Computational Fluid Dynamics

Exercise 4 – Time integration

Description

Consider the heat equation

$$\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2}, \quad 0 \le x \le 1, \quad 0 \le t \le 4,$$

with boundary conditions

$$T(0,t) = 0 \text{ and } \frac{\partial T}{\partial x}(1,t) = 0,$$

and initial condition

$$T(x,0) = x.$$

This equation is solved with a finite-difference method on a grid with N grid points ($N = 10$ or 20). The Neumann condition is discretized with a mirror point. The quantity $T(1,t)$ is monitored. As time-integration method the generalized Crank-Nicolson method is used. The discretized heat equation reads

$$\frac{T_j^{n+1} - T_j^n}{\delta t} = (1-\omega)\frac{T_{j+1}^n - 2T_j^n + T_{j-1}^n}{h^2} + \omega\frac{T_{j+1}^{n+1} - 2T_j^{n+1} + T_{j-1}^{n+1}}{h^2}.$$

The simulation program is available as Matlab file `cfd_4.m`.

Required files

This exercise requires the file `cfd_4.m`.

Typing the command `cfd_4` in Matlab asks for the input of the required parameters for this exercise: N, ω `(omega)` and δt.

Questions to be solved on the computer

1) First consider the fully explicit method ($\omega = 0$). Vary the number of grid points as $N = 10$ and $N = 20$. Solve the heat equation for various time steps, as indicated in the table below. Monitor whether the time integration is stable. In Question 4 you are asked to explain the stability limit on the time step.

time step	0.001 stable?	0.00125 stable?	0.00126 stable?	0.0025 stable?	0.005 stable?	0.0051 stable?	0.01 stable?
N=10							
N=20							

2) Next, use the Crank-Nicolson method ($\omega = 0.5$) and the generalized Crank-Nicolson method with $\omega = 0.6$. Use $N = 10$. Set the time step δt at 0.05 and 0.5; see the table below. For the larger time step the solution shows clear oscillations. Make a rough estimate for the damping factor (defined as the quotient of two successive amplitudes) of the oscillations.

time step	$\delta t = 0.05$	$\delta t = 0.5$	
	stable?	stable?	damping
$\omega = 0.5$			
$\omega = 0.6$			

3) Finally, again for $N = 10$, investigate the fully implicit method ($\omega = 1$). Use the same time steps as in Question 2. Observe whether or not the solution shows oscillations.

time step	$\delta t = 0.05$		$\delta t = 0.5$	
	stable?	oscillations?	stable?	oscillations?
$\omega = 1.0$				

Questions to be solved by pencil-and-paper

4) First consider the fully explicit method ($\omega = 0$). Carry out a Fourier analysis of this method. Determine the maximum allowable time step for $N = 10$ and $N = 20$. Compare these stability limits with the empirical observations in Question 1.

5) Carry out a Fourier stability analysis of the generalized Crank-Nicolson method, and determine the amplification factor. Show that the generalized Crank-Nicolson method is unconditionally stable for $\omega \geq 0.5$. Investigate how the amplification factor behaves when $\delta t \to \infty$. Now explain the oscillations visible in Question 2. How are these oscillations influenced when ω is increased from 0.5 to 0.6? Give a possible explanation why the observed amplification factors in Question 2 are somewhat different from the theoretical amplification factors computed in this Question.

6) Derive theoretically for which values of the Crank-Nicolson parameter ω the solution is wiggle-free. As a special case, explain why the fully implicit method ($\omega = 1$) does not show oscillations (see Question 3). Hint: Use the concept of a positive operator.