Projection based model reduction of multi-agent systems using graph partitions

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The University of Tokyo, 2014
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2. Leader-Follower Multi-Agent Systems
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5. $H_2$ Analysis of the Model Reduction Error and Almost Equitable Partitions
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General problem: given a large scale networked multi-agent system with leaders and followers, approximate it by a lower order network, while preserving (some of) the network structure.

Does the approximated network preserve important properties such as consensus?

How good is the approximation? Can one compute (upper and lower bounds on) the error?
Graph theory

- Weighted directed graph $G = (V, E, A)$, $V = \{1, 2, \ldots, n\}$, $E$ set of arcs, adjacency matrix $A$.
- Laplacian of $G$: $L = D - A$, $D = \text{diag}(d_1, d_2, \ldots, d_n)$, $d_i = \sum_j a_{ij}$.
- Incidence matrix of $G$: $R = [r_{ij}]$ with
  
  $$r_{ij} = \begin{cases} 
  1 & \text{if vertex } i \text{ is the head of arc } j \\
  -1 & \text{if vertex } i \text{ is the tail of arc } j \\
  0 & \text{otherwise}
  \end{cases}$$

  for $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, k$ ($k$ is the total number of arcs).
- Incidence matrix undirected graph: assign arbitrary orientation to the edges and take the incidence matrix of the corresponding directed graph.
- Edge weight matrix: $W = \text{diag}(w_1, w_2, \ldots, w_k)$ with $w_j$ is the weight associated to the edge (arc) $j$, $j = 1, 2, \ldots, k$.
- For undirected graphs Laplacian $L$ then $L = RWR^\top$. 
Leader follower multi-agent systems

$G = (V, E, A)$ weighted undirected graph, $A = [a_{ij}]$, Assume $G$ connected graph.

$V_L = \{v_1, v_2, \ldots, v_m\} \subset V$ leaders, $V_F := V \setminus V_L$ followers.

Leader-follower multi-agent system:

$$\dot{x}_i = \begin{cases} 
\sum_{j=1}^{n} a_{ij}(x_j - x_i) & \text{if } i \in V_F \\
\sum_{j=1}^{n} a_{ij}(x_j - x_i) + u_\ell & \text{if } i \in V_L
\end{cases}$$

$x_i \in \mathbb{R}$ state of agent $i$, $u_\ell \in \mathbb{R}$ external input applied to agent $i = v_\ell$. In compact form

$$\dot{x} = -Lx + Mu,$$

with $L$ the Laplacian of $G$, $M$ is given by

$$M_{i\ell} = \begin{cases} 
1 & \text{if } i = v_\ell \\
0 & \text{otherwise.}
\end{cases}$$
\[
\dot{x} = -Lx + Mu
\]

\[
L = \begin{bmatrix}
5 & 0 & 0 & 0 & 0 & 0 & -5 & 0 & 0 & 0 \\
0 & 5 & 0 & 0 & -3 & -2 & 0 & 0 & 0 & 0 \\
0 & 0 & 6 & -1 & -2 & -3 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 6 & -5 & 0 & 0 & 0 & 0 & 0 \\
0 & -3 & -2 & -5 & 25 & -2 & -6 & -7 & 0 & 0 \\
-5 & -2 & -3 & 0 & -2 & 25 & -6 & -7 & 0 & 0 \\
0 & 0 & 0 & 0 & -6 & -6 & 15 & -1 & -1 & -1 \\
0 & 0 & 0 & 0 & -7 & -7 & -1 & 15 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 \\
\end{bmatrix}, \quad M = \begin{bmatrix}
1 & 0 \\
0 & 0 \\
0 & 0 \\
0 & 0 \\
0 & 0 \\
0 & 0 \\
0 & 0 \\
0 & 0 \\
0 & 0 \\
0 & 0 \\
\end{bmatrix}.
\]
Problem Statement

**Problem**

Approximate the leader-follower multi-agent system

\[ \dot{x} = -Lx + Mu \]

by a reduced order multi-agent system

\[ \dot{\hat{x}} = -\hat{L}x + \hat{M}u \]

such that the spatial structure of the network is preserved.

**Idea**

We want to do this by reducing the size of the network graph through clustering of the agents/vertices.
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Petrov-Galerkin Projections

Consider an arbitrary system

\[ \dot{x} = Ax + Bu, \ y = Cx \]

with state space \( \mathbb{R}^n \). Let \( W, V \in \mathbb{R}^{n \times r} \) (with \( r < n \)) such that \( W^\top V = I \). Then \( VW^\top \) is a projection.

A reduced order model (projected model) is then given by

\[ \dot{\hat{x}} = W^\top AV\hat{x} + W^\top Bu, \ y = CV\hat{x} \]

with reduced order state space \( \mathbb{R}^r \).

General reduction framework: Krylov based, truncation methods, moment matching use this projection with appropriate choice of matrices \( V \) and \( W \).

Idea

Put idea of clustering of network graph into the Petrov-Galerkin framework.
Graph partitions

- $V = \{1, 2, \ldots, n\}$ vertex set of a graph $G$.
- Any nonempty subset of $V$ is called a cell of $V$.
- Collection of cells, given by $\pi = \{C_1, C_2, \ldots, C_r\}$, called a partition of $V$ if $\bigcup_i C_i = V$ and $C_i \cap C_j = \emptyset$ whenever $i \neq j$.
- For a cell $C \subset V$, the characteristic vector of $C$ is the $n$-dimensional column vector $p(C)$ with

$$p_i(C) = \begin{cases} 1 & \text{if } i \in C, \\ 0 & \text{otherwise.} \end{cases}$$

- For a partition $\pi = \{C_1, C_2, \ldots, C_r\}$, the characteristic matrix of $\pi$ is

$$P(\pi) = [p(C_1) \ p(C_2) \ \cdots \ p(C_r)].$$
Vertex set: $V = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$.

Cell: any nonempty subset of $V$

Partition: $\pi = \{\{1, 2, 3, 4\}, \{5, 6\}, \{7\}, \{8\}, \{9, 10\}\}$

Characteristic matrix:

$$P(\pi) = \begin{bmatrix}
1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1
\end{bmatrix}^T.$$
A direct application of Petrov-Galerkin projections will destroy the spatial structure of the network (e.g. balanced truncation, moment matching).

Instead: choose a graph partition, characteristic matrix $P(\pi)$.

$$W := P(\pi)(P^\top(\pi)P(\pi))^{-1}$$

$$V := P(\pi).$$

Corresponding reduced order system:

$$\dot{\hat{x}} = -\hat{L}\hat{x} + \hat{M}u,$$

$$\hat{L} = (P^\top P)^{-1}P^\top LP$$
$$\hat{M} = (P^\top P)^{-1}P^\top M$$
\( \hat{L} \) is the Laplacian of a weighted \textit{directed} graph, \( \hat{G} = (\hat{V}, \hat{E}, \hat{A}) \).

Each cell of \( \pi \) in \( G \) becomes a vertex in \( \hat{G} \).

There is an arc from vertex \( p \) to vertex \( q \) in \( \hat{G} \) if and only if there exist \( i \in C_p \) and \( j \in C_q \) with \( p \neq q \) such that \( \{i, j\} \in E \).

Therefore, \( \hat{G} \) is a \textit{symmetric} \textit{directed} graph, i.e. \( (i, j) \in \hat{E} \iff (j, i) \in \hat{E} \).

Relationship between \( A \) and \( \hat{A} = [\hat{a}_{pq}] \) given by

\[
\hat{a}_{pq} = \frac{1}{|C_p|} \sum_{i \in C_p, j \in C_q} a_{ij},
\]

for \( p \neq q \).

The input weights in \( \hat{M} \) depend on the cardinality of the leader cells.
Relationship between the original and the reduced order model

The reduced order model: \( \dot{x} = -\hat{L}\dot{x} + \hat{M}u \)

\( \hat{L} = \begin{bmatrix} 5 & -5 & 0 & 0 & 0 \\ -10 & 23 & -6 & -7 & 0 \\ 0 & -12 & 15 & -1 & -2 \\ 0 & -14 & -1 & 15 & 0 \\ 0 & 0 & -1 & 0 & 1 \end{bmatrix} \), \( \hat{M} = \begin{bmatrix} 0.25 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \)
Relationship between the original and the reduced order model

- **Original graph:**

- **Reduced graph:**
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Preservation of consensus

- We have assumed that our weighted undirected graph $G$ is connected. Hence the multi-agent system $\dot{x} = -Lx$ reaches consensus, i.e. for all $i,j \in V$ we have $x_i(t) - x_j(t) \to 0$ as $t \to \infty$.

- Since $\hat{L} = (P^T P)^{-1} P^T L P$, we see that 
  
  \[(P^T P)^{1/2} \hat{L} (P^T P)^{-1/2} = (P^T P)^{-1/2} P^T L P (P^T P)^{-1/2}\]

  Thus $\hat{L}$ is similar to a symmetric matrix, and hence has only real eigenvalues.

- Moreover: the eigenvalues $\hat{\lambda}_i$ of $\hat{L}$ interlace the eigenvalues $\lambda_i$ of $L$: 
  \[\lambda_i \leq \hat{\lambda}_i \leq \lambda_{n-r+i} \text{ for } i = 1, 2, \ldots, r.\]

- In particular $\lambda_2 \leq \hat{\lambda}_2$ so the reduced order system $\dot{\hat{x}} = -\hat{L}\hat{x}$ reaches consensus with rate of convergence at least as fast as the original network.
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Recall $G = (V, E, A)$ undirected weighted graph. Incidence matrix $R$, edge weighting matrix $W$.

Original model: $\dot{x} = -Lx + Mu$

We add output variables as: $y = W^{1/2}R^Tx$

$\|y\|^2 = x^T L x$

Example:

\[
y = \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix}^{1/2} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}
\]

Output variables after projection: $y = W^{1/2}R^TP\hat{x}$

In this example:

\[
y = \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix}^{1/2} \begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix}
\]
Model reduction error

- **Original model:**
  \[
  \dot{x} = -Lx + Mu \\
y = W^{\frac{1}{2}} R^\top x
  \]

- **Reduced model:**
  \[
  \hat{x} = -\hat{L}\hat{x} + \hat{M}u \\
y = W^{\frac{1}{2}} \hat{R}^\top x
  \]

- Transfer matrices \( S(s) = W^{\frac{1}{2}} R^\top (sl + L)^{-1} M \) and \( \hat{S}(s) = W^{\frac{1}{2}} \hat{R}^\top (sl + \hat{L})^{-1} \hat{M} \)

- Approximation error \( \| S - \hat{S} \|_2 \) (\( \mathcal{H}_2 \)-norm). Note \( S \) and \( \hat{S} \) are in \( \mathcal{H}_2 \).

- Problem: for a given partition, find an expression for this error. Find a priori upper or lower bounds.

- First for *almost equitable partitions*.
Almost equitable partitions

**Definition**

Let \( G = (V, E) \) be a weighted undirected graph. Recall \( a_{ij} \) indicates the weight associated to the edge \( \{i, j\} \). Partition \( \pi = \{C_1, C_2, \ldots, C_r\} \) called an *almost equitable partition* (AEP) of \( G \) if for each \( p, q \in \{1, 2, \ldots, r\} \) with \( p \neq q \) there exists \( d_{pq} \in \mathbb{R} \) such that 

\[
\sum_{j \in N(i, C_q)} a_{ij} = d_{pq}
\]

for all \( i \in C_p \).
An almost equitable partition can be best described by an example:

\[ \pi = \{\{1, 2, 3, 4\}, \{5, 6\}, \{7\}, \{8\}, \{9, 10\}\} \] is an almost equitable partition
Let \( \pi = \{C_1, C_2, \ldots, C_r\} \) be an AEP of \( G \), \( V_L = \{v_1, v_2, \ldots, v_m\} \) leader vertices.

Let \( k_i \) be s.t. \( v_i \in C_{k_i} \) (\( i = 1, 2 \ldots, m \)).

**Theorem**

The (normalized) model reduction error is given by

\[
E(\pi) = \frac{\|S - \hat{S}\|_2^2}{\|S\|_2^2} = \frac{\sum_{i=1}^{m}(1 - \frac{1}{|C_{k_i}|})}{m(1 - \frac{1}{n})}
\]
Let $V_L = \{1, 7\}$

Then,

$$E(\pi) = \frac{\sum_{i=1}^{m} (1 - \frac{1}{|C_{k_i}|})}{m \left(1 - \frac{1}{n}\right)} = \frac{(1 - \frac{1}{4}) + (1 - \frac{1}{1})}{2 \left(1 - \frac{1}{10}\right)} = \frac{5}{12}$$
Outline of the proof

- \( S(s) = W^{\frac{1}{2}} R^\top (sl + L)^{-1} M \) and \( \hat{S}(s) = W^{\frac{1}{2}} \hat{R}^\top (sl + \hat{L})^{-1} \hat{M} \).
- \( \Delta(s) := S(s) - \hat{S}(s) \). Recall \( E(\pi) = \|\Delta\|_2^2 / \|S\|_2^2 \).
- \( \pi \) almost equitable partition \( \Leftrightarrow \text{im}(P) \) is \( L \)-invariant \( \Rightarrow \Delta^\top (-s) \hat{S}(s) = 0 \).
- Using this: \( \|S\|_2^2 = \|\hat{S}\|_2^2 + \|\Delta\|_2^2 \).
- \( \|S\|_2^2 = \text{trace}(M^\top X_1 M) \) and \( \|\hat{S}\|_2^2 = \text{trace}(\hat{M}^\top Y_1 \hat{M}) \), where \( X_1 \) and \( Y_1 \) are the Grammians of \((W^{\frac{1}{2}} R^\top, L)\) and \((W^{\frac{1}{2}} \hat{R}^\top, \hat{L})\), respectively.
- \( X_1 = \frac{1}{2} I_n - \frac{1}{2n} 1_n 1_n^\top \), \( Y_1 = P^T X_1 P \).

\[ \Rightarrow \|S\|_2^2 = \frac{m}{2} (1 - \frac{1}{n}) \] and \[ \|\hat{S}\|_2^2 = \frac{1}{2} \sum_{i=1}^{m} \frac{1}{|C_{k_i}|} - \frac{m}{2n} \]

\[ \|\Delta\|_2^2 = \frac{m}{2} (1 - \frac{1}{n}) - \frac{1}{2} \sum_{i=1}^{m} \frac{1}{|C_{k_i}|} + \frac{m}{2n} = \frac{1}{2} \sum_{i=1}^{m} (1 - \frac{1}{|C_{k_i}|}) \]
Remarks

- $0 \leq E(\pi) \leq 1$
- $E(\pi) = 0$ for $\pi = \{\{1\}, \{2\}, \ldots, \{n\}\}$.
- $E(\pi) = 1$ for $\pi = \{V\}$.
- $E(\pi)$ determined by the cardinalities of those cells in $\pi$ that contain the leaders.

Corollary

If $\pi$ is an almost equitable partition with the property that $\{v_i\} \in \pi$ for all $i = 1, 2 \ldots, m$ (i.e. each leader is the unique element in its cell) then $E(\pi) = 0$. 
What if we start with an *arbitrary* (not necessarily AEP) partition? In that case the closed form expression fails. But: can we make an estimate of $E(\pi)$?

$\pi = \{\{1, 2, 3, 4, 5, 6\}, \{7, 8\}, \{9, 10\}\}$ is not an almost equitable partition.
Let $G = (V, E, A)$ be a weighted undirected graph. Partial ordering on all partitions of $G$. Let $\pi_1$ and $\pi_2$ be two partitions. We say $\pi_1 \leq \pi_2$ if every cell in $\pi_1$ is a subset of a cell in $\pi_2$: $\pi_1$ is finer than $\pi_2$, $\pi_2$ is coarser than $\pi_1$.

In terms of the characteristic matrices: $\pi_1 \leq \pi_2$ if and only if $\text{im}(P(\pi_2)) \subseteq \text{im}(P(\pi_1))$.

Using the fact that for any AEP $\pi_0$, $\text{im}(P(\pi_0))$ is $L$-invariant we obtain:

**Theorem**

Let $\pi_0$ be an AEP of $G$. Then for every partition $\pi$ that is coarser than $\pi_0$ we have $E(\pi_0) \leq E(\pi)$. 
Let $\pi$ be an arbitrary partition of $G$. Idea: find a finer AEP of $G$ which is "as close as possible" to $\pi$.

Define $\Pi_{\text{AEP}}(\pi) := \{\pi_0 \mid \pi_0 \text{ AEP and } \pi_0 \leq \pi\}$.

Since "\(\leq\)" is a partial ordering, this set contains a unique maximal element $\pi^*_{\text{AEP}}(\pi)$, the maximal almost equitable partition finer than $\pi$. Thus we get:

**Corollary**

Let $\pi$ be a partition of $G$ and let $\pi^*_{\text{AEP}}(\pi)$ be the maximal AEP finer than $\pi$. Then $E(\pi^*_{\text{AEP}}(\pi)) \leq E(\pi)$. 
Let $G$ be a weighted undirected graph with vertex set $V = \{1, 2, \ldots, n\}$ and $\pi$ a partition. How to compute $\pi_{\text{AEP}}^*(\pi)$?

Let $\Pi$ be the set of all partitions of $G$. Define a map $\Psi : \mathbb{R}^{n\times \bullet} \to \Pi$ as follows: for $X \in \mathbb{R}^{n\times \bullet}$, $i, j \in V$ are in the cell $\Psi(X)$ if and only if the $i$th and $j$th row of $X$ are the same.

Example:

$$
\Psi\left( \begin{bmatrix}
2 & 0 & 4 & 1 \\
2 & 0 & 4 & 1 \\
0 & 5 & 2 & 0 \\
\end{bmatrix}\right) = \begin{array}{c}
1  \\
2 \\
3 \\
\end{array}
$$

$$
\Psi\left( \begin{bmatrix}
2 & 0 & 4 & 1 & 1 \\
0 & 5 & 2 & 0 & 9 \\
2 & 0 & 4 & 1 & 1 \\
\end{bmatrix}\right) = \begin{array}{c}
1  \\
2 \\
3 \\
\end{array}
$$
Theorem

Let $\pi$ be a partition of $G$. Define a sequence of partitions $\{\pi_k\}$ by

$$
\begin{align*}
\pi_0 &= \pi \\
\pi_{k+1} &= \Psi([ P(\pi_k) \quad LP(\pi_k) ]) \\
\end{align*}
$$

$(k = 1, 2, \ldots)$. Then

1. $\pi_0 \succeq \pi_1 \succeq \pi_2, \ldots$. \\
2. There exists $q$ with $0 \leq q \leq n$ such that $\pi_q = \pi_{q+\ell}$ for all $\ell > 0$. \\
3. For any such $q$ we have $\pi_q = \pi_{\text{AEP}}(\pi)$. \\

\( \pi = \{\{1, 2, 3, 4, 5, 6\}, \{7, 8\}, \{9, 10\}\} \) (not an almost equitable partition).
Example

\[
P(\pi) = \begin{bmatrix}
1 & 0 & 0 \\
1 & 0 & 0 \\
1 & 0 & 0 \\
1 & 0 & 0 \\
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
0 & 0 & 1 \\
\end{bmatrix}, \quad L(P(\pi)) = \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
13 & -13 & 0 \\
13 & -13 & 0 \\
-12 & 14 & -2 \\
-14 & 14 & 0 \\
0 & 0 & 1 \\
0 & 0 & 1 \\
\end{bmatrix}
\]

By inspection we see that
\[\pi_1 = \Psi \left( \begin{bmatrix} P(\pi) & LP(\pi) \end{bmatrix} \right) = \{1, 2, 3, 4\}, \{5, 6\}, \{7\}, \{8\}, \{9, 10\},\]
which is the partition we studied before:
This partition is almost equitable, and therefore equal to $\pi^*_\text{AEP}(\pi)$.

Recall that with $V_L = \{1, 7\}$ we had computed $E(\pi^*_\text{AEP}(\pi)) = \frac{5}{12}$. Let $E(\pi)$ be the error associated with $\pi$. Conclusion: $E(\pi) \geq \frac{5}{12}$. 
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Directions for future research

- Extension to directed weighted graphs.
- Instead of a lower bound on the error associated with an arbitrary partition we would like to find upper bounds.
- Extension to the $\mathcal{H}_\infty$ norm of the error transfer matrix.
- Extension to the discrete time case.
- Extension to multi-agent systems with arbitrary homogeneous (heterogeneous) linear dynamics at the vertices.
- Combination of our clustering method with model reduction of the dynamics of the vertices.
THANKS FOR YOUR HOSPITALITY!