

Towards a Non-Reflecting Boundary Condition for Wave Simulations in Offshore Applications

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1 INTRODUCTION

The numerical modeling of wave propagation in offshore applications is a formidable challenge even today when we have highly capable numerical schemes and computational power at our disposal. The waves travel in a huge and unbounded domain while the phenomena of interest are only around the structures. Therefore, the infinite domain is truncated via artificial boundaries. This results in a compact computational domain and a residual infinite domain. One of the bottlenecks is what type of boundary condition should be imposed on these introduced boundaries. It should be chosen such that the solution in the small truncated domain coincides with that in the original infinite domain.

Several methods for preventing reflection from the artificial boundaries have been discussed by Wellens (2012). Damping zones cannot dissipate outgoing waves and leave incoming waves unaffected at the same time. This does not suit offshore applications. Matching procedures to external solutions was discontinued because non-linear analytical theories that account for diffraction and radiation as a result of objects in the domain do not exist. Non-reflecting boundary conditions (NRBCs) suit the simulation of extreme free surface wave impact events better. NRBCs are studied in a vast amount of literature, we refer to Tsynkov (1998) and Givoli (2004) for more detailed reviews. However, not many have been specifically designed for use in our application.

A set of requirements for the design of a NRBC in ComFLOW have been formulated in Wellens (2012) and Duz (2015): (1) the problem in the domain including boundary condition (BC) on the artificial boundaries is well-posed; (2) the BC is compatible with the numerical scheme used inside the domain; (3) the application of the BC does not increase computational cost significantly; (4) the BC allows waves to travel in and out of the domain over the same boundary at the same time.

This paper derives a NRBC and implements it in our in-house CFD simulation tool ComFLOW. This work is organized as follows. The mathematical model governing the fluid flow is briefly described in Section 2. Section 3 derives the formulation of our NRBC in a three-dimensional domain, followed by its numerical implementation in Section 4. The well-posedness of the NRBC is discussed in Section 5. The last Section presents the numerical results of the reflection properties of our boundary condition along with the related discussions.

2 MATHEMATICAL MODEL

In our methods, the Navier-Stokes equations are solved on a staggered Cartesian grid, which keep the grid lines aligned with the coordinate axes. The free surface is tracked by the Volume of Fluid (VoF) approach in combination with a local height function.

To obtain a solution to the system of partial differential equations, boundary conditions need to be imposed. At solid boundaries such as structures or bottoms, Dirichlet conditions are applied. At the inflow boundaries, the incoming wave is prescribed. To complete the mathematical model, the NRBC is applied at the outflow boundaries. This NRBC will be derived in the next Section.

3 NRBC IN A THREE-DIMENSIONAL DOMAIN

Since we are looking at waves and waves are well described by potential theory, our boundary condition here is derived from the argument of potential theory.

3.1 Factorization of wave equation

The starting point of the derivation of the NRBC is often the planar wave equation given by:

$$\left(\partial^2 / \partial t^2 - c_0^2 \Delta\right) \phi_w = 0 \quad (1)$$

in which ϕ_w denotes the wave potential and c_0 represents the wave velocity. Inspired by Keys (1985), we proceed by factoring Eq. (1) as:

$$(\partial / \partial t - c_0 \vec{b} \cdot \nabla) \cdot (\partial / \partial t + c_0 \vec{b} \cdot \nabla) \phi_w = 0 \quad (2)$$

where \vec{b} is an arbitrary vector with the unit length. There are two ways to choose \vec{b} . The first one is

$$\vec{b} = \vec{n} \quad (3)$$

However, $\vec{n} \cdot \nabla$ in Eq. (2) does not see waves parallel to the boundary any more.

An alternative is to choose \vec{b} in the direction of the wave:

$$\vec{b} = \vec{e}_k \quad (4)$$

The vector \vec{e}_k splits into components as:

$$\vec{e}_k = \cos \theta \vec{n} + \sin \theta \vec{\tau} \quad (5)$$

where θ is the angle of incident waves with \vec{n} the normal direction of the boundary and $\vec{\tau}$ is the tangential of the domain boundary.

3.2 Non-reflecting boundary condition

Substitution of the relation (4) into Eq. (2) results in:

$$(\partial / \partial t - c_0 \vec{e}_k \cdot \nabla) \cdot (\partial / \partial t + c_0 \vec{e}_k \cdot \nabla) \phi_w = 0 \quad (6)$$

Introducing the variable ψ as:

$$\psi = (\partial / \partial t + c_0 \vec{e}_k \cdot \nabla) \phi_w \quad (7)$$

Eq. (6) turns to:

$$(\partial / \partial t - c_0 \vec{e}_k \cdot \nabla) \psi = 0 \quad (8)$$

By means of the method of characteristic, the term ψ is constant along the characteristic line defined by

$$dt / d\bar{x} = -c_0 \vec{e}_k.$$

If $-c_0 \vec{e}_k < 0$, $\psi = (\partial / \partial t + c_0 \vec{e}_k \cdot \nabla) \phi_w$ corresponds to the undesirable reflection from the boundary.

Accordingly, the boundary condition reads:

$$(\partial / \partial t + c_0 \vec{e}_k \cdot \nabla) \phi_w = R^{in} \quad (9)$$

in which R^{in} is the characteristic from the incoming waves. If no incoming waves exist, Eq. (9) becomes:

$$(\partial / \partial t + c_0 \vec{e}_k \cdot \nabla) \phi_w = 0 \quad (10)$$

Note that this boundary condition allows waves to travel in and out of the domain over the same boundary.

This is especially required when there is a structure in the domain.

4 NUMERICAL DISCRETIZATION OF NRBC IN COMFLOW

4.1 Discretisation of NRBC

Since the governing equations inside the computational domain are written in terms of the velocity component and pressure, the boundary condition (10) must be interpreted using the same variables. We employ a staggered grid arrangement for the flow variables inside the grid cells, therefore, the location of the outflow boundary must be specified accordingly.

All flow variables required here can be calculated by taking derivatives of the wave potential. Exploiting the potential concept and the linearized Bernoulli equation, we obtain the following expressions for the derivatives of the wave potential with respect to time and space:

$$\partial \phi_w / \partial n = u_b, \partial \phi_w / \partial \tau = v_b \quad (11)$$

$$\partial \phi_w / \partial t = -gz - p_b / \rho \quad (12)$$

The subscript b indicates that the quantity is defined at the boundary. Both the velocity and pressure are defined at the same position in space to avoid phase shift errors between flow variables. Following the same reasoning, these variables are also defined at the same instant in time, which is the new time level. Substitution of the relations (5), (11) and (12) into the boundary condition (10) gives:

$$-gz - p_b / \rho + c_0 \cos \theta u_b + c_0 \sin \theta v_b = 0 \quad (13)$$

This formulates the NRBC in our model.

4.2 Extension of NRBC

The NRBC in this abstract is only designed for one wave component, while in reality a wave is often composed by superposition of a number of components. Each individual component has its own wave number and propagation direction. Thus, the phase speed needs to be approximated to account for the dispersive properties of the waves. Better approximation of the dispersion relation leads to a lower reflection. The wave number k can be found by taking derivatives of the solution in space. By taking the second-order derivatives of ϕ_w in z – direction, we obtain:

$$\partial^2 \phi_w / \partial z^2 = k^2 \phi_w \quad (14)$$

Now a rational polynomial in z is introduced to approximate the dispersion relation as:

$$c_0 \approx \sqrt{gh} [a_0 + a_1(kh)^2] / [1 + b_1(kh)^2] \quad (15)$$

Here the coefficients a_0, a_1 and b_1 can be chosen such that different kh – ranges of the dispersion relation are approximated well.

Substituting the relations (14) and (15) into the boundary condition (13) gives:

$$(1 + b_1 h^2 \partial^2 / \partial z^2)(-gz - p_b / \rho) + \sqrt{gh}(a_0 + a_1 h^2 \partial^2 / \partial z^2) \cos \theta_{u_b} + \sqrt{gh}(a_0 + a_1 h^2 \partial^2 / \partial z^2) \sin \theta_{v_b} = 0 \quad (16)$$

The results with approximated phase velocity will be given in the future.

Regarding the directional effects of the waves, a higher-order boundary condition serves better. This may be part of future development.

5 WELL-POSEDNESS OF NRBC

We analyse the well-posedness by studying the reflection coefficient at the boundary in case an approximate phase speed c_a is used for waves which propagate with (exact) phase speed c_0 .

For propagating waves, the reflection coefficient is derived by Wellens (2012) and reads:

$$R = [c_a(kh) - c_0(kh)] / [c_a(kh) + c_0(kh)] \quad (17)$$

The phase velocity c_0 follows from the dispersion relation:

$$c_0 = \sqrt{g} \sqrt{k \tanh(|k|h) / (|k|h)} \quad (18)$$

Both c_a and c_0 are positive reals. Hence, the reflection coefficient is clearly smaller than 1.

In case of evanescent and spurious modes \vec{k} is complex-valued. Considering $k_x = ik_{xI}$ with real k_{xI} , dispersion relation becomes for the special case $k_y = 0$:

$$c_0 = \sqrt{g} \sqrt{k \tan k_{xI} h / (k_{xI} h)} \quad (19)$$

The reflection coefficient R for evanescent waves with real c_a becomes complex with a modulus equal to 1, irrespective whether c_a is positive or negative. Hence, from the reflective point of view, the evanescent modes are ‘innocent’.

The spurious modes, i.e. $n\pi < kh < (1/2 + n)\pi$, require further attention since here c_0 is real and positive, which does not necessarily apply to c_a . For the reflection coefficient to be smaller than 1 in modulus, c_a must be positive. Negative values are only allowed in the wave-number region of the evanescent waves.

6 VERIFICATION AND DISCUSSION

In this Section, the performance of the NRBC is tested. Less reflection implies better performance. The reflection coefficients for different ' kh ' values are obtained from regular wave simulations with an NRBC at the downstream end of the domain and are compared to the theoretical reflection coefficients.

The regular wave imposed at the inflow boundary has a wave period of $T = 4s$ and a wave height of $H = 0.1m$. We choose various water depths linked to different ' kh ' values. Fig. 1 shows the comparison of the numerical surface elevation at the outflow boundary of the domain with that theoretical value in the case where the water depth $h = 5m$.

To investigate the reflection coefficients of our NRBC, for each ' kh ' two simulations are carried out. First, a

wave simulation is performed in an sufficiently long domain, i.e. the domain length is chosen such that reflected waves cannot reach the measurement location during the simulation time. The second simulation is carried in a smaller domain. The simulation differs from the previous one in the domain length and the boundary procedure at the outflow end of the domain. Measurement of the surface elevation, taken at the exact same position (the outflow boundary of the smaller domain), are compared to measurements in the larger domain. The difference is attributed to the boundary procedure.

The surface elevations at the measurement location in the larger domain is subtracted from the surface elevations in the domain where NRBC is applied. The results are normalized by the amplitude of waves in the larger domain, yielding a reflection signal. Finally, the reflection coefficients from the numerical results are compared to the theoretical values, defined in Eq. (17). This is displayed in Fig. 2.

The NRBC performs best for a single ' kh ' value. Both the theoretical and numerical reflection coefficient are zero. Away from this ' kh ' value the performance quickly becomes worse.

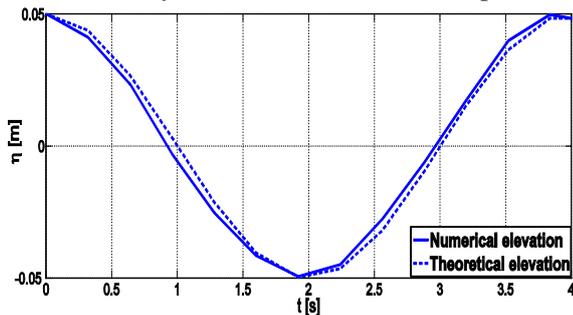


Fig.1 Surface elevations at the outflow boundary

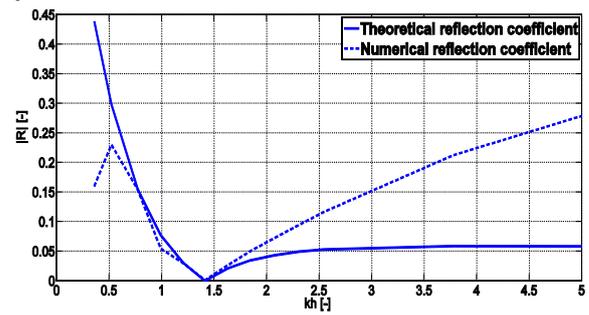


Fig.2 Reflection coefficients at the outflow boundary

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