Robust Synchronization of Uncertain Linear Multi-Agent Systems

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Outline

1. Synchronization

2. Robust synchronization

3. Computation of robustly synchronizing protocols

4. Guaranteed robust synchronization radius

5. Future research
Agent dynamics

Multi-agent network with $p$ agents, communication topology represented by the network graph. Vertices represent the agents, edges represent the communication. Identical nominal dynamics of the agents. Agent $i$ has dynamics

$$\dot{x}_i = Ax_i + Bu_i, \quad y_i = Cx_i, \quad i = 1, 2\ldots, p$$

$(A, B)$ is stabilizable, $(C, A)$ is detectable. State $x_i \in \mathbb{R}^n$, input $u_i \in \mathbb{R}^m$, output $y_i \in \mathbb{R}^q$
Synchronization of multi-agent systems

Communication topology

Undirected or directed graph with \( p \) edges, with Laplacian

\[
L = D - A
\]

Degree matrix \( D = \text{diag}(d_1, d_2 \ldots d_p) \) with \( d_i \) the number of edges into node \( i \).

Adjacency matrix \( A = (a_{ij}) \) with

\[
a_{ij} = \begin{cases} 
1 & \text{if there is an edge from } j \text{ into } i \\
0 & \text{otherwise}
\end{cases}
\]

Neighbouring set of agent \( i \) is \( \mathcal{N}_i := \{j \mid \text{there is an edge from } j \text{ into } i\} \).

Information of agent \( i \) about its neighbours is \( \sum_{j \in \mathcal{N}_i} (y_i - y_j) \).
The synchronization problem is the problem of finding a protocol that makes the network synchronized. We consider dynamic protocols of the form

\[ \dot{w}_i = Aw_i + BF \sum_{j \in N_i} (w_i - w_j) + G \left( \sum_{j \in N_i} (y_i - y_j) - Cw_i \right), \quad u_i = Fw_i. \]

Structure of the protocol: combination of observer for the \( i \)th relative state \( \sum_{j \in N_i} (x_i - x_j) \) and static feedback of the estimate \( w_i \) of this relative state.

Error \( e_i := w_i - \sum_{j \in N_i} (x_i - x_j) \) satisfies \( \dot{e}_i = (A - GC)e_i \)

Note: design parameters are the gains \( F \) and \( G \).
Network dynamics

Interconnecting the agents using this protocol yields the closed loop dynamics of the overall network. Denote $\mathbf{x} = \text{col}(x_1, x_2, \ldots, x_p)$, $\mathbf{w} = \text{col}(w_1, w_2, \ldots, w_p)$. Network dynamics:

$$
\begin{pmatrix}
\dot{x} \\
\dot{w}
\end{pmatrix} =
\begin{pmatrix}
I \otimes A \\
L \otimes GC \\
I \otimes (A - GC) + (L \otimes BF)
\end{pmatrix}
\begin{pmatrix}
x \\
w
\end{pmatrix}
$$

Definition

The network is said to be synchronized by the dynamic protocol if for all $i, j = 1, 2, \ldots, p$ we have $x_i(t) - x_j(t) \to 0$ and $w_i(t) - w_j(t) \to 0$ as $t \to \infty$. 
Synchronization of multi-agent systems

Laplacian eigenvalues

**Undirected** graph: $L$ real symmetric. Graph connected $\iff L$ has rank $p - 1$. In that case the eigenvalue 0 has multiplicity one. Remaining $p - 1$ eigenvalues: $0 < \lambda_2 \leq \lambda_3 \leq \ldots \leq \lambda_p$.

**Directed** graph: $L$ complex eigenvalues. Graph contains a spanning tree $\iff L$ has rank $p - 1$. In that case the eigenvalue 0 has multiplicity one. Remaining $p - 1$ eigenvalues: $\lambda_2, \lambda_3, \ldots, \lambda_p$ and $\Re(\lambda_i) > 0$.

Theorem

The protocol $\dot{w}_i = Aw_i + BF \sum_{j \in \mathcal{N}_i} (w_i - w_j) + G (\sum_{j \in \mathcal{N}_i} (y_i - y_j) - Cw_i)$, $u_i = Fw_i$. synchronizes the network if and only if for $i = 2, 3, \ldots, p$ the matrices

$$
\begin{pmatrix}
A & \lambda_i BF \\
GC & A - GC + \lambda_i BF
\end{pmatrix}
$$

are Hurwitz.
Corollary

The protocol synchronizes the network if and only if the single linear system

\[ \dot{x} = Ax + Bu, \quad y =Cx \]

is internally stabilized by all \( p - 1 \) feedback controllers

\[ \dot{w} = Aw + Bu + G(y - Cw), \quad u = \lambda_i Fw, \quad i = 2, 3, \ldots, p. \]

This holds if and only if \( A - GC \) and \( A + \lambda_i BF \) \((i = 2, 3, \ldots, p)\) are Hurwitz. Such \( F \) and \( G \) exist if and only if \((C, A)\) is detectable and \((A, B)\) is stabilizable.
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Additive uncertainty

Idea: agent dynamics is uncertain (→ heterogeneity) Here: additive perturbations.

Nominal transfer matrix of agent $i$: $G(s) = C(sI - A)^{-1}B$ perturbed to $G(s) + \Delta_i(s)$, $\Delta_i \in R \mathcal{H}_\infty$.

This means that the dynamics of agent $i$ is perturbed to the system obtained by interconnecting

$$\dot{x}_i = Ax_i + Bu_i, \; y_i = Cx_i + d_i, \; z_i = u_i$$

with $d_i = \Delta_i z_i$.

Uncertainty radius: $\eta > 0$ given, perturbations $\Delta_i \in R \mathcal{H}_\infty$ with $\|\Delta_i\|_\infty \leq \eta$.

The dynamics of agent $i$ is any system with transfer matrix of the form $G + \Delta_i$ with $\|\Delta_i\|_\infty \leq \eta$. 
Robust synchronization

Definition

Given $\eta > 0$, the problem of robust synchronization is to find a dynamic protocol such that for all $i$ and for all $\Delta_i \in R \mathcal{H}_\infty$ with $\|\Delta_i\|_\infty \leq \eta$ the network is synchronized, i.e. for all $i,j = 1,2\ldots,p$ we have $x_i(t) - x_j(t) \to 0$ and $w_i(t) - w_j(t) \to 0$ as $t \to \infty$.

Weighted protocol

For robust synchronization we modify the earlier protocol to include a weighting factor on $L$:

$$\dot{w}_i = Aw_i + BF \sum_{j \in \mathcal{N}_i} \frac{1}{N} (w_i - w_j) + G\left( \sum_{j \in \mathcal{N}_i} \frac{1}{N} (y_i - y_j) - Cw_i \right), \quad u_i = Fw_i.$$

$N$ is a positive real number that, next to $F$ and $G$, needs to be determined.
Theorem

Let $\eta > 0$. The following two statements are equivalent:

1. The weighted dynamic protocol (depending on $F, G$ and $N$) synchronizes the network with perturbed agent dynamics

\[ \dot{x}_i = Ax_i + Bu_i, \quad y_i = Cx_i + d_i, \quad z_i = u_i, \quad d_i = \Delta_i z_i \]

for all $\Delta_i \in \mathbb{R} \mathcal{H}_\infty$ with $\|\Delta_i\|_\infty \leq \eta$

2. the single perturbed linear system

\[ \dot{x} = Ax + Bu, \quad y = Cx + d, \quad z = u, \quad d = \Delta z \]

is internally stabilized for all $\Delta \in \mathbb{R} \mathcal{H}_\infty$ with $\|\Delta\|_\infty \leq \eta$ by all $p - 1$ feedback controllers

\[ \dot{w} = Aw + Bu + G(y - Cw), \quad u = \frac{1}{N} \lambda_i F w, \quad i = 2, 3, \ldots, p \]
Robust synchronization

Simultaneous $\mathcal{H}_\infty$-controllers

By the small gain theorem: given $\eta > 0$, find $N \in \mathbb{R}$, and gain matrices $F$ and $G$ such that all $p-1$ controllers solve the $\mathcal{H}_\infty$-control problem for the system $\dot{x} = Ax + Bu$, $y = Cx + d$, $z = u$:

- the interconnection of the system with the $i$th controller is internally stable
- $\|G_i\|_\infty < \frac{1}{\eta}$, where $G_i$ is the closed loop transfer matrix from $d$ to $z$ of the interconnection of the system with the $i$th controller.

In the sequel, we will explain how to obtain such $N$, $F$ and $G$. 
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Riccati equation and inequality

Associated with \((A, B, C)\) consider the ARE

\[
A^\top P + PA - \gamma PBB^\top P = 0
\]

together with the strict Riccati inequality

\[
AQ + QA^\top - QC^\top CQ < 0.
\]

\(\gamma > 0\) to be determined. Let \(P(\gamma) \geq 0\) be the maximal real symmetric solution. \(A - \gamma BB^\top P(\gamma)\) is Hurwitz (assuming \(A\) has no eigenvalues on the imaginary axis). Let \(Q > 0\) be any solution of the inequality.
Choose \( N > \frac{\lambda_2}{\lambda_p} \), equivalently \((\frac{\lambda_p}{N})^2 < \frac{\lambda_2}{N}\). Choose \( \gamma \) such that
\[
(\frac{\lambda_p}{N})^2 < \gamma < \frac{\lambda_2}{N}
\]
Let \( \eta > 0 \) be such that
\[
\eta < \frac{1}{\sqrt{\rho(P(\gamma)Q)}}.
\]
Define
\[
F := -B^\top P(\gamma), \quad G := (I - \eta^2 QP(\gamma))^{-1} QC^\top
\]
Then the dynamic protocol with this \( N, F \) and \( G \) synchronizes the network for all perturbations \( \Delta_i \in \mathbb{R} H_\infty \) \((i = 1, 2 \ldots, p)\) with \( \| \Delta_i \|_\infty \leq \eta \).
Let $\lambda_2, \lambda_2, \ldots, \lambda_p$ the nonzero eigenvalues of $L$. Within this set let $\lambda_m$ have minimal real part, $\lambda_M$ have maximal modulus, and $\lambda_\ell$ have maximal argument, i.e.:

$$\text{Re}(\lambda_m) = \min_{i=2,\ldots,p} \text{Re}(\lambda_i),$$

$$|\lambda_M| = \max_{i=2,\ldots,p} |\lambda_i|,$$

$$\text{Arg}(\lambda_\ell) = \max_{i=2,\ldots,p} \text{Arg}(\lambda_i).$$

Note: $-\pi/2 < \text{Arg}(\lambda_i) < \pi/2$. 

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Main theorem, directed graphs

For $N > 0$, define

$$f_N := \frac{\text{Re}(\lambda_m)}{|\lambda_M|^2} N.$$

Choose $N$ such that

$$f_N > 1, \text{ and } f_N + \frac{1}{f_N} > 2 + 4 \tan^2(\text{Arg}(\lambda_\ell))$$

(such $N$ always exists!). Choose

$$\gamma = \frac{1}{2} \left( \frac{|\lambda_M|^2}{N^2} + \frac{\text{Re}(\lambda_m)}{N} \right)$$

Let $P(\gamma)$ maximal solution of the ARE, $Q > 0$ any solution of the strict Riccati inequality,
Let $\eta > 0$ be such that

$$\eta < \frac{1}{\sqrt{\rho(P(\gamma)Q)}}$$

Define

$$F := -B^T P(\gamma), \quad G := (I - \eta^2 QP(\gamma))^{-1} QC^T$$

Then the dynamic protocol with this $N, F$ and $G$ synchronizes the network for all perturbations $\Delta_i \in \mathcal{RH}_\infty$ ($i = 1, 2, \ldots, p$) with $\|\Delta_i\|_\infty \leq \eta$. 
Computation of robustly synchronizing protocols

Example

Directed cycle graph with three vertices:

\[
L = \begin{pmatrix}
1 & 0 & -1 \\
-1 & 1 & 0 \\
0 & -1 & 1 \\
\end{pmatrix}
\]

Nonzero eigenvalues: \( \lambda_2 = \frac{3}{2} + j\frac{1}{2}\sqrt{3} \), \( \lambda_3 = \frac{3}{2} - j\frac{1}{2}\sqrt{3} \). \( \text{Re}(\lambda_m) = \frac{3}{2} \), \( |\lambda_M|^2 = 3 \) and \( \tan^2(\text{Arg}(\lambda_\ell)) = \frac{1}{3} \). Thus \( f_N = \frac{N}{2} \).

Conditions on \( N \): \( \frac{N}{2} > 1 \) and \( \frac{N}{2} + \frac{2}{N} > \frac{10}{3} \), equivalently \( N > 6 \).

As an example take \( N = 10 \). Then take \( \gamma = \frac{1}{2} \left( \frac{3}{N^2} + \frac{3}{2N} \right) = 0.09 \), etc.
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Question: for a given a multi-agent system,

- what is its guaranteed robust synchronization radius, i.e. the supremum over all values of $\eta > 0$ such that a suitable weighted dynamic protocol achieves synchronization for all $\Delta_i$ with $\|\Delta_i\|_\infty \leq \eta$

- How does it depend on the network graph?

Here: only the undirected graph case. It turns out that a guaranteed radius can be found that is proportional to the quotient $\lambda_2/\lambda_p$ of the second smallest and largest eigenvalue of the Laplacian:
Theorem, undirected graphs

Let $P^+$ and $Q^+$ be the maximal real symmetric solutions of the Riccati equations

$$A^TP + PA - PB^TBP = 0, \quad AQ + QA^T - QC^TCQ = 0$$

Then for each $\eta > 0$ that satisfies

$$\eta < \frac{\lambda_2}{\lambda_p} \frac{1}{\sqrt{\rho(P^+Q^+)}}$$

there exists a dynamic protocol achieving synchronization for all perturbations $\Delta_i \in \mathcal{RH}_\infty$ with $\|\Delta_i\|_\infty \leq \eta$. 

Guaranteed robust synchronization radius
Guaranteed robust synchronization radius

Some special cases

- **Complete graphs** $\lambda_2 = \lambda_p = p$. Take $N > p$, and $\frac{p^2}{N^2} < \gamma < \frac{p}{N}$. $\frac{\lambda_2}{\lambda_p} = 1$, which is maximal.

- **Star graphs** $\lambda_2 = 1$ and $\lambda_p = p$. Take $N > p^2$ and $\frac{p^2}{N^2} < \gamma < \frac{1}{N}$. $\frac{\lambda_2}{\lambda_p} = \frac{1}{p}$, decreases with increasing number of agents.

- **Line graphs** $\lambda_2 = 2(1 - \cos \frac{\pi}{p})$ and $\lambda_p = 2(1 + \cos \frac{\pi}{p})$. For large $p$ we have $\lambda_2 \approx 0$ and $\lambda_p \approx 4$. $N$ will then be very large, while $\gamma$ will be very small. $\frac{\lambda_2}{\lambda_p}$ is small for large $p$.

- **Cycle graphs** $\lambda_2 = 2(1 - \cos \frac{2\pi}{p})$ and

\[
\lambda_p = \begin{cases} 
4 & p \text{ even} \\
2(1 + \cos \frac{\pi}{p}) & p \text{ odd} 
\end{cases}
\]

For large $p$ we have $\lambda_2 \approx 0$ and $\lambda_p \approx 4$. $\frac{\lambda_2}{\lambda_p}$ will be small for large $p$. 

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Future research

- Different types of uncertainty: coprime factor uncertainty, multiplicative uncertainty
- Nominal linear dynamics but nonlinear perturbations
- Extension to multi-agent systems with nonlinear nominal dynamics

THANK YOU!