). From Eq. (3) we conclude that \( N_{(w,c)}(\mathcal{P}) \) is stabilizable.

**Definition 4.4.** Let \( \mathcal{P} \in \mathbb{L}^{w+c+\tau} \). Assume \( v \) is free in \( \mathcal{P} \). Then \( \mathcal{P} \land \mathcal{E} \) is called a regulator for \( \mathcal{P} \) with respect to \( \mathcal{E} \) if

\[ (w, c, 0) \in \mathcal{P} \land \mathcal{E} \] is detectable from \((c, v)\) in \( \mathcal{P} \).

The interconnection of the plant with the exosystem is shown schematically in Fig. 5 and is given by

\[
\mathcal{P} \land \mathcal{E} \equiv \{ (w, c, v) | (w, c, v) \in \mathcal{P}, v \in \mathcal{E} \}. \]
5. Solution to the asymptotic tracking and regulation problem

As a first step in resolving Problem 1, we will show that without loss of generality we can assume that in \( \mathcal{P} \land \mathcal{E} \), the interconnection of plant and exosystem, \( \nu \) is observable from \((w, \psi)\), equivalently, \( \mathcal{E} \land \mathcal{N}(\mathcal{P}) = 0 \).

Let
\[
R_1 \left( \frac{d}{dt} \right) w + R_2 \left( \frac{d}{dt} \right) c + R_3 \left( \frac{d}{dt} \right) v = 0, \quad \text{and}
\]
\[
V \left( \frac{d}{dt} \right) v = 0
\]
be minimal representations of \( \mathcal{P} \) and \( \mathcal{E} \) respectively, where \( V \) is square and nonsingular. Factorize
\[
\begin{bmatrix} R_2 \cr V \end{bmatrix} = \begin{bmatrix} K_2 \cr V' \end{bmatrix} D,
\]
where \( D \) is square and nonsingular and \( \begin{bmatrix} K_2 \cr V' \end{bmatrix} \) has full column rank for all \( \lambda \in \mathbb{C} \). Define
\[
\mathcal{P}' := \left\{ (w, \psi, \nu) \mid R_1 \left( \frac{d}{dt} \right) w 
\right. \\

+ R_2 \left( \frac{d}{dt} \right) c + R_3 \left( \frac{d}{dt} \right) v = 0 \},
\]
and
\[
\mathcal{E}' := \left\{ \nu \mid V' \left( \frac{d}{dt} \right) v = 0 \right\}. \tag{10}
\]
We have
\[
\mathcal{P} \land \mathcal{E} = \ker \begin{bmatrix} R_1 \left( \frac{d}{dt} \right) & R_2 \left( \frac{d}{dt} \right) & R_3 \left( \frac{d}{dt} \right) \\
0 & C \left( \frac{d}{dt} \right) & 0 \end{bmatrix}, \tag{11}
\]
and
\[
{\mathcal{P}' \land \mathcal{E}'} = \ker \begin{bmatrix} R_1 \left( \frac{d}{dt} \right) & R_2 \left( \frac{d}{dt} \right) & R_3 \left( \frac{d}{dt} \right) \\
0 & C \left( \frac{d}{dt} \right) & 0 \end{bmatrix}.
\]

It is easy to see that \( \nu \) is observable from \((w, \psi)\) in \( \mathcal{P} \land \mathcal{E} \) (use the fact that \( \begin{bmatrix} K_2 \cr V' \end{bmatrix} \) has full column rank for all \( \lambda \in \mathbb{C} \)).

Let \( \mathcal{E} \in \mathcal{E}' \). The following theorem shows that for the solvability of Problem 1 the assumption \( \mathcal{E} \land \mathcal{N}(\mathcal{P}) = 0 \) can indeed be made without loss of generality:

**Theorem 5.1.** Let \( \mathcal{P}, \mathcal{E}, \mathcal{P}' \) and \( \mathcal{E}' \) be given by Eqs. (7)–(10), respectively. Then \( \mathcal{E} \) is a regulator for \( \mathcal{P} \) with respect to \( \mathcal{E} \) if and only if \( \mathcal{E}' \) is a regulator for \( \mathcal{P}' \) with respect to \( \mathcal{E}' \).

**Proof.** Let \( C \left( \frac{d}{dt} \right) c = 0 \) be a minimal representation of \( \mathcal{E} \). We have
\[
\mathcal{P} \land \mathcal{E} = \ker \begin{bmatrix} R_1 \left( \frac{d}{dt} \right) & R_2 \left( \frac{d}{dt} \right) & R_3 \left( \frac{d}{dt} \right) \\
0 & C \left( \frac{d}{dt} \right) & 0 \end{bmatrix}, \tag{12}
\]
and
\[
\mathcal{P}' \land \mathcal{E}' = \ker \begin{bmatrix} R_1 \left( \frac{d}{dt} \right) & R_2 \left( \frac{d}{dt} \right) & R_3 \left( \frac{d}{dt} \right) \\
0 & C \left( \frac{d}{dt} \right) & 0 \end{bmatrix}. \tag{13}
\]

From the above it is easy to see that the interconnection \( \mathcal{P} \land \mathcal{E} \) is regular, \( \nu \) is free in \( \mathcal{P} \land \mathcal{E} \), and \( \mathcal{N}(w, \psi)(\mathcal{P} \land \mathcal{E}) \) is stable if and only if \( \begin{bmatrix} R_1 \cr R_2 \cr C \end{bmatrix} \) is square, nonsingular and Hurwitz. In turn, this holds if and only if the interconnection \( \mathcal{P}' \land \mathcal{E}' \) is regular, \( \nu \) is free in \( \mathcal{P}' \land \mathcal{E}' \), and \( \mathcal{N}(w, \psi)(\mathcal{P}' \land \mathcal{E}') \) is stable. In order to proceed we now show \((\mathcal{P} \land \mathcal{E} \land \mathcal{E}')_w = (\mathcal{P}' \land \mathcal{E}' \land \mathcal{E}')_w\).
that the controller

Example 5.4. Asymptotic tracking and regulation brings forward the 'internal

\( \mathbf{E}(\xi) = f(\xi)\xi^n \) (internal model principle). Therefore \( \mathbf{E} \) is a regulator if and only if \( c_1 \) has at least an \( n \)-fold root in \( 0 \), equivalently, the controller transfer function has at least an \( n \)-fold pole in \( 0 \).

As regulation is an asymptotic property, intuitively the stable part of the exosystem does not affect regulation. Indeed, in the following theorem, we show that we can get the general problem to the case when the exosystem is anti-stable.

Theorem 5.5. Let \( \mathbb{P} \in \mathbb{L}^{\mathbb{R}^c+y} \) and \( \mathbf{E} \in \mathbb{L}^{\mathbb{E}_{\text{aut}}} \). Assume \( v \) is free in \( \mathbb{P} \). Let \( \mathbf{E} = \mathbf{E}_1 \oplus \mathbf{E}_2 \) where \( \mathbf{E}_1 \in \mathbb{L}^{\mathbb{E}_{\text{aut}}} \) is stable and \( \mathbf{E}_2 \in \mathbb{L}^{\mathbb{E}_{\text{aut}}} \) is anti-stable. Let \( \mathbf{E} \in \mathbb{L}^{\mathbb{E}'} \). Then the following statements are equivalent:

1. \( \mathbf{E} \) is a regulator for \( \mathbb{P} \) with respect to \( \mathbb{E} \).
2. \( \mathbf{E} \) is a regulator for \( \mathbb{P} \) with respect to \( \mathbb{E}_w \).

Proof. Before turning to the actual proof of this theorem, we will first prove the following three lemmas.

Lemma 5.6. Let \( \mathbb{P} \in \mathbb{L}^{\mathbb{R}^c+y} \) and \( \mathbb{E} \in \mathbb{L}^\mathbb{E}_{\text{aut}} \). Assume \( v \) is free in \( \mathbb{P} \). Let \( \mathbb{E} = \mathbb{E}_1 \oplus \mathbb{E}_2 \) with \( \mathbb{E}_1, \mathbb{E}_2 \in \mathbb{L}^{\mathbb{E}_{\text{aut}}} \). Then

\((\mathbb{P} \cap \mathbb{E}_1) + (\mathbb{P} \cap \mathbb{E}_2) = (\mathbb{P} \cap \mathbb{E})\).

Proof. As \( \mathbf{E} = \mathbf{E}_1 \oplus \mathbf{E}_2 \) the inclusion \((\mathbb{P} \cap \mathbb{E}_1) + (\mathbb{P} \cap \mathbb{E}_2) \subseteq \mathbb{P} \cap \mathbb{E} \) is straightforward. To prove the converse inclusion let \( (w,v) \in \mathbb{P} \cap \mathbb{E} \). Then there exist \( v_1 \in \mathbb{E}_1 \) and \( v_2 \in \mathbb{E}_2 \) such that \( v = v_1 + v_2 \). Since \( v \) is free in \( \mathbb{P} \), there exists \( w_1 \) such that \( (w_1, v_1) \in \mathbb{P} \cap \mathbb{E}_1 \subseteq \mathbb{P} \cap \mathbb{E} \). Define \( w_2 := w - w_1 \). By linearity, we have \( (w_2, v_2) = (w_1, v_1) \) \( (w_2, v_2) \in \mathbb{P} \cap \mathbb{E}_2 \). Consequently, \( (w,v) = (w_1, v_1) + (w_2, v_2) \in (\mathbb{P} \cap \mathbb{E}_1) + (\mathbb{P} \cap \mathbb{E}_2) \). This implies \((\mathbb{P} \cap \mathbb{E}_1) + (\mathbb{P} \cap \mathbb{E}_2) \subseteq \mathbb{P} \cap \mathbb{E} \).

Lemma 5.7. Let \( \mathbb{P} \in \mathbb{L}^{\mathbb{R}^c+y} \) and let \( \mathbb{E}_w \in \mathbb{L}^{\mathbb{E}_{\text{aut}}} \) be stable. If \( \mathbb{N}_w(\mathbb{P}) \) is stable then \( \mathbb{P} \cap \mathbb{E}_w \) is stable.

Proof. Let \( R_1 \left( \frac{d}{dt} \right)^2 + R_2 \left( \frac{d}{dt} \right) + \mathbb{S} \left( \frac{d}{dt} \right) = 0 \) be minimal representations of \( \mathbb{P} \) and \( \mathbb{E}_w \), respectively, where \( \mathbb{S} \) is Hurwitz. We have

\[ \mathbb{P} \cap \mathbb{E}_w = \ker \left[ \begin{bmatrix} R_1 & R_2 & \mathbb{S} \\ 0 & R_2 & \mathbb{S} \end{bmatrix} \right] \]

and

\[ \mathbb{N}_w(\mathbb{P}) = \ker \left[ R_1 \left( \frac{d}{dt} \right) \right] \]

The stability of \( \mathbb{N}_w(\mathbb{P}) \) implies that \( R_1(\lambda) \) has full column rank for all \( \lambda \in \mathbb{C}^+ \), which in turn implies that \( \frac{\mathbb{R}_1(\lambda)}{\mathbb{S}(\lambda)} \) has full column rank for all \( \lambda \in \mathbb{C}^+ \). Therefore \( \mathbb{P} \cap \mathbb{E}_w \) is stable.

Lemma 5.8. Let \( \mathbb{P} \in \mathbb{L}^{\mathbb{R}^c+y} \) and \( \mathbf{E} \in \mathbb{L}^\mathbb{E}_{\text{aut}} \). Assume \( v \) is free in \( \mathbb{P} \). Let \( \mathbf{E} = \mathbf{E}_1 \oplus \mathbf{E}_2 \) with \( \mathbf{E}_1, \mathbf{E}_2 \in \mathbb{L}^\mathbb{E}_{\text{aut}} \) is stable and \( \mathbf{E}_2 \in \mathbb{L}^\mathbb{E}_{\text{aut}} \) is anti-stable. Let \( \mathbf{E} \in \mathbb{L}^\mathbb{E} \) be such that \( v \) is free in \( \mathbb{P} \cap \mathbb{E} \). Then the following statements are equivalent:

1. \( (\mathbb{P} \cap \mathbb{E} \cap \mathbf{E})_w \) is stable.
2. \( (\mathbb{P} \cap \mathbf{E}_1 \cap \mathbf{E}_2)_w \) is stable.

Proof. (\( (1) \Rightarrow (2) \)) As \( \mathbf{E}_1 \subseteq \mathbb{E} \) we have \( \mathbb{P} \cap \mathbb{E}_w \mathbb{E}_2 \subseteq \mathbb{P} \cap \mathbb{E} \mathbb{E}_w \mathbf{E}_2 \) which implies \((\mathbb{P} \cap \mathbb{E}_1 \cap \mathbf{E}_2)_w \subseteq (\mathbb{P} \cap \mathbb{E} \cap \mathbf{E})_w \). Therefore, the stability of \((\mathbb{P} \cap \mathbb{E} \cap \mathbf{E})_w \) implies that \((\mathbb{P} \cap \mathbb{E}_1 \cap \mathbf{E}_2)_w \) is stable.

(\( (2) \Rightarrow (1) \)) We have \((\mathbb{P} \cap \mathbb{E}_1 \cap \mathbf{E}_2)_w = ((\mathbb{P} \cap \mathbb{E}_1 \cap \mathbf{E}_2)_w \cap \mathbf{E}_2)_w \) stable. From Theorem 5.2 we must have the stability of \( \mathbb{N}_w \).
Finally, by combining these lemmas we arrive at:

**Proof of Theorem 5.5.** It is evident from Lemma 5.8 and Definition 4.4 that $\mathcal{E}$ is a regulator for $\mathcal{P}$ with respect to $\mathcal{E}$ if and only if $\mathcal{E}$ is a regulator for $\mathcal{P}$ with respect to $\mathcal{E}$. □

Based on Theorems 5.1 and 5.5, without loss of generality we hereafter make the following assumptions:

**Assumptions.** A1. $\mathcal{E} \in \mathcal{C}_{aut}$ is an anti-stable system, and
A2. $v$ is observable from $(w, c)$ in $\mathcal{P} \wedge c$, i.e., $\mathcal{E} \cap \mathcal{N}_{v}(\mathcal{P}) = 0$.

The following theorem is the main result of this paper. It provides a complete solution to Problem 1.

**Theorem 5.9.** Let $\mathcal{P} \in \mathcal{C}_{aut}^{x+y}$ with system variable $(w, c, v)$. Assume $v$ is free in $\mathcal{P}$. Let $\mathcal{E} \in \mathcal{C}_{aut}$ with system variable $v$. Assume $\mathcal{E}$ is anti-stable and $v$ is observable from $(w, c)$ in $\mathcal{P} \wedge c$. Then there exists a regulator for $\mathcal{P}$ with respect to $\mathcal{E}$ if and only if the following conditions hold:

1. $(w, v)$ is detectable from $c$ in $\mathcal{P} \wedge c$.
2. $\mathcal{N}_{v}(\mathcal{P})$ is stabilizable, and
3. there exists a polynomial matrix $X \in \mathbb{R}[\bar{\xi}]^{x+y}$ such that $(0, X \mathcal{E}, v, v) \in \mathcal{P}$ for all $v \in \mathcal{E}$.

**Proof.** Let

$$
\begin{align*}
R_1 \begin{pmatrix} \frac{d}{dt} w \\ \frac{d}{dt} c \end{pmatrix} + R_2 \begin{pmatrix} \frac{d}{dt} c \\ \frac{d}{dt} v \end{pmatrix} v &= 0, \quad \text{and} \\
V \begin{pmatrix} \frac{d}{dt} v \end{pmatrix} v &= 0 
\end{align*}
$$

be minimal representations of $\mathcal{P}$ and $\mathcal{E}$, respectively.

(only if)

1. We easily see that $(w, 0, v) \mid (w, 0, v) \in \mathcal{P} \wedge c \subseteq \mathcal{P} \wedge c$. It then follows from Definition 4.4 that $lim_{t \to +\infty} \langle w(t) , 0 \rangle = 0$ for all $w, v \in \mathcal{P} \wedge c$. Hence, if $(w, 0, v) \in \mathcal{P} \wedge c$ then $w$ is a stable Bohl function. As $v$ is observable from $(w, c) \in \mathcal{P} \wedge c$, $v$ is a stable Bohl function for all $(w, 0, v) \in \mathcal{P} \wedge c$. Therefore we have $lim_{t \to +\infty} \langle w(t), v(t) \rangle = 0$ for all $(w, 0, v) \in \mathcal{P} \wedge c$, in other words, $(w, v)$ is detectable from $c$ in $\mathcal{P} \wedge c$. This proves condition 1.

2. Let $\mathcal{E} = \ker(\mathcal{C}(\frac{d}{dt}))$ be a minimal representation of a regulator for $\mathcal{P}$ with respect to $\mathcal{E}$. From Definition 4.4 and using Theorem 4.3, $\mathcal{N}_{v}(\mathcal{P})$ is stabilizable. This proves condition 2.

3. In order to show that condition 3. is necessary for the existence of a regulator we make use of the internal model principle given in Theorem 5.2.

We have

$$
\mathcal{P} \wedge c = \ker \begin{pmatrix} R_1 \begin{pmatrix} \frac{d}{dt} w \\ \frac{d}{dt} c \end{pmatrix} & R_2 \begin{pmatrix} \frac{d}{dt} c \\ \frac{d}{dt} v \end{pmatrix} \\ 0 & C \begin{pmatrix} \frac{d}{dt} v \end{pmatrix} \end{pmatrix} 
\end{pmatrix}.
$$

The facts that $v$ is free in $\mathcal{P} \wedge c$ and that $\mathcal{N}_{v}(\mathcal{P} \wedge c)$ is stable imply that $R_1 \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ is Hurwitz. There exists a unimodular matrix $U$ such that

$$
U \begin{pmatrix} \tilde{R}_1 & R_2 & \frac{d}{dt} \tilde{R}_3 \\ 0 & C & \frac{d}{dt} \tilde{R}_3 \end{pmatrix} = \begin{pmatrix} \tilde{R}_1 & 0 & \tilde{R}_3 \\ R_2 & \tilde{R}_2 & \tilde{R}_3 \end{pmatrix},
$$

where $\tilde{R}_1$ and $\tilde{R}_2$ are Hurwitz. Therefore we have

$$
\mathcal{P} \wedge c = \ker \begin{pmatrix} \begin{pmatrix} \frac{d}{dt} \tilde{R}_1 & 0 & \frac{d}{dt} \tilde{R}_3 \\ \frac{d}{dt} \tilde{R}_2 & \frac{d}{dt} \tilde{R}_3 & \frac{d}{dt} \tilde{R}_3 \end{pmatrix} \\ 0 & 0 & \tilde{V} \begin{pmatrix} \frac{d}{dt} \end{pmatrix} \end{pmatrix}
\end{pmatrix}.
$$

From Eqs. (20) and (23), we have $\mathcal{P} \wedge c \wedge c$

$$
\mathcal{E} = \ker \begin{pmatrix} \begin{pmatrix} \frac{d}{dt} \tilde{R}_1 & 0 & \frac{d}{dt} \tilde{R}_3 \\ \frac{d}{dt} \tilde{R}_2 & \frac{d}{dt} \tilde{R}_3 & \frac{d}{dt} \tilde{R}_3 \end{pmatrix} \\ \begin{pmatrix} \tilde{R}_1 & 0 & \tilde{R}_3 \\ \tilde{R}_2 & \tilde{R}_2 & \tilde{R}_3 \end{pmatrix} \end{pmatrix}.
$$

It is easy to see that this matrix has full row rank. Then we have $(\mathcal{P} \wedge c \wedge c)(\mathcal{E})(w,v) = (\mathcal{P} \wedge c \wedge c)(w,v) \wedge c = \ker \begin{pmatrix} \frac{d}{dt} \tilde{R}_1 & \frac{d}{dt} \tilde{R}_3 \\ \frac{d}{dt} \tilde{R}_2 & \frac{d}{dt} \tilde{R}_3 \\ \frac{d}{dt} \tilde{R}_2 & \frac{d}{dt} \tilde{R}_3 \end{pmatrix}$. From Theorem 5.2, the internal model principle, the fact that $(\mathcal{P} \wedge c \wedge c)(w,v)$ is stable implies that $\mathcal{E} \subseteq \mathcal{N}_{v}(\mathcal{P} \wedge c)(w,v)$. Hence from Eqs. (20) and (25) there exists a polynomial matrix $\tilde{Y}_1$ such that

$$
\tilde{R}_1 = \tilde{Y}_1 \mathcal{V}.
$$

Using Eqs. (26) and (27) we have

$$
\begin{pmatrix} 0 \\ \tilde{R}_2 \end{pmatrix} X + \begin{pmatrix} \tilde{R}_1 \\ \tilde{R}_2 \end{pmatrix} \mathcal{V} = \begin{pmatrix} \tilde{Y}_1 \\ \tilde{Y}_2 \end{pmatrix} \mathcal{V}.
$$

Pre-multiplying both sides by $U^{-1}$ in the above equation, we obtain

$$
\begin{bmatrix}
R_2 \\
C
\end{bmatrix} X + \begin{bmatrix}
R_1 \\
0
\end{bmatrix} = \begin{bmatrix}
Y_1 \\
Y_2
\end{bmatrix} V
$$

(29)

where $\begin{bmatrix}
Y_1 \\
Y_2
\end{bmatrix} := U^{-1} \begin{bmatrix}
Y_1 \\
Y_2
\end{bmatrix}$. Then we have

$$R_2 X + R_3 = Y_1 V,
$$

(30)
and $\mathcal{C}X = Y_1 V$. Therefore, in order to be a regulator, the controller $\mathcal{C}$ must have the internal model of $\mathcal{E}$ in the form of $\mathcal{C}X = Y_1 V$.

Since $\mathcal{E} = \ker(V \left( \frac{d}{dt} \right))$, from Eq. (30), $\begin{bmatrix} R_2 & R_3 \end{bmatrix} \begin{bmatrix}
X \left( \frac{d}{dt} \right) v
\end{bmatrix} = 0$
holds for all $v \in \mathcal{E}$, i.e.,

$$0, X \left( \frac{d}{dt} \right) v, v \in \mathcal{P}.
$$

(if) Let $\mathcal{P}$ be given by Eq. (19). There exists a unimodular matrix $T$ such that

$$T \begin{bmatrix} R_1 & R_2 & R_3 \end{bmatrix} = \begin{bmatrix} R_{11} & R_{12} & R_{13} \\
0 & R_{22} & R_{23} \end{bmatrix},
$$

where $R_{11}$ has full row rank. Therefore we have

$$\mathcal{P} = \ker \begin{bmatrix}
R_{11} & R_{12} & R_{13} \\
0 & R_{22} & R_{23}
\end{bmatrix},
$$

(32)

$$\mathcal{N}_{(w,c)}(\mathcal{P}) = \ker \begin{bmatrix}
R_{11} & R_{12} & R_{13} \\
0 & R_{22} & R_{23}
\end{bmatrix},
$$

(33)

$$\mathcal{N}_{(w,c)}(\mathcal{P})_c = \ker \begin{bmatrix}
R_{22}
\end{bmatrix}, \quad \text{and}
$$

(34)

$$\mathcal{P} \land_c \mathcal{E} = \ker \begin{bmatrix}
R_{11} & R_{12} & R_{13} \\
0 & R_{22} & R_{23}
\end{bmatrix}.
$$

(35)

There exists a polynomial matrix $X \in \mathbb{R}[\xi]^{n \times r}$ such that $0, X \left( \frac{d}{dt} \right) v, v \in \mathcal{P}$ for all $v \in \mathcal{E}$. Hence $V \left( \frac{d}{dt} \right) v = 0$ implies

$$\begin{bmatrix}
R_{12} \\
R_{22}
\end{bmatrix} X \left( \frac{d}{dt} \right) v + \begin{bmatrix}
R_{13} \\
R_{23}
\end{bmatrix} v = 0.
$$

(36)

Therefore there exists a polynomial matrix $Y = \begin{bmatrix} Y_1 \\
Y_2
\end{bmatrix}$ such that

$$\begin{bmatrix}
R_{12} \\
R_{22}
\end{bmatrix} X + \begin{bmatrix}
R_{13} \\
R_{23}
\end{bmatrix} = \begin{bmatrix}
Y_1 \\
Y_2
\end{bmatrix} V.
$$

(37)

This implies

$$R_{22} X + R_{23} = Y_2 V.
$$

(38)

From Eq. (33), the fact that $\mathcal{N}_{(w,c)}(\mathcal{P})$ is stabilizable implies that $\begin{bmatrix} R_{11}(\lambda) \\
0 & R_{22}(\lambda)
\end{bmatrix}$ has full row rank for all $\lambda \in \mathbb{C}^+$, which in turn implies that $R_{23}(\lambda)$ has full row rank for all $\lambda \in \mathbb{C}^+$. From Eq. (34), we conclude that $\mathcal{N}_{(w,c)}(\mathcal{P})_c$ is stabilizable. From Proposition 3.3 there exists a $\mathcal{C} \in \mathbb{C}^{r \times s}$ such that $\mathcal{N}_{(w,c)}(\mathcal{P})_c \land \mathcal{C}$ is stable and regular. Factor $R_{22}$ as $R_{22} = DK$ where $D$ is Hurwitz and $K(\lambda)$ has full row rank for all $\lambda \in \mathbb{C}$. Let $S$ be such that $\begin{bmatrix} K \end{bmatrix}$ is unimodular.

Then for an arbitrary polynomial matrix $F$ and an arbitrary Hurwitz polynomial matrix $H$ of suitable dimensions, it is easy to verify that

$$C = FR_{22} + HS
$$

(39)

serves as a stabilizing controller for $(\mathcal{N}_{(w,c)}(\mathcal{P}))_c$. Note that $\begin{bmatrix} R_{22} \\
C
\end{bmatrix}$ is Hurwitz for all $C$ given by the Eq. (39).

From Eq. (35), $(w, v)$ is detectable from $c$ in $\mathcal{P} \land_c \mathcal{E}$ implies that $\begin{bmatrix} R_{11}(\lambda) \\
0 & R_{22}(\lambda)
\end{bmatrix}$ has full column rank for all $\lambda \in \mathbb{C}^+$. This implies that $R_{11}$ is square nonsingular and $R_{22}$ has full column rank for all $\lambda \in \mathbb{C}^+$ (use the fact that $V$ is anti-Hurwitz) we conclude that $\begin{bmatrix} R_{22}(\lambda) \\
V(\lambda)
\end{bmatrix}$ has full column rank for all $\lambda \in \mathbb{C}$.

Hence there exists a solution $(F, M)$ of the equation

$$FR_{23} + MV = HS.
$$

(40)

We now prove that any controller given by $C = \ker(C \left( \frac{d}{dt} \right))$ where $C = FR_{22} + HS$ with $F$ satisfying Eq. (40) serves as a regulator. The following identities hold true.

$$C \mathcal{X} = FR_{22} X + HSX$$

$$= FR_{22} X + FR_{23} + MV(\text{from Eq. (40)})$$

$$= F(2R_{22} X + R_{23}) + MV$$

$$= FV_{Y_2} V + MV(\text{from Eq. (38)})$$

$$= (FV_{Y_2} + M)V.
$$

Then, we define $W := FV_{Y_2} + M$ to rewrite the above equality as

$$C \mathcal{X} = WV.
$$

(41)

We also have

$$\mathcal{P} \land_c \mathcal{E} = \ker \begin{bmatrix}
R_{11} & R_{12} & R_{13} \\
0 & R_{22} & R_{23}
\end{bmatrix}.$$

(42)

$$\mathcal{N}_{(w,c)}(\mathcal{P} \land_c \mathcal{E}) = \ker \begin{bmatrix}
0 & R_{22} \\
0 & C
\end{bmatrix}.$$

(43)

As $C$ is chosen such that $\begin{bmatrix} R_{22} \\
C
\end{bmatrix}$ is Hurwitz, $\begin{bmatrix} R_{11} \\
0 & R_{22}
\end{bmatrix}$ is square, nonsingular and Hurwitz. Hence, the interconnection $\mathcal{P} \land_c \mathcal{E}$ is regular from Eq. (42), and $\mathcal{N}_{(w,c)}(\mathcal{P} \land_c \mathcal{E})$ is stable from Eq. (43).

It also follows that $v$ is free in $\mathcal{P} \land_c \mathcal{E}$. We have

$$\mathcal{P} \land_c \mathcal{E} \land_c \mathcal{C} = \begin{bmatrix}
R_{11} & R_{12} & R_{13} \\
0 & R_{22} & R_{23}
\end{bmatrix}.$$

(44)
Substituting Eq. (37) into the above equation yields
\[ \mathcal{P} \land _v \mathcal{E} \]
\[ = \begin{pmatrix} R_{11} \left( \frac{d}{dt} \right) w + R_{12} \left( \frac{d}{dt} \right) \left( c - X \left( \frac{d}{dt} \right) v \right) \\ + Y_1 V \left( \frac{d}{dt} \right) v = 0, \\
R_{22} \left( \frac{d}{dt} \right) \left( c - X \left( \frac{d}{dt} \right) v \right) + Y_2 V \left( \frac{d}{dt} \right) v = 0, \\
C \left( \frac{d}{dt} \right) \left( c - X \left( \frac{d}{dt} \right) v \right) + CV \left( \frac{d}{dt} \right) v = 0, \\
V \left( \frac{d}{dt} \right) v = 0 \end{pmatrix} . \]

It further follows from Eq. (41) that
\[ \mathcal{P} \land _v \mathcal{E} \]
\[ = \begin{pmatrix} R_{11} \left( \frac{d}{dt} \right) w + R_{12} \left( \frac{d}{dt} \right) \left( c - X \left( \frac{d}{dt} \right) v \right) \\ + Y_1 V \left( \frac{d}{dt} \right) v = 0, \\
R_{22} \left( \frac{d}{dt} \right) \left( c - X \left( \frac{d}{dt} \right) v \right) + Y_2 V \left( \frac{d}{dt} \right) v = 0, \\
C \left( \frac{d}{dt} \right) \left( c - X \left( \frac{d}{dt} \right) v \right) + CV \left( \frac{d}{dt} \right) v = 0, \\
V \left( \frac{d}{dt} \right) v = 0 \end{pmatrix} . \]

From the above, we see that, for all \((w, c, v) \in \mathcal{P} \land _v \mathcal{E} \land _c \mathcal{E},\)
\[ (w, c - X \left( \frac{d}{dt} \right) v) \text{ belongs to ker} \begin{pmatrix} R_{11} \left( \frac{d}{dt} \right) & R_{12} \left( \frac{d}{dt} \right) \\ 0 & R_{22} \left( \frac{d}{dt} \right) \end{pmatrix} , \]
\[ \text{ker} \begin{pmatrix} R_{11} \left( \frac{d}{dt} \right) & R_{12} \left( \frac{d}{dt} \right) \\ 0 & R_{22} \left( \frac{d}{dt} \right) \end{pmatrix} \]
\[ \text{is Hurwitz, } \lim_{t \to \infty} (w(t), c(t) - X \left( \frac{d}{dt} \right) v(t)) \]
\[ = 0 \text{ holds for all } (w, c, v) \in \mathcal{P} \land _v \mathcal{E} \land _c \mathcal{E}. \]

Thus, condition 3. requires that the behaviors \(\mathcal{N}(\mathcal{P}) \times \mathcal{E} \) and \(\mathcal{N}(\mathcal{E}) \land _c \mathcal{E} \land _c \mathcal{E} \) are isomorphic. The behaviors \(\mathcal{N}(\mathcal{P}) \land _v \mathcal{E} \) and \(\mathcal{N}(\mathcal{P}) \times \mathcal{E} \) are shown schematically in Fig. 7(a) and (b), respectively. Note that \(\mathcal{N}(\mathcal{P}) \land _c \mathcal{E} \) is the \((c, v)\)-behavior in the interconnected system of the plant and the exosystem, while \(\mathcal{N}(\mathcal{P}) \times \mathcal{E} \) is the \((c, v)\)-behavior in the disconnected system of the plant and the exosystem. Thus the condition can be interpreted as requiring that the behavior obtained after disconnecting the plant and the exosystem with \((w, v) = (0, 0)\) are isomorphic.

We will now outline an algorithmic procedure that, starting with polynomial kernel representations of \(\mathcal{P} \in \mathcal{L}^{p+c+c} \) and \(\mathcal{E} \in \mathcal{L}^{c+c} \), checks whether a regulator for \(\mathcal{P} \) with respect to \(\mathcal{E} \) exists. If there exists a regulator, the algorithm also gives a procedure to construct one.

**Algorithm 1.** Let \(R_1 \left( \frac{d}{dt} \right) w + R_2 \left( \frac{d}{dt} \right) c + R_3 \left( \frac{d}{dt} \right) v = 0 \) and \(V \left( \frac{d}{dt} \right) v = 0 \) be minimal kernel representations of \(\mathcal{P} \) and \(\mathcal{E} \), respectively, where \([R_1 \quad R_2] \) has full row rank and \(V \) is square and nonsingular. Then,
\[ (1) \text{ if } [R_1(\lambda) \quad R_2(\lambda)] \text{ has full row rank for all } \lambda \in \mathbb{C}^+ \text{ continue further, else declare there exists no regulator for } \mathcal{P} \text{ with respect to } \mathcal{E} . \]
\[ (2) \text{ if } R_1(\lambda) \text{ has full column rank for all } \lambda \in \mathbb{C}^+ \text{ continue further, else declare there exists no regulator for } \mathcal{P} \text{ with respect to } \mathcal{E} . \]
\[ (3) \text{ if } [R_1(\lambda) \quad V(\lambda)] \text{ has full column rank for all } \lambda \in \mathbb{C} \text{ continue further, else factorize } \]
\[ [R_1 \quad R_2] = [R_1' \quad R_2] D, \]
where \(D \) is square and nonsingular and \([R_1(\lambda) \quad V(\lambda)] \) has full column rank for all \(\lambda \in \mathbb{C} . \)
\[ \text{ Assign } R_3 = R_1' \text{ and } V = V' . \]
\[ (4) \text{ if } [R_1(\lambda) \quad R_2(\lambda)] \text{ has full column rank for all } \lambda \in \mathbb{C}^+ \text{ continue further, else declare there exists no regulator for } \mathcal{P} \text{ with respect to } \mathcal{E} . \]
(5) If $V$ is anti-Hurwitz continue further, else factorize $V = \sum U_1 U_2$ where $U_1, U_2$ are unimodular matrices and $\sum$ is Hurwitz and $\sum$ is anti-Hurwitz. Assign $V = \sum U_2$.

(6) Solve
\[ R_2 X + R_3 = YV \] (48)
for $(X, Y)$. If there exists no solution, declare there exists no regulator for $P$ with respect to $E$, else continue further.

(7) Choose a unimodular matrix $T$ such that
\[ T \begin{bmatrix} R_1 & R_2 & R_3 \end{bmatrix} = \begin{bmatrix} R_{11} & R_{12} & R_{13} \\ 0 & R_{22} & R_{23} \end{bmatrix}, \] (49)
where $R_{11}$ has full row rank. Factor $R_{22}$ as $R_{22} = D_1 K$ where $D_1$ is Hurwitz and $K(\lambda)$ has full row rank for all $\lambda \in \mathbb{C}$. Choose $S$ such that $[k]$ is unimodular.

(8) Solve
\[ \begin{bmatrix} F & M \end{bmatrix} \begin{bmatrix} R_{23} \\ V \end{bmatrix} = H S X \] (50)
for $(F, M)$, where $H$ is an arbitrary Hurwitz polynomial matrix.

(9) Define $C := FR_{22} + HS$. Then the controller $C$ defined by
\[ C \left( \frac{d}{dt} \right) c = 0 \] is a regular, admissible, stabilizing controller for $P$ with respect to $E$.

In order to illustrate the theory developed so far in this paper we now present some worked-out examples.

**Example 5.12.** Let the system $P$, with to-be-regulated variable $w$, interconnection variable $(c_1, c_2)$ and disturbance variable $v$, be given by
\[ \begin{bmatrix} 1 & \frac{d}{dt} + 3 & 1 & \frac{d}{dt} + 1 \\ \frac{d}{dt} + 2 & 0 & 0 & \frac{d}{dt} + 4 \end{bmatrix} \begin{bmatrix} w \\ c_1 \\ c_2 \\ v \end{bmatrix} = 0. \]

Let the exosystem $E$ with system variable $v$ be given by
\[ \frac{d}{dt} v = 0. \] (51)

Then $\mathcal{N}_{(w, c_1, c_2)}(P)$ and $P \land v, E$ are given by
\[ \begin{bmatrix} 1 & \frac{d}{dt} + 3 & 1 \\ \frac{d}{dt} + 2 & 0 & 0 \end{bmatrix} \begin{bmatrix} w \\ c_1 \\ c_2 \end{bmatrix} = 0. \]

and
\[ \begin{bmatrix} 1 & \frac{d}{dt} + 3 & 1 & \frac{d}{dt} + 1 \\ \frac{d}{dt} + 2 & 0 & 0 & \frac{d}{dt} + 4 \end{bmatrix} \begin{bmatrix} w \\ c_1 \\ c_2 \\ v \end{bmatrix} = 0. \]

respectively.

(1) It is easy to see that $w$ is detectable from $(c_1, c_2, v)$ in $P$ and $\mathcal{N}_{(w, c_1, c_2)}(P)$ is stabilizable. Therefore from Theorem 4.3 there exists a regular, admissible, stabilizing controller for $P$. It is easy to verify that $C = \{(c_1, c_2) | c_1 = 0\}$ is a regular, admissible, stabilizing controller for $P$.

(2) It is also easy to see that $E$ is an anti-stable system, $v$ is observable from $(w, c_1, c_2)$ in $P \land v, E$ and $(w, v)$ is detectable from $(c_1, c_2)$ in $P \land v, E$. There exists a polynomial matrix $\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \in \mathbb{R}[\xi]^{2 \times 1}$ such that $0, X_1 \left( \frac{d}{dt} \right) v, X_2 \left( \frac{d}{dt} \right) v, v \in P$ for all $v \in E$ if and only if there exist polynomial matrices $\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \in \mathbb{R}[\xi]^{2 \times 1}$ and $\begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} \in \mathbb{R}[\xi]^{2 \times 1}$ satisfying the equation
\[ \begin{bmatrix} \xi + 3 & 1 \\ 0 & \xi + 4 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} + \begin{bmatrix} \xi + 1 \\ \xi + 4 \end{bmatrix} = \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} (\xi - 1). \] (52)

As $\xi + 4 = Y_2 (\xi - 1)$ is not solvable for $Y_2 \in \mathbb{R}[\xi]$, Eq. (52) is also not solvable. Therefore from Theorem 5.9 there does not exist a regulator for $P$ with respect to $E$.

**Example 5.13.** Let the system $P$, with to-be-regulated variable $w$, interconnection variable $(c_1, c_2)$ and disturbance variable $v$, and the exosystem $E$ with system variable $v$ be given by
\[ \begin{bmatrix} R_{11} & \frac{d}{dt} \\ 0 & \frac{d}{dt} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} v \end{bmatrix}, \]
\[ V \left( \frac{d}{dt} \right) v = 0, \]
where $R_{11} = \xi + 2, R_{12} = \begin{bmatrix} 0 & 1 \end{bmatrix}, R_{13} = \xi + 1, R_{22} = \begin{bmatrix} \xi - 2 & -1 \end{bmatrix}, R_{23} = -\xi$ and $V = \xi - 1$.

(1) It is easy to see that $w$ is detectable from $(c_1, c_2, v)$ in $P$ and $\mathcal{N}_{(w, c_1, c_2)}(P)$ is stabilizable. Therefore from Theorem 4.3 there exists a regular, admissible, stabilizing controller for $P$. It is easy to verify that $C = \{(c_1, c_2) | c_1 = 0\}$ is a regular, admissible, stabilizing controller for $P$. 

It is also easy to see that \( E \) is an anti-stable system, \( v \) is observable from \((w, c_1, c_2) \in \mathcal{P} \land E\) and \( (w, v) \) is detectable from \((c_1, c_2) \in \mathcal{P} \land E\). There exists a polynomial matrix \( X_1, X_2 \in \mathbb{R}(\xi)^{1 \times 1} \) such that \( (0, X_1, \mathbb{R}(\xi)^{1 \times 1} v, X_2, \mathbb{R}(\xi)^{1 \times 1} v) \in \mathcal{P} \) for all \( \xi \in \mathbb{R}(\xi)^{1 \times 1} \) and \( \exists \xi \in \mathbb{R}(\xi)^{1 \times 1} \) satisfying the equation

\[
\begin{bmatrix}
0 & -1 & 1
\end{bmatrix}
\begin{bmatrix}
X_1 & X_2
\end{bmatrix}
+ \begin{bmatrix}
\xi + 1 & -\xi
\end{bmatrix}
\begin{bmatrix}
Y_1 & Y_2
\end{bmatrix}
(\xi - 1). \tag{53}
\]

It is easy to see that \( \begin{bmatrix}
X_1 & X_2
\end{bmatrix}
= \begin{bmatrix}
1 & 1
\end{bmatrix}
\) and \( \begin{bmatrix}
Y_1 & Y_2
\end{bmatrix}
= \begin{bmatrix}
0 & 0
\end{bmatrix}
\) is a solution to Eq. (53). Therefore, from Theorem 5.9 there exists a regulator for \( \mathcal{P} \) with respect to \( E \). We note here that the controller \( C = \{(c_1, c_2) | c_1 = 0\} \) is a regular, admissible, stabilizing controller for \( \mathcal{P} \) but not a regulator for \( \mathcal{P} \) with respect to \( E \).

Now we use Algorithm-1 to construct a regular, admissible, stabilizing controller of \( \mathcal{P} \) which also acts as a regulator for \( \mathcal{P} \) with respect to \( E \). As the conditions in steps 1–6 of Algorithm-1 are already satisfied, we here start from step 7. of Algorithm-1.

7. As \( R_2(\lambda) \) has full row rank for all \( \lambda \in \mathbb{C} \), we have \( K = R_2 \) and \( D = 1 \). Then \( S \) defined by \( S := \begin{bmatrix} 1 & 0 \end{bmatrix} \) satisfies the condition that \( \begin{bmatrix} K \\ S \end{bmatrix} \) is unimodular.

8. For the choice \( H = 1 \), we have \( HSX = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = 1 \). Then the solution to Eq. (50) is given by \( \begin{bmatrix} F \\ M \end{bmatrix} = \begin{bmatrix} -1 & 1 \end{bmatrix} \).

9. Then \( C = FR_2 + HS = -1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & -1 \end{bmatrix} \). The controller defined by

\[ C = \{(c_1, c_2) | -\frac{d}{dt}c_1 + 3c_1 - c_2 = 0 \}\]

is a regulator for \( \mathcal{P} \) with respect to \( E \).

**Remark 5.14.** In the problem formulation of this paper, the plant contains three kinds of variables, namely the variable \( w \) to be regulated, the control variable \( c \), and the exogenous variable \( v \). A possible extension is to include in the plant an additional variable, called \( w' \) that (like \( c \)) only needs to be taken to rest if the exogenous variable \( v \) is equal to zero. In this new setup, our plant \( \mathcal{P} \) has variables \((w', w, c, v)\), where \( v \) is free. The aim it to find a “modified regulator” for \( \mathcal{P} \), which is defined to be a regular controller \( C \in \mathbb{L}(\xi) \) such that

\[
\begin{align*}
(1) & \text{ } v \text{ is free in } \mathcal{P} \land C,
(2) & \text{ } (w', w, c, 0) \in \mathcal{P} \land C \implies (w'(t), w(t), c(t)) \to 0 \text{ as } t \to \infty,
(3) & \text{ } (w', w, c, v) \in \mathcal{P} \land C \implies (w(t)) \to 0 \text{ as } t \to \infty.
\end{align*}
\]

It can be shown that this new condition can be reduced to the problem studied in this paper. This can be done by eliminating the new variable \( w' \) from \( \mathcal{P} \), thus obtaining the system \((\mathcal{P})_{w, c, v} \in \mathbb{L}(\xi) \land C \) can then be shown that \( C \in \mathbb{L}(\xi) \) is a “modified regulator” for the extended plant \( \mathcal{P} \) if and only if \( C \) is a regulator (in the sense of this paper) for the projected plant \((\mathcal{P})_{w, c, v} \). For details, we refer to Fiaz (2010).

The above allows us to apply the results of this paper to the classical regulator problem in the input-state-output setting (see Francis, 1977; Francis & Wonham, 1975). In that case, apart from the to be regulated output \( w \), the control variable \((u, y) \) and the exogenous variable \( v \), the plant contains the state variable \( x \) which has to driven to zero if the exogenous signal \( v \) is equal to zero. The classical results in this context can thus be reobtained by applying the results from this paper. Again, for details we refer to Fiaz (2010).

6. Conclusions

In this paper we have formulated and resolved the problem of asymptotic tracking and regulation in a completely representation-free manner. We have used the theory of behavioral control for this purpose. In the behavioral context, controllers act on the plant using general interconnection, without a priori input-output partitions. Given a plant and an exosystem, we have established necessary and sufficient conditions for the existence of a regulator only in terms of the plant and exosystem dynamics.

**References**


Dr. Shaik Fiaz received the Dutch Institute of Systems and Control (DISC) Best Ph.D. Thesis award for the best PhD thesis defended in 2010 in The Netherlands in the area of systems and control.

K. Takaba received his B.Eng., M.Eng., and Dr.Eng. degrees from Kyoto University, Japan, in 1989, 1991, and 1996, respectively. In 1991, he joined the faculty of Kyoto University, where he is currently an Associate Professor of Department of Applied Mathematics and Physics. His research interests include robust/optimal control, behavioral approach to systems and control, constrained control systems, and networked control systems. He is a member of IFAC and IEEE.

H.L. Trentelman is a full professor in Systems and Control at the Johann Bernoulli Institute for Mathematics and Computer Science of the University of Groningen in The Netherlands. From 1991 to 2008, he served as an associate professor and later as an adjoint professor at the same institute. From 1985 to 1991, he was an assistant professor, and later, an associate professor at the Mathematics Department of the University of Technology at Eindhoven, The Netherlands. He obtained his Ph.D. degree in Mathematics at the University of Groningen in 1985. His research interests are the behavioral approach to systems and control, robust control, model reduction, multi-dimensional linear systems, hybrid systems, analysis and control of networked systems, and the geometric theory of linear systems. He is a co-author of the textbook “Control Theory for Linear Systems” (Springer, 2001). Dr. Trentelman is an associate editor of the IEEE Transactions on Automatic Control and of Systems and Control Letters, and is past associate editor of the SIAM Journal on Control and Optimization.