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Thermodynamics of Hidden Processes

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Outline

- Processes on $\mathbb{Z}$ ($d = 1$)
- Fields on $\mathbb{Z}^d$ ($d > 1$)

Example

- **Input:** Markov chain $\{X_n\}$, $X_n \in \{-1, 1\}$, with the transition probability matrix

$$\mathbf{P} = \begin{pmatrix} 1-p & p \\ p & 1-p \end{pmatrix}.$$

- **Noise/Channel:** Bernoulli random variables $\{Z_n\}$, $Z_n \in \{-1, 1\}$, with

$$\mathbb{P}(Z_n = -1) = \delta,\ \mathbb{P}(Z_n = 1) = 1 - \delta.$$

- **Output:** $Y_n = X_n \cdot Z_n, \quad \forall n.$

$$X_n, Z_n \in \{0, 1\},\ Y_n = X_n \oplus Z_n = X_n + Z_n \mod 2.$$

Process $\{Y_n\}$ is known as

$\bigstar_1$ a hidden Markov chain [PR/ST/IT]
$\bigstar_2$ a 1-block factor ($Y_n = \phi((X_n, Z_n))$) [DS]
$\bigstar_3$ a Markov source [IT]
$\bigstar_4$ a lumped/aggregated Markov chain [PR]
$\bigstar_5$ a ”*copy with noise*” [SP]
$\bigstar_6$ a fuzzy process [SP]

Thermodynamics of $\{Y_n\}$:

Every mathematician builds for himself his own theoretical physics. R. Dobrushin, quotation from Ya. Sinai: *Mathematicians and Physicists = Cats and Dogs?*

$\checkmark_1$ Understand (describe) the structure of conditional probabilities

$$\begin{aligned}
\mathbb{P}(Y_0|Y_{>0}) &:= \mathbb{P}(Y_0|Y_1, Y_2, \ldots) \\
\mathbb{P}(Y_0|Y_{<0}) &:= \mathbb{P}(Y_0|Y_{-1}, Y_{-2}, \ldots) \\
\mathbb{P}(Y_0|Y_{\neq 0}) &:= \mathbb{P}(Y_0|Y_k, k \in \mathbb{Z} \setminus \{0\})
\end{aligned}$$

$\checkmark_2$ Variational principle for $\mathbb{P}$

$\checkmark_3$ Entropy of $\mathbb{P}$.

Assumptions: $\mathbf{P} = (p_{ij}) > 0$ and $\Pi = (\pi_{ik}) > 0$.

$p_{ij} = \mathbb{P}(X_{n+1} = j | X_n = i)$, $\pi_{ik} = \mathbb{P}(Y_n = k | X_n = i)$.

Proposition I

For every $y = (y_k)_{k \in \mathbb{Z}}$

$$\mathbb{P}(y_0 | y_{-\infty}^{-1}) := \lim_{n \to \infty} \mathbb{P}(y_0 | y_{-n}^{-1}),$$

$$\sup_{y, \tilde{y}} \left| \mathbb{P}(y_0 | y_{-n}^{-1} y_{-\infty}^{-n-1}) - \mathbb{P}(y_0 | y_{-n}^{-1} \tilde{y}_{-\infty}^{-n-1}) \right| \leq (1 - \lambda)^n.$$

Proposition II

There exist constants α, β such that for all y

$$0 < \alpha \leq \mathbb{P}(y_0 | y_{-\infty}^{-1}) \leq \beta < 1$$

Proposition I:

$$\sup_{y,\tilde{y}} \left| \mathbb{P}(y_0 | y_{-n}^{-1} y_{-\infty}^{-n-1}) - \mathbb{P}(y_0 | y_{-n}^{-1} \tilde{y}_{-\infty}^{-n-1}) \right| = o(n)$$

$\Longleftrightarrow$ **quasi-locality (continuity) of conditional probabilities**.

Proposition II:

$$0 < \alpha \leq \mathbb{P}(y_0 | y_{-\infty}^{-1}) \leq \beta < 1$$

$\Longleftrightarrow$ $\mathbb{P}$ is **uniformly non-null** or **finite energy condition**.

Proposition II is easy:

$$\mathbb{P}(y_0|y_{-n}^{-1}) = \frac{\sum_{x_{-n}^{0}} \mathbb{P}(x_{-n}^{0})\mathbb{P}(y_{-n}^{0}|x_{-n}^{0})}{\sum_{x_{-n}^{-1}} \mathbb{P}(x_{-n}^{-1})\mathbb{P}(y_{-n}^{-1}|x_{-n}^{-1})}$$

$$= \sum_{x_{-n}^{0}} \mathbb{P}(x_0|x_{-1})\mathbb{P}(y_0|x_0)w(x_{-n}^{-1}; y_{-n}^{-1}),$$

where $\sum_{x_{-n}^{-1}} w(x_{-n}^{-1}; y_{-n}^{-1}) = 1$. Hence,

$$\min_{i,j} p_{ij} \min_{j,l} \pi_{jl} \leq \mathbb{P}(y_0|y_{-n}^{-1}) \leq \max_{i,j} p_{ij} \max_{j,l} \pi_{jl}$$

Idea of the proof using Markov property:

$$(\ldots, Y_{-n-2}, Y_{-n-1}) \perp_{X_{-n}} (Y_{-n+1}, Y_{-n+2}, \ldots),$$
$$(\ldots, X_{-n-2}, X_{-n-1}) \perp_{X_{-n}} (X_{-n+1}, X_{-n+2}, \ldots).$$

$$\begin{aligned}\mathbb{P}(Y_0 = y_0 | Y_{-n-m}^{-1} = y_{-n-m}^{-1}, X_{-n} = x_{-n}) \\ = \mathbb{P}(Y_0 = y_0 | Y_{-n-m}^{-1} = y_{-n+1}^{-1}, X_{-n} = x_{-n}).\end{aligned}$$

$$\inf_{x_{-n}} \mathbb{P}(y_0 | y_{-n+1}^{-1}, x_{-n}) \leq \mathbb{P}(y_0 | y_{-n-m}^{-1}) \leq \sup_{\tilde{x}_{-n}} \mathbb{P}(y_0 | y_{-n+1}^{-1}, \tilde{x}_{-n})$$

$$\sup_{y, x_{-n}, \tilde{x}_{-n}} |\mathbb{P}(y_0 | y_{-n+1}^{-1}, x_{-n}) - \mathbb{P}(y_0 | y_{-n+1}^{-1}, \tilde{x}_{-n})| \to 0.$$

Propositions I & II are equivalent to Law($\{Y_n\}$) being

1. Gibbs
2. chain with complete connections
3. g-measure
4. uniform martingale
5. . . .

Proposition I:7 proofs in 50 years

- Harris (1955), Birch (1962):
  independent coupling of Markov chains
- Baum & Petrie (1970):
  direct
- Lörinczi, Maes & vande Velde (1998):
  disagreement percolation on $\mathbb{Z}^d$
- Fernandez, Ferrari, Galves (2001):
  maximal coupling/regeneration property of Markov chains
- Han & Marcus (2006):
  contraction principle + perturbation method

λ-championship

Decay rate $\sim (1-\lambda)^n$.
In our example, for $p = 0.4$ and $\delta = 0.1$

Harris	$\lambda_H \approx 2.777 \cdot 10^{-2}$
Birch	$\lambda_B \approx 1.505 \cdot 10^{-5}$
Baum & Petrie	$\lambda_{BP} = 0.06$
Lörinczi *et al*	$\lambda_{LMvV} = \ldots < 1$
Han & Marcus	$\lambda_{HM} = 0.42$
Fernandez *et al*	$\lambda_{FFG} = 0.8$

$$\lambda_{FFG} = 2p \quad \textit{for} \quad p < \frac{1}{2} \text{ independent of } \delta!$$

Slide added after the Banff meeting!

During the workshop, Yuval Peres explained how the results of [1,2] could be used to derive the following estimate

$$\lambda \geq 2p.$$

In fact, the above inequality is a corollary of the following rather deep result

$$\begin{aligned} \sup |\mathbb{P}[Y_0 = y_0 | Y_{1,\ldots,n-1}, X_n = 1] - P[Y_0 = y_0 | Y_{1,\ldots,n-1}, X_n = -1]| \\ \leq \sup |\mathbb{P}[Y_0 = y_0 | X_n = 1] - P[Y_0 = y_0 | X_n = -1]| \end{aligned}$$

[1] C. Kenyon, E. Mossel, Y. Peres, *Glauber dynamics on trees and hyperbolic graphs.* 42nd IEEE Symposium on Foundations of Computer Science (Las Vegas, NV, 2001), IEEE Computer Soc., Los Alamitos, CA, 568–578. (**2001**)

[2] N. Berger, C. Kenyon, E. Mossel, Y. Peres, *Glauber dynamics on trees and hyperbolic graphs.* Probab. Theory

Related Fields 131 no.3 311–340. (**2005**)

Yet another interesting fact

$$Y_n = X_n \cdot Z_n$$

For processes more general than Markov chains

$$\sup_w |\mathbb{P}(Y_0 = w_0 | Y_{-n}^{-1} = w_{-n}^{-1}) - \mathbb{P}(X_0 = w_0 | X_{-n}^{-1} = w_{-n}^{-1})| \leq C\delta.$$

[Collet, Galves, Leonardi. Preprint 2007]

One-sided versus two-sided conditional probabilities

$$\mathbb{P}(y_0|y_{-\infty}^{-1}) \quad \leftrightsquigarrow \quad \mathbb{P}(y_0|y_{-\infty}^{-1}, y_{+1}^{+\infty})$$

You have “reasonable” one-sided description of cond. prob., can you get reasonable two-sided description.

In $d = 1$, **under strong uniqueness conditions**,
R. Fernandez & G. Maillard ('04 & '05) showed that you can switch between one-sided and two-sided representations.

Denoising

Given $Y_1, \ldots, Y_n$, estimate $X_1, \ldots, X_n$, i.e., **produce**

$$\hat{X}_i = \hat{X}_i(Y_1^n), \quad i = 1, \ldots, n,$$

minimizing *expected normalized cumulative loss*, e.g.,

$$\mathbb{E}\left(\frac{1}{n}\sum_{i=1}^{n} L(X_i, \hat{X}_i)\right),$$

where $L : \mathtt{A} \times \mathtt{A} \longrightarrow \mathbb{R}_+$ is a *loss function*.

T.Weissman, E.Ordentlich, G.Seroussi, S.Verdu, M.Weinberger
Universal discrete denoising: known channel (2005).

If we know

$$\mathbb{P}(X_i = a | Y_1^n = y_1^n), \quad a \in A,$$

then

$$\hat{X}_i = \operatorname*{argmin}_x \sum_{a \in A} L(a, x) \mathbb{P}(X_i = a | Y_1^n = y_1^n).$$

$\mathbb{P}(X_i = a | Y_1^n = y_1^n)$ can be expressed in terms of (generalized) inverse Π^{-1} and

$$\left\{ \mathbb{P}(Y_i = b | Y^{n \setminus i} = y^{n \setminus i}),\ b \in B \right\}.$$

DUDE: Estimate $\mathbb{P}(Y_i = b | Y_{i-k}^{i-1} = y_{i-k}^{i-1}, Y_{i+1}^{i+k} = y_{i+1}^{i+k})$.

What can thermodynamics do for you?

- Structure & formalism
- existence of divergence (relative entropy rate)

$$\mathbf{D}(\mathbb{Q}|\mathbb{P}) = \lim_{n\to\infty} \frac{1}{n+1} \sum_{y_0^n} \mathbb{Q}(y_0^n) \log \frac{\mathbb{Q}(y_0^n)}{\mathbb{P}(y_0^n)}.$$

- Large deviations & Central Limit Theorems

THEOREM. (FERNANDEZ-FERRARI-GALVES)
Let $\mathbb{P}$ be a stationary measure of the process $\{Y_n\}$, $Y_n \in S$, such that

- $\mathbb{P}(Y_0 = y_0 | Y_{-\infty}^{-1} = y_{-\infty}^{-1})$ are quasi-local (continuous),
- $\mathbb{P}(Y_0 = y_0 | Y_{-\infty}^{-1} = y_{-\infty}^{-1}) \in [\alpha, \beta] \subset (0, 1)$.

Then $\mathbb{P}$ is a countable mixture of Markov chains (CMMC)

$$\mathbb{P}(Y_0 = y_0 | Y_{-\infty}^{-1} = y_{-\infty}^{-1}) = \lambda_0 \mathbb{P}_0(y_0) + \sum_{k \geq 1} \lambda_k \mathbb{P}_k(y_0 | y_{-k}^{-1}),$$

where

$$\lambda_k \geq 0, \quad \text{and} \sum_{k \geq 0} \lambda_k = 1;$$

$\mathbb{P}_0$ is a measure on S, and $\mathbb{P}_k$ are k-step Markov measures.

$$\mathbb{P}(Y_0 = y_0 | Y_{-\infty}^{-1} = y_{-\infty}^{-1}) = \lambda_0 \mathbb{P}_0(y_0) + \sum_{k \geq 1} \lambda_k \mathbb{P}_k(y_0 | y_{-k}^{-1})$$

- Easy generating scheme:
  1. Take a uniform random variable $t \sim U[0,1]$;
  2. Find unique k such that

     $$\sum_{i=0}^{k-1} \lambda_i \leq t < \sum_{i=0}^{k} \lambda_i;$$

  3. Generate Y_0 according to $\mathbb{P}_k(\cdot | Y_{-k}^{-1})$.
- Representation is not unique!

O. Zuk, E. Domany, I. Kanter & M. Aizenman, (2006)
Taylor Series Expansions for the Entropy Rate of Hidden Markov Processes:

$$h(\mathcal{Y}) = \lim_{n\to\infty} H(Y_0|Y_{-n}^{-1}) = \sum_{k=0}^{+\infty} f_k \delta^k.$$

Application of this method also gives expansions for

$$\mathbb{P}(y_0|y_{-\infty}^{-1}) = \sum_{k=0}^{\infty} g_k(y^0_{-k-1})\delta^k,$$

$$g_0 = \frac{1}{2}\Big(1 + (1-2p)y_0y_1\Big), \quad g_1 = -\frac{(1-2p)y_0y_1}{1+(1-2p)y_1y_2}, \ldots$$

[S. van Wijk, MSc thesis, TU Eindhoven, 2007]

$$g_2 =$$

$$\frac{2(1-2p)y_0y_1[(1-2p)y_2y_3-(1-2p)y_1y_2\{3-(1-2p)y_2y_3\}+1]}{((1-2p)y_1y_2+1)^2((1-2p)y_2y_3+1)}$$

$$g_3 =$$

$$\frac{16\lambda^2 y_0 y_1^2 y_2\left(y_1 y_2^2 y_3^2 y_4 \lambda^3 - y_2 y_3(y_1(y_2+y_4)-y_3y_4)\lambda^2-(y_1y_2+y_3(y_2-y_4))\lambda+1\right)}{(\lambda y_1 y_2+1)^3(\lambda y_2 y_3+1)^2(\lambda y_3 y_4+1)}$$

$$\lambda = (1-2p)$$

$$\xi_i = y_i y_{i+1}$$

Suppose $\{X_n\}$, $X_n \in \mathcal{A}$, and Law($\{X_n\}$) is Gibbs,

$$Y_n \sim \mathbb{P}(\cdot | X_n) \quad \text{or} \quad Y_n = \phi(X_n).$$

Is the Law($\{Y_n\}$) Gibbs?

If $\{X_n\}$ is **fully supported**, then $\{Y_n\}$ is **expected** to be Gibbs.

Example: Walters (1986), van den Berg (mid 1990's)

- $\{X_n\}$ iid Bernoulli:

$$\mu_p(X_n = 1) = p, \quad \mu_p(X_n = -1) = 1 - p.$$

- $Y_n = X_{n-1} \cdot X_n = \phi(X_n^{hb})$, where $X_n^{hb} = (X_{n-1}, X_n)$, but now μ_p^{hb} is not fully supported on $(\{-1,1\}^2)^{\mathbb{Z}}$.
- $\nu = \mu_p \circ \phi^{-1}$ law of $\{Y_n\}$.
- ν is (highly) non-Gibbs

  DS $\nu = \mu_p \circ \phi^{-1} = \mu_{1-p} \circ \phi^{-1} \Rightarrow$ troubles;

  SM ν has **everywhere discontinuous cond.prob.'s**.

J.-R. Chazottes, **E. Ugalde**: sufficient conditions to be Gibbs.

Part II: Random Fields on $\mathbb{Z}^d$

"... ergodic theory inherits from statistical mechanics not only its name, but also an obligation to analyze spatially extended systems with multi-dimensional symmetry groups." K. Schmidt

Gibbs random fields

Definition. A measure μ on $\Omega = A^{\mathbb{Z}^d}$ is Gibbs for potential $\Phi = \{\Phi(V,\cdot)\}_{V\subset\mathbb{Z}^d \text{ is finite}}$ (denoted $\mu \in \mathcal{G}_\Omega(\Phi)$) if

$$\mu(x_\Lambda | x_{\Lambda^c}) = \frac{\exp\Big(-H_\Lambda^\Phi(x_\Lambda x_{\Lambda^c})\Big)}{\sum_{z_\Lambda} \exp\Big(-H_\Lambda^\Phi(z_\Lambda x_{\Lambda^c})\Big)} \quad (\mu - a.e.),$$

where

$$H_\Lambda^\Phi(x) = \sum_{V\cap\Lambda\neq\emptyset} \Phi(V,x) = \sum_{V\cap\Lambda\neq\emptyset} \Phi(V,x_V).$$

$$\mu(x_\Lambda | x_{\Lambda^c}) = \frac{\exp\Big(-H_\Lambda^\Phi(x_\Lambda x_{\Lambda^c})\Big)}{\sum_{z_\Lambda} \exp\Big(-H_\Lambda^\Phi(z_\Lambda x_{\Lambda^c})\Big)} =: \gamma_\Lambda(x_\Lambda | x_{\Lambda^c}) \quad (\mu - a.e.).$$

- Hamiltonian is absolutely convergent: $\Phi \in \mathfrak{B}_1(\Omega)$;
- $\gamma_\Lambda(x_\Lambda | x_{\Lambda^c})$ is **quasi-local**: for $V \nearrow \mathbb{Z}^d$

$$\sup_x \sup_{z,w} \Big| \gamma_\Lambda(x_\Lambda | x_{V \setminus \Lambda} z_{V^c}) - \gamma_\Lambda(x_\Lambda | x_{V \setminus \Lambda} w_{V^c}) \Big| \to 0,$$

- γ is **uniformly non-null**: for every $\Lambda \Subset \mathbb{Z}^d$, $\exists \alpha_\Lambda, \beta_\Lambda$:

$$0 < \alpha_\Lambda \leq \gamma_\Lambda(x_\Lambda | x_{\Lambda^c}) \leq \beta_\Lambda < 1 \quad \forall x \in \Omega.$$

Fuzzy or Hidden Gibbs states

- μ on $\Omega = A^{\mathbb{Z}^d}$ is Gibbs ($\mu \in \mathcal{G}_\Omega(\Phi)$);
- $\phi : A^{\mathbb{Z}^d} \to B^{\mathbb{Z}^d}$ ($|A| > |B|$),

$$y_k = \phi(x_k) \quad \forall k \in \mathbb{Z}^d.$$

- $\nu = \mu \circ \phi^{-1}$ is a measure on $B^{\mathbb{Z}^d}$. **Is it Gibbs**?
- ν is always **uniformly non-null**.
- ν might not be **quasi-local**.
- If not Gibbs, how non-Gibbs?
- Which "Gibbs" results are valid for ν?

Dobrushin's reconstruction program

Gibbs $\subsetneq$ almost Gibbs $\subsetneq$ weakly Gibbs

$$\nu(y_\Lambda | y_{\Lambda^c}) = \frac{\exp\Big(-H^\Psi_\Lambda(y_\Lambda y_{\Lambda^c})\Big)}{\sum_{z_\Lambda} \exp\Big(-H^\Psi_\Lambda(z_\Lambda y_{\Lambda^c})\Big)} = \gamma_\Lambda(y_\Lambda | y_{\Lambda^c}) \quad (\nu - a.e.).$$

- $\gamma_\Lambda(y_\Lambda | y_{\Lambda^c})$ is **not** everywhere continuous, but only for $y \in \Omega_0 \subset \Omega$

  $$\sup_z \Big| \gamma_\Lambda(y_\Lambda | y_{V \setminus \Lambda} y_{V^c}) - \gamma_\Lambda(y_\Lambda | y_{V \setminus \Lambda} z_{V^c}) \Big| \to 0,$$

  as $V \nearrow \mathbb{Z}^d$, and $\nu(\Omega_0) = 1$.
- Hamiltonian is **not** absolutely convergent, but ν-a.s;

Variational Principle for Gibbs states

If $\mu \in \mathcal{G}_\Omega(\Phi)$, then for any λ, $\mathbf{D}(\lambda|\mu)$ exists:

$$\mathbf{D}(\lambda|\mu) = \lim_{n\to\infty} \frac{1}{|\Lambda_n|} \sum_{x_{\Lambda_n}} \lambda(x_{\Lambda_n}) \log \frac{\lambda(x_{\Lambda_n})}{\mu(x_{\Lambda_n})}, \quad \Lambda_n = [0,n]^d,$$

Moreover, $\mathbf{D}(\lambda|\mu) = 0 \iff \lambda \in \mathcal{G}_\Omega(\Phi)$.

Variational Principle for Fuzzy Gibbs states

If $\mu \in \mathcal{G}_\Omega(\Phi)$, $\nu = \mu \circ \phi^{-1}$, then for any ρ, $\mathbf{D}(\rho|\nu)$ exists. Moreover,

$$\mathbf{D}(\rho|\nu) = 0 \iff \rho = \lambda \circ \pi^{-1} \text{ for some } \lambda \in \mathcal{G}_\Omega(\Phi).$$

Potts model

q-spins: $\Omega = \{1, \ldots, q\}^{\mathbb{Z}^d}$; **Parameter:** inverse temperature β.

$$\gamma_{q,\beta}(x_\Lambda | x_{\Lambda^c}) = \frac{1}{Z_{q,\beta}^{\Lambda, x_{\Lambda^c}}} \exp\left(2\beta \sum_{\substack{i \in \Lambda, \\ j \in \Lambda \cup \partial_+ \Lambda: \\ |i-j|=1}} \mathbb{I}[x_i = x_j] \right) = \gamma_{q,\beta}(x_\Lambda | x_{\partial_+ \Lambda})$$

Theorem (Aizenman-Chayes-Chayes-Newman)

There exists critical $\beta_c = \beta_c(q, d) \in (0, +\infty)$:

- **[High T]** $\beta < \beta_c$ there is a unique Gibbs measure,
- **[Low T]** $\beta > \beta_c$ there are multiple Gibbs measures (phase transition).

Fuzzy Potts model

$$\phi : \{1, \ldots, q\}^{\mathbb{Z}^d} \to \{1, \ldots, s\}^{\mathbb{Z}^d} \ (q > s)$$

$$y_{\mathbf{k}} = \phi(x_{\mathbf{k}}) = \begin{cases} \text{[redacted]} & \text{if } x_{\mathbf{k}} \in \{1, \ldots, r_1\}, \\ 2, & \text{if } x_{\mathbf{k}} \in \{r_1 + 1, \ldots, r_1 + r_2\}, \\ \ldots & \\ s, & \text{if } x_{\mathbf{k}} \in \{r_1 + \ldots + r_{s-1} + 1, \ldots, q\}. \end{cases}$$

Maes-vande Velde: $q = sn$ with $r_1 = \ldots = r_s = n$;
Haggstrom: general $r = (r_1, \ldots, r_s)$.

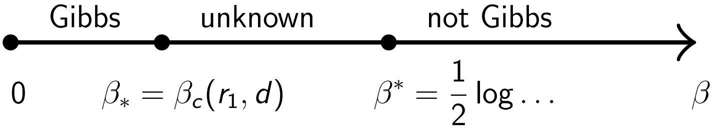
Notation: w.l.o.g. $r_1 = \min\{r_i : \ r_i > 1\}$.

Theorem. (Haggström, 2003)

Let $\mu_{q,\beta}$, ϕ be as above. Then $\nu_{q,\beta,\phi} = \mu_{q,\beta} \circ \phi^{-1}$

- is Gibbs if $\beta < \beta_c(r_1, d)$;
- is not Gibbs if $\beta > \dfrac{1}{2} \log \dfrac{1 + (r_1 - 1) p_c(d)}{1 - p_c(d)}$.

Moreover, $\nu_{q,\beta,\phi}$ is never Markov of any finite order.



Conjecture

$\nu_{q,\beta,\phi}$ is not Gibbs for $\beta > \beta_c(r_1, d)$.

Loss of Gibbsianity

Scenario (van Enter–Fernandez–Sokal, 1993)

loss of Gibsianity occurs when there is a **hidden** *phase transition in the original system conditioned on the image spins.*

Let $\mu \in \mathcal{G}_X(\Phi)$, $\nu = \mu \circ \phi^{-1}$, fiber $\Omega_y = \{x : \phi(x) = y\}$.

- if $|\mathcal{G}_{\Omega_y}(\Phi)| = 1$ for all y, then ν **is Gibbs**;
- if $|\mathcal{G}_{\Omega_y}(\Phi)| \geq 2$ for **some** y, then ν is **not Gibbs**.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