The port-Hamiltonian approach to physical system modeling and control

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Part I : Network Modeling and Analysis
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Part III : Distributed-Parameter Systems (Hans Zwart)

Part I : Network Modeling and Analysis

1. Motivation and context
2. From passive systems to port-Hamiltonian systems
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From passive systems to port-Hamiltonian systems

A square nonlinear system

$$\Sigma : \begin{array}{rcll} \dot{x} & = & f(x) + g(x)u, & u \in \mathbb{R}^m \\ \\ y & = & h(x), & y \in \mathbb{R}^m \end{array}$$

where $x \in \mathbb{R}^n$ are coordinates for an n-dimensional state space $\mathcal{X}$, is **passive** if there exists a **storage function** $H : \mathcal{X} \to \mathbb{R}$ with $H(x) \geq 0$ for every x, such that

$$H(x(t_2)) - H(x(t_1)) \leq \int_{t_1}^{t_2} u^T(t)y(t)dt$$

for all solutions $(u(\cdot), x(\cdot), y(\cdot))$ and times $t_1 \leq t_2$.
The system is **lossless** if $\leq$ is replaced by $=$.

If H is *differentiable* then 'passive' is equivalent to

$$\frac{d}{dt}H \leq u^T y$$

which reduces to (Willems, Hill-Moylan)

$$\begin{aligned} \tfrac{\partial^T H}{\partial x}(x) f(x) &\leq 0 \\ \\ h(x) &= g^T(x) \tfrac{\partial H}{\partial x}(x) \end{aligned}$$

while in the lossless case $\leq$ is replaced by $=$.

In the *linear* case

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned}$$

is passive if there exists a *quadratic* storage function $H(x) = \frac{1}{2}x^T Q x$, with $Q = Q^T \geq 0$ satisfying the LMIs

$$A^T Q + QA \leq 0, \quad C = B^T Q$$

Every *linear* passive system with storage function $H(x) = \frac{1}{2}x^T Qx$, satisfying

$$\ker Q \subset \ker A$$

can be rewritten as a linear **port-Hamiltonian system**

$$\begin{aligned} \dot{x} &= (J-R)Qx + Bu, \quad J = -J^T, \quad R = R^T \geq 0 \\ y &= B^T Qx, \end{aligned}$$

in which case the storage function $H(x) = \frac{1}{2}x^T Qx$ is called the **Hamiltonian**.

- **Passive linear systems are thus port-Hamiltonian with non-negative Hamiltonian.**
- *Conversely* **every port-Hamiltonian system with non-negative Hamiltonian is passive.**

Mutatis mutandis 'most' nonlinear lossless systems can be written as a port-Hamiltonian system

$$\begin{aligned} \dot{x} &= J(x)\tfrac{\partial H}{\partial x}(x) + g(x)u \\ y &= g^T(x)\tfrac{\partial H}{\partial x}(x) \end{aligned}$$

with $J(x) = -J^T(x)$ and $\frac{\partial H}{\partial x}(x)$ the *column vector* of partial derivatives. Note that

$$\dot{x} = J(x)\frac{\partial H}{\partial x}(x)$$

is the internal Hamiltonian dynamics known from physics, which in classical mechanics can be written as

$$\begin{aligned} \dot{q} &= \tfrac{\partial H}{\partial p}(q,p) \\ \dot{p} &= -\tfrac{\partial H}{\partial q}(q,p) \end{aligned}$$

with the Hamiltonian H the total (kinetic + potential) energy.

Similarly, most nonlinear passive systems can be written as a port-Hamiltonian system (with dissipation)

$$\begin{aligned} \dot{x} &= [J(x) - R(x)]\tfrac{\partial H}{\partial x}(x) + g(x)u \\ y &= g^T(x)\tfrac{\partial H}{\partial x}(x) \end{aligned}$$

with $R(x) = R^T(x) \geq 0$ specifying the energy dissipation

$$\frac{d}{dt}H = -\frac{\partial^T H}{\partial x}(x)R(x)\frac{\partial H}{\partial x}(x) + u^T y \leq u^T y$$

However, in *network modeling* it is the **other way around**: one derives the system in port-Hamiltonian form (and if the Hamiltonian $H \geq 0$ then it is the storage function of a passive system).

The matrix $J(x)$ corresponds to the internal **power-conserving interconnection** structure of physical systems due to:

- Basic conservation laws such as Kirchhoff's laws.
- Powerless constraints; kinematic constraints.
- Transformers, gyrators, exchange between different types of energy.

The matrix $R(x)$ corresponds to the **internal energy dissipation** in the system (due to resistors, damping, viscosity, etc.)

Main message: **start with port-Hamiltonian models instead of passive models**. Closer to physical modeling, and capturing more information than just the energy-balance of passivity.

A bit of port-based network modeling

The **passivity** framework considers a system component, and its power-exchange with other system components:

$$\frac{d}{dt}H \leq u^T y$$

The feedback interconnection of two passive systems

$$\tfrac{d}{dt}H_1 \leq u_1^T y_1, \qquad \tfrac{d}{dt}H_2 \leq u_2^T y_2$$

$$u_1 = -y_2 + v_1, \qquad u_2 = y_1 + v_2$$

leads to an interconnected system that is **again passive**, since

$$\frac{d}{dt}(H_1 + H_2) \leq u_1^T y_1 + u_2^T y_2 = v_1^T y_1 + v_2^T y_2$$

The feedback interconnection is a typical example of a *power-conserving interconnection* (total power is zero):

$$u_1^T y_1 + u_2^T y_2 + v_1^T y_1 + v_2^T y_2 = 0$$

In port-based modeling, e.g. bond graphs, one looks at the system as the power-conserving interconnection of ideal basic system components: (energy-) storage elements, resistive elements, transformers, gyrators, constraints, etc.

- The Hamiltonian of the resulting port-Hamiltonian system is the sum of the energies of the storage elements.
- The J- and B-matrix is determined by the transformers, gyrators, constraints, and the power-conserving interconnection.
- The R-matrix is determined by the resistive elements, and the way they are connected.

An k dimensional **storage element** is determined by a k-dimensional state vector $x = (x_1, \cdots, x_k)$ and a Hamiltonian $H(x_1, \cdots, x_k)$ (energy storage), defining the lossless system

$$\begin{aligned} \dot{x}_i &= f_i, \qquad i = 1, \cdots, k \\ e_i &= \frac{\partial H}{\partial x_i}(x_1, \cdots, x_k) \\ \frac{d}{dt}H &= \sum_{i=1}^{k} f_i e_i \end{aligned}$$

Such a k- dimensional storage component is written in vector notation as a port-Hamiltonian system with $J = 0$, $R = 0$, and $B = I$:

$$\begin{aligned} \dot{x} &= f \\ e &= \frac{\partial H}{\partial x}(x) \end{aligned}$$

The elements of x are called **energy variables**, those of $\frac{\partial H}{\partial x}(x)$ **co-energy variables**. Furthermore the elements of f are **flow variables**, and of e **effort variables**.

Note that $e^T f$ is power.

Example: The ubiquitous mass-spring-damper system:

Two storage elements:

- **Spring** Hamiltonian $H_s(q) = \frac{1}{2}kq^2$ (potential energy)

$$\begin{aligned} \dot{q} &= f_s & &= \text{velocity} \\ e_s &= \frac{dH_s}{dq}(q) = kq & &= \text{force} \end{aligned}$$

- **Mass** Hamiltonian $H_m(p) = \frac{1}{2m}p^2$ (kinetic energy)

$$\begin{aligned} \dot{p} &= f_m & &= \text{force} \\ e_m &= \frac{dH_m}{dp}(p) = \frac{p}{m} & &= \text{velocity} \end{aligned}$$

interconnected by

$$f_s = e_m = y, \quad f_m = -e_s + u$$

(power-conserving since $f_s e_s + f_m e_m = uy$) yields the port-Hamiltonian system

$$\begin{bmatrix} \dot{q} \\ \dot{p} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial H}{\partial q}(q,p) \\ \frac{\partial H}{\partial p}(q,p) \end{bmatrix} (q,p) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{\partial H}{\partial q}(q,p) \\ \frac{\partial H}{\partial p}(q,p) \end{bmatrix}$$

with

$$H(q,p) = H_s(q) + H_m(p)$$

Addition of the damper

$$e_d = \frac{dR}{df_d} = cf_d, \quad R(f_d) = \frac{1}{2}cf_d^2 \quad \text{(Rayleigh function)}$$

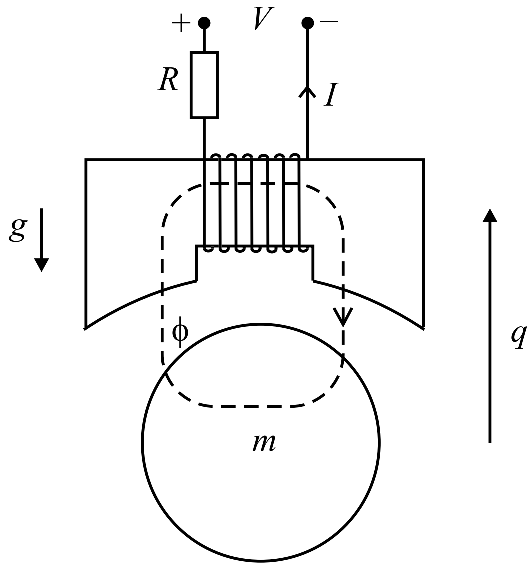
via the extended interconnection

$$f_s = e_m = f_d = y, \quad f_m = e_s - e_d + u$$

leads to the **mass-damper-spring system** in port-Hamiltonian form

$$\begin{aligned} \begin{bmatrix} \dot{q} \\ \dot{p} \end{bmatrix} &= \left(\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & c \end{bmatrix} \right) \begin{bmatrix} \frac{\partial H}{\partial q}(q,p) \\ \frac{\partial H}{\partial p}(q,p) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \\ y &= \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{\partial H}{\partial q}(q,p) \\ \frac{\partial H}{\partial p}(q,p) \end{bmatrix} \end{aligned}$$

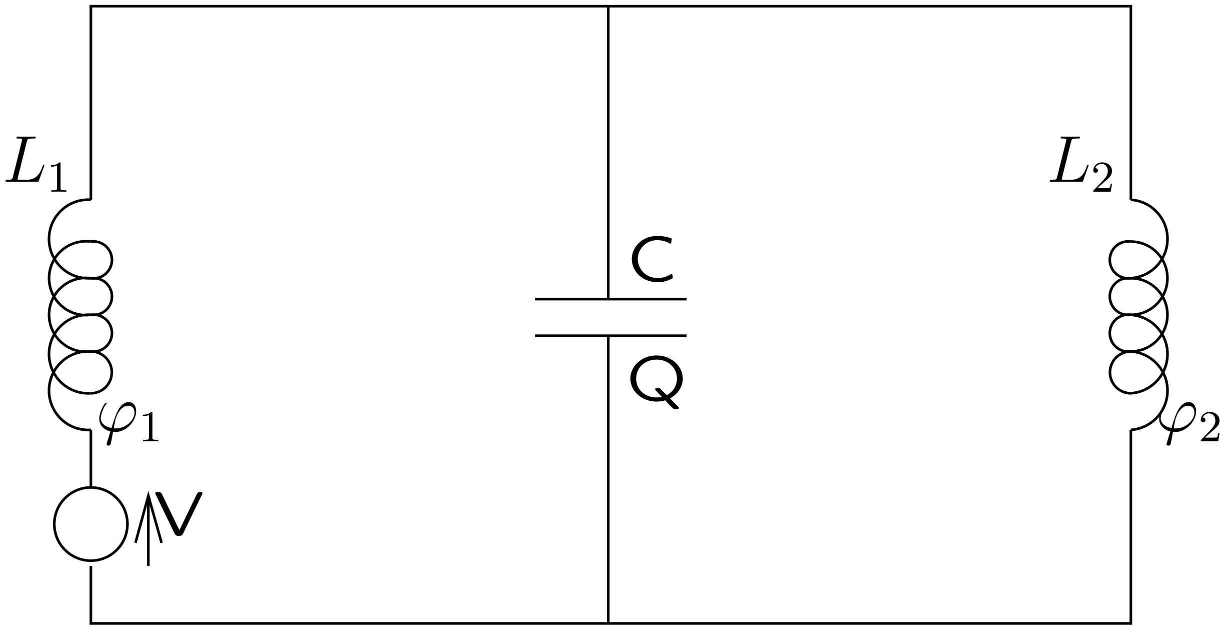
Example: Electro-mechanical systems



$$\begin{bmatrix} \dot{q} \\ \dot{p} \\ \dot{\varphi} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -\frac{1}{R} \end{bmatrix} \begin{bmatrix} \frac{\partial H}{\partial q}(q,p,\phi) \\ \frac{\partial H}{\partial p}(q,p,\phi) \\ \frac{\partial H}{\partial \varphi}(q,p,\phi) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} V, \qquad I = \frac{\partial H}{\partial \varphi}(q,p,\phi)$$

Coupling electrical/mechanical domain via Hamiltonian $H(q,p,\phi)$.

Example: LC circuits. Two inductors with magnetic energies $H_1(\varphi_1), H_2(\varphi_2)$ (φ_1 and φ_2 magnetic flux linkages), and capacitor with electric energy $H_3(Q)$ (Q charge). V denotes the voltage of the source.



Question: *How to write this circuit as a port-Hamiltonian system in a modular way?*

Hamiltonian equations for the components of the LC-circuit:

$$\begin{array}{lrcll} \textit{Inductor } 1 & \dot{\varphi}_1 & = & f_1 & \text{(voltage)} \\ \text{(current)} & e_1 & = & \frac{\partial H_1}{\partial \varphi_1} & \\ \textit{Inductor } 2 & \dot{\varphi}_2 & = & f_2 & \text{(voltage)} \\ \text{(current)} & e_2 & = & \frac{\partial H_2}{\partial \varphi_2} & \\ \textit{Capacitor} & \dot{Q} & = & f_3 & \text{(current)} \\ \text{(voltage)} & e_3 & = & \frac{\partial H_3}{\partial Q} & \end{array}$$

All are port-Hamiltonian systems with $J = 0$ and $g = 1$.

If the elements are *linear* then the Hamiltonians are *quadratic*, e.g. $H_1(\varphi_1) = \frac{1}{2L_1}\varphi_1^2$, and $\frac{\partial H_1}{\partial \varphi_1} = \frac{\varphi_1}{L_1} =$ current , etc.

Kirchhoff's interconnection laws in $f_1, f_2, f_3, e_1, e_2, e_3, f = V, e = I$ are

$$\begin{bmatrix} -f_1 \\ -f_2 \\ -f_3 \\ e \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 0 \\ -1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ e_3 \\ f \end{bmatrix}$$

Substitution of eqns. of components yields port-Hamiltonian system

$$\begin{aligned} \begin{bmatrix} \dot{\varphi}_1 \\ \dot{\varphi}_2 \\ \dot{Q} \end{bmatrix} &= \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial H}{\partial \varphi_1} \\ \frac{\partial H}{\partial \varphi_2} \\ \frac{\partial H}{\partial Q} \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} f \\ e &= \frac{\partial H}{\partial \varphi_1} \end{aligned}$$

with $H(\varphi_1, \varphi_2, Q) := H_1(\varphi_1) + H_2(\varphi_2) + H_3(Q)$ total energy.

However, this class of port-Hamiltonian systems is **not closed under interconnection**:

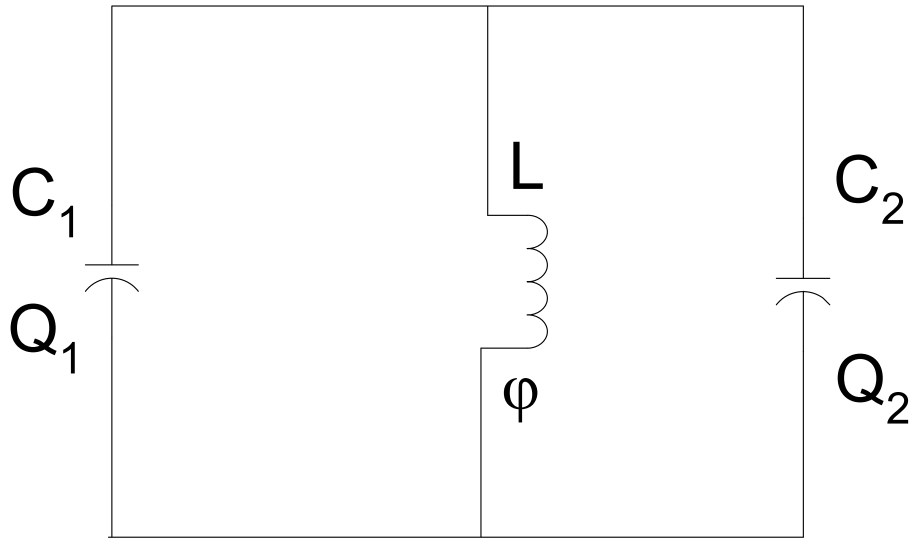


Figure 1: Capacitors and inductors swapped

Interconnection leads to **algebraic constraints** between the state variables Q_1 and Q_2.

How to model DAEs as port-Hamiltonian systems ?

Intermezzo: what is the appropriate generalization of the skew-symmetric mapping J ? Answer: **Dirac structures**. ('From skew-symmetric *mappings* to skew-symmetric *relations*')

Power is defined by

$$P = e(f) =: < e \mid f >= e^T f, \quad (f, e) \in \mathcal{V} \times \mathcal{V}^*.$$

where the linear space $\mathcal{V}$ is called the space of *flows* f (e.g. currents), and $\mathcal{V}^*$ the space of *efforts* e (e.g. voltages). Symmetrized form of power is the indefinite *bilinear form* $\ll, \gg$ on $\mathcal{V} \times \mathcal{V}^*$:

$$\ll (f^a, e^a), (f^b, e^b) \gg \quad := \quad < e^a \mid f^b > + < e^b \mid f^a >,$$

$$(f^a, e^a), (f^b, e^b) \in \mathcal{V} \times \mathcal{V}^*.$$

Definition 1 (Weinstein, Courant, Dorfman) *A (constant) Dirac structure is a subspace*

$$\mathcal{D} \subset \mathcal{V} \times \mathcal{V}^*$$

such that

$$\mathcal{D} = \mathcal{D}^{\perp},$$

where $\perp$ *denotes orthogonal complement with respect to the bilinear form* $\ll, \gg$.

For a *finite-dimensional* space $\mathcal{V}$ this is equivalent to

(i) $< e \mid f > e^T f = 0$ for all $(f, e) \in \mathcal{D}$,

(ii) $\dim \mathcal{D} = \dim \mathcal{V}$.

For any skew-symmetric map $J : \mathcal{V}^* \to \mathcal{V}$ its **graph**

$$\{(f, e) \in \mathcal{V} \times \mathcal{V}^* \mid f = Je\}$$

is a Dirac structure !

For many systems, especially those with 3-D mechanical components, the interconnection structure will be *modulated* by the energy or geometric variables.

This leads to the notion of *non-constant* Dirac structures on *manifolds*.

Definition 2 *Consider a smooth manifold* M*. A Dirac structure on* M *is a vector subbundle* $\mathcal{D} \subset TM \oplus T^*M$ *such that for every* $x \in M$ *the vector space*

$$\mathcal{D}(x) \subset T_x M \times T_x^* M$$

is a Dirac structure as before.

Geometric definition of a port-Hamiltonian system

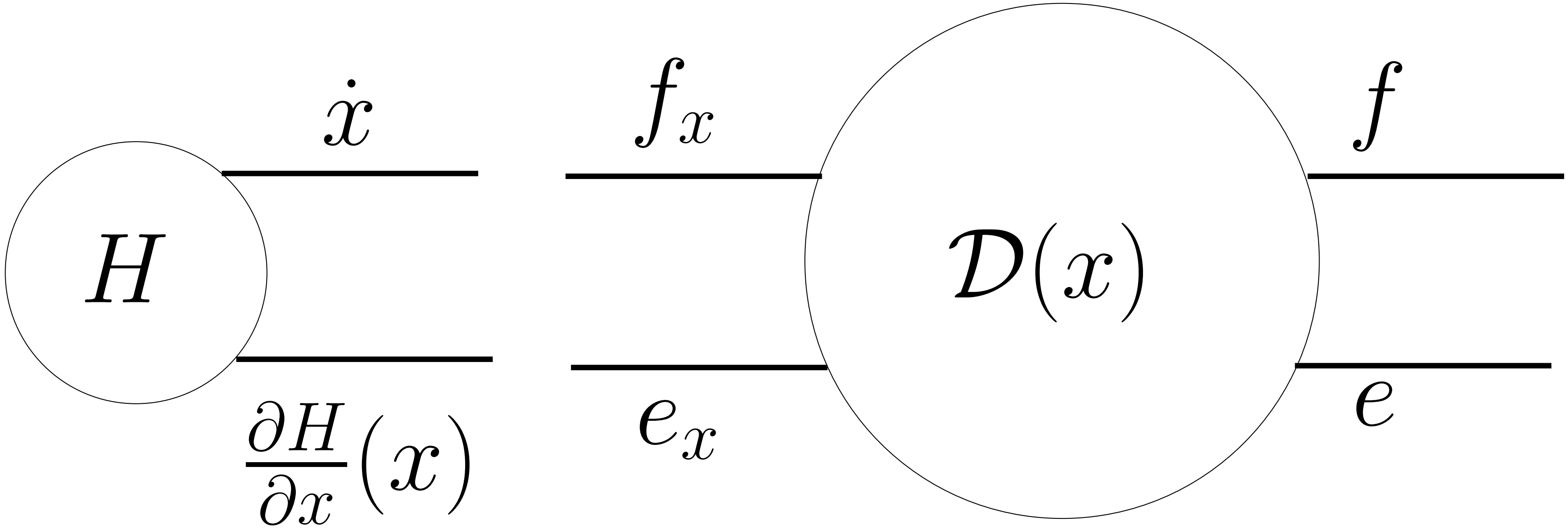


Figure 2: Port-Hamiltonian system

The dynamical system defined by the DAEs

$$(-\dot{x}(t) = f_x(t), \frac{\partial H}{\partial x}(x(t)) = e_x(t), f(t), e(t)) \in \mathcal{D}(x(t)), \quad t \in \mathbb{R}$$

is called a **port-Hamiltonian system**.

Particular case is a Dirac structure $\mathcal{D}(x) \subset T_x\mathcal{X} \times T_x^*\mathcal{X} \times \mathcal{F} \times \mathcal{F}^*$ given as the graph of the skew-symmetric map

$$\begin{bmatrix} f_x \\ e \end{bmatrix} = \begin{bmatrix} -J(x) & -g(x) \\ g^T(x) & 0 \end{bmatrix} \begin{bmatrix} e_x \\ f \end{bmatrix},$$

leading ($f_x = -\dot{x}, e_x = \frac{\partial H}{\partial x}(x)$) to a port-Hamiltonian system as before

$$\begin{aligned} \dot{x} &= J(x)\frac{\partial H}{\partial x}(x) + g(x)f, \quad x \in \mathcal{X}, f \in \mathbb{R}^m \\ e &= g^T(x)\frac{\partial H}{\partial x}(x), \qquad\qquad e \in \mathbb{R}^m \end{aligned}$$

Energy-dissipation is included by terminating some of the ports by static resistive elements

$$f_R = -F(e_R), \qquad \text{where } e_R^T F(e_R) \geq 0, \quad \text{for all } e_R.$$

If $H \geq 0$ then the system stays passive with respect to the remaining flows and efforts f and e:

$$\frac{d}{dt}H \leq e^T f$$

This leads, e.g. for linear damping, to input-state-output port-Hamiltonian systems in the form

$$\begin{cases} \dot{x} &= [J(x) - R(x)]\frac{\partial H}{\partial x}(x) + g(x)f \\ e &= g^T(x)\frac{\partial H}{\partial x}(x) \end{cases}$$

where $J(x) = -J^T(x)$, $R(x) = R^T(x) \geq 0$ are the interconnection and damping matrices, respectively.

Example: Mechanical systems with kinematic constraints

Constraints on the generalized velocities $\dot{q}$:

$$A^T(q)\dot{q} = 0.$$

This leads to **constrained** Hamiltonian equations

$$\begin{aligned} \dot{q} &= \frac{\partial H}{\partial p}(q,p) \\ \dot{p} &= -\frac{\partial H}{\partial q}(q,p) + A(q)\lambda + B(q)u \\ 0 &= A^T(q)\frac{\partial H}{\partial p}(q,p) \\ y &= B^T(q)\frac{\partial H}{\partial p}(q,p) \end{aligned}$$

with $H(q,p)$ total energy, and λ the **constraint forces**.
By elimination of the constraints and constraint forces one derives a port-Hamiltonian model *without constraints*.

Can be extended to general *multi-body systems*.

Intermezzo: Relation with classical Hamiltonian equations

$$\dot{x} = J\frac{\partial H}{\partial x}(x)$$

with constant or ' integrable'J- matrix admits coordinates $x = (q, p, r)$ in which

$$J = \begin{bmatrix} 0 & I & 0 \\ -I & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad \begin{array}{rcl} \dot{q} & = & \frac{\partial H}{\partial p}(q,p,r) \\ \dot{p} & = & -\frac{\partial H}{\partial q}(q,p,r) \\ \dot{r} & = & 0 \end{array}$$

For constant or integrable Dirac structure one gets Hamiltonian DAEs

$$\begin{array}{rcl} \dot{q} & = & \frac{\partial H}{\partial p}(q,p,r,s) \\ \dot{p} & = & -\frac{\partial H}{\partial q}(q,p,r,s) \\ \dot{r} & = & 0 \\ 0 & = & \frac{\partial H}{\partial s}(q,p,r,s) \end{array}$$

Some properties of port-Hamiltonian systems

Port-Hamiltonian systems modeling encodes more information than energy-balance.

The Dirac structure determines all the *Casimir functions* (conserved quantities which are independent of H).
Example: In the first LC circuit the total flux $\phi_1 + \phi_2$ is a conserved quantity that is solely determined by the circuit topology.
(In Part II this will be used for *set-point control*.)
Furthermore, the Dirac structure directly determines the *algebraic constraints*.
Example: In the second LC-circuit the state variables Q_1 and Q_2 are related by

$$\frac{Q_1}{C_1} = \frac{Q_2}{C_2}$$

Any power-conserving interconnection of port-Hamiltonian systems is again port-Hamiltonian

- The resulting Hamiltonian is the sum of the Hamiltonians of the individual systems.
- The Dirac structure is determined by the Dirac structures of the individual systems, and the way they are interconnected.
- The resistive structure is determined by the resistive structures of the individual systems.

Conclusion: **port-Hamiltonian systems theory provides a modular framework for modeling and analysis of complex multi-physics lumped-parameter systems.**

Network modeling is prevailing in modeling and simulation of lumped-parameter physical systems (multi-body systems, electrical circuits, electro-mechanical systems, hydraulic systems, robotic systems, etc.), with many advantages:

- Modularity and flexibility. Re-usability ('libraries').
- Multi-physics approach.
- Suited to design/control.

Disadvantage of network modeling: it generally leads to a large set of DAEs, *seemingly without any structure*.
Port-based modeling and port-Hamiltonian system theory identifies the underlying structure of network models of physical systems, to be used for analysis, simulation and control.

Distributed-parameter port-Hamiltonian systems

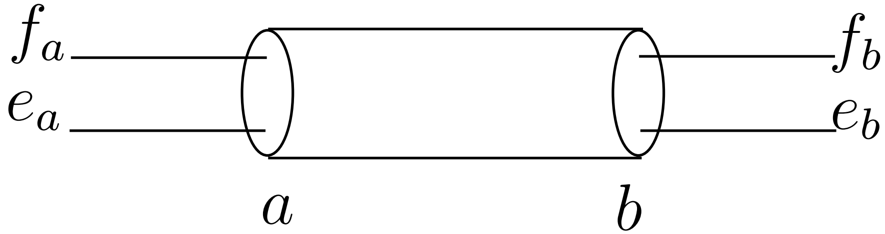


Figure 3: **Simplest example: Transmission line**

Telegrapher's equations define the boundary control system

$$
\begin{aligned}
\frac{\partial Q}{\partial t}(z,t) &= -\frac{\partial}{\partial z} I(z,t) &&= -\frac{\partial}{\partial z}\frac{\phi(z,t)}{L(z)} \\
\frac{\partial \phi}{\partial t}(z,t) &= -\frac{\partial}{\partial z} V(z,t) &&= -\frac{\partial}{\partial z}\frac{Q(z,t)}{C(z)} \\
f_a(t) &= V(a,t), \quad e_1(t) &&= I(a,t) \\
f_b(t) &= V(b,t), \quad e_2(t) &&= I(b,t)
\end{aligned}
$$

Transmission line as port-Hamiltonian system

Define *internal* flows $f_x = (f_E, f_M)$ and efforts $e_x = (e_E, e_M)$:

electric flow	$f_E : [a,b] \to \mathbb{R}$
magnetic flow	$f_M : [a,b] \to \mathbb{R}$
electric effort	$e_E : [a,b] \to \mathbb{R}$
magnetic effort	$e_M : [a,b] \to \mathbb{R}$

together with *external* boundary flows $f = (f_a, f_b)$ and boundary efforts $e = (e_a, e_b)$. Define the *infinite-dimensional Dirac structure*

$$\begin{bmatrix} f_E \\ f_M \end{bmatrix} = \begin{bmatrix} 0 & \frac{\partial}{\partial z} \\ \frac{\partial}{\partial z} & 0 \end{bmatrix} \begin{bmatrix} e_E \\ e_M \end{bmatrix}$$

$$\begin{bmatrix} f_{a,b} \\ e_{a,b} \end{bmatrix} = \begin{bmatrix} e_{E|a,b} \\ e_{M|a,b} \end{bmatrix}$$

This defines a Dirac structure on the space of *internal* flows and efforts and *boundary* flows and efforts.

Substituting (as in the lumped-parameter case)

$$\left.\begin{array}{rcl} f_E & = & -\frac{\partial Q}{\partial t} \\ f_M & = & -\frac{\partial \varphi}{\partial t} \end{array}\right\} f_x = -\dot{x}$$

$$\left.\begin{array}{rcl} e_E & = & \frac{Q}{C} = \frac{\partial \mathcal{H}}{\partial Q} \\ e_M & = & \frac{\varphi}{L} = \frac{\partial \mathcal{H}}{\partial \varphi} \end{array}\right\} e_x = \frac{\partial H}{\partial x}$$

with, for example, quadratic energy density

$$\mathcal{H}(Q,\varphi) = \frac{1}{2}\frac{Q^2}{C} + \frac{1}{2}\frac{\varphi^2}{L}$$

we recover the telegrapher's equations.

Of course, the telegrapher's equations can be rewritten as the linear **wave equation**

$$\frac{\partial^2 Q}{\partial t^2} = -\frac{\partial}{\partial z}\frac{\partial I}{\partial t} = -\frac{\partial}{\partial z}\frac{\partial}{\partial t}\frac{\phi}{L} =$$

$$-\frac{\partial}{\partial z}\frac{1}{L}\frac{\partial \phi}{\partial t} = \frac{\partial}{\partial z}\frac{1}{L}\frac{\partial}{\partial z}\frac{Q}{C} = \frac{1}{LC}\frac{\partial^2 Q}{\partial z^2}$$

(if $L(z), C(z)$ do not depend on z), or similar expressions in ϕ, I or V.

The same equations hold for a *vibrating string*, or for a *compressible gas/fluid* in a one-dimensional pipe (see Part III).

Basic question:

Which of the boundary variables f_a, f_b, e_a, e_b can be considered to be inputs, and which outputs ? **See Part III**.

Example 2: Shallow water equations; distributed-parameter port-Hamiltonian system with non-quadratic Hamiltonian

The dynamics of the water in an open-channel canal can be described by

$$\partial_t \begin{bmatrix} h \\ v \end{bmatrix} + \begin{bmatrix} v & h \\ g & v \end{bmatrix} \partial_z \begin{bmatrix} h \\ v \end{bmatrix} = 0$$

with $h(z,t)$ the height of the water at position z, and $v(z,t)$ the velocity (and g gravitational constant).

This can be written as a port-Hamiltonian system by recognizing the total energy

$$H(h,v) = \frac{1}{2} \int_a^b [hv^2 + gh^2] dz$$

yielding the co-energy functions[a]

$$\begin{aligned} e_h &= \frac{\partial \mathcal{H}}{\partial h} = \tfrac{1}{2}v^2 + gh \quad \text{Bernoulli function} \\ e_v &= \frac{\partial \mathcal{H}}{\partial v} = hv \quad \text{mass flow} \end{aligned}$$

It follows that the shallow water equations can be written, similarly to the telegraphers equations, as

$$\frac{\partial h}{\partial t}(z,t) = -\frac{\partial}{\partial z}\frac{\partial \mathcal{H}}{\partial v}$$

$$\frac{\partial v}{\partial t}(z,t) = -\frac{\partial}{\partial z}\frac{\partial \mathcal{H}}{\partial h}$$

with boundary variables $-hv_{|a,b}$ and $(\frac{1}{2}v^2 + gh)_{|a,b}$.

[a] Daniel Bernoulli, born in 1700 in Groningen as son of Johann Bernoulli, professor in mathematics at the University of Groningen and forerunner of the Calculus of Variations (the *Brachistochrone problem*).

Paying tribute to history:

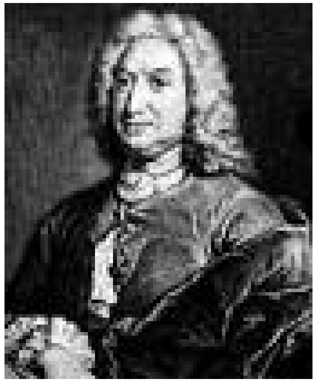


Figure 4: Johann Bernoulli, professor in Groningen 1695-1705.

Figure 5: Daniel Bernoulli, born in Groningen in 1700.

We obtain the energy balance

$$\frac{d}{dt}\int_a^b [hv^2 + gh^2]dz = -(hv)(\frac{1}{2}v^2 + gh)|_a^b$$

which can be rewritten as

$$-v(\tfrac{1}{2}gh^2)|_a^b - v(\tfrac{1}{2}hv^2 + \tfrac{1}{2}gh^2))|_a^b =$$

velocity $\times$ pressure $+$ energy flux through the boundary

Conservation laws

All examples sofar have the same structure

$$\frac{\partial \alpha_1}{\partial t}(z,t) = -\frac{\partial}{\partial z}\frac{\partial \mathcal{H}}{\partial \alpha_2} = -\frac{\partial}{\partial z}\beta_2$$

$$\frac{\partial \alpha_2}{\partial t}(z,t) = -\frac{\partial}{\partial z}\frac{\partial \mathcal{H}}{\partial \alpha_1} = -\frac{\partial}{\partial z}\beta_1$$

with boundary variables $\beta_1|_{\{a,b\}}, \beta_2|_{\{a,b\}}$, corresponding to two coupled **conservation laws**:

$$\frac{d}{dt}\int_a^b \alpha_1 = -\int_a^b \frac{\partial}{\partial z}\beta_2 = \beta_2(a) - \beta_2(b)$$

$$\frac{d}{dt}\int_a^b \alpha_2 = -\int_a^b \frac{\partial}{\partial z}\beta_1 = \beta_1(a) - \beta_1(b)$$

(In the transmission line, α_1 and α_2 is charge- and flux-density, and β_1, β_2 voltage V and current I, respectively.)

For some purposes it is illuminating to rewrite the equations in terms of the co-energy variables β_1, β_2:

$$\begin{bmatrix} \frac{\partial \beta_1}{\partial t} \\ \frac{\partial \beta_2}{\partial t} \end{bmatrix} = \begin{bmatrix} \frac{\partial^2 H}{\partial \alpha_1^2} & \frac{\partial^2 H}{\partial \alpha_1 \alpha_2} \\ \frac{\partial^2 H}{\partial \alpha_2 \alpha_1} & \frac{\partial^2 H}{\partial \alpha_2^2} \end{bmatrix} \begin{bmatrix} \frac{\partial \alpha_1}{\partial t} \\ \frac{\partial \alpha_2}{\partial t} \end{bmatrix} = - \begin{bmatrix} \frac{\partial^2 H}{\partial \alpha_1^2} & \frac{\partial^2 H}{\partial \alpha_1 \alpha_2} \\ \frac{\partial^2 H}{\partial \alpha_2 \alpha_1} & \frac{\partial^2 H}{\partial \alpha_2^2} \end{bmatrix} \begin{bmatrix} \frac{\partial \beta_2}{\partial z} \\ \frac{\partial \beta_1}{\partial z} \end{bmatrix}$$

For the transmission line this yields

$$\begin{bmatrix} \frac{\partial V}{\partial t} \\ \frac{\partial I}{\partial t} \end{bmatrix} = - \begin{bmatrix} 0 & \frac{1}{C} \\ \frac{1}{L} & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial V}{\partial z} \\ \frac{\partial I}{\partial z} \end{bmatrix}$$

The matrix is called the **characteristic matrix**, whose eigenvalues are the characteristic velocities $\frac{1}{\sqrt{LC}}$ and $-\frac{1}{\sqrt{LC}}$ corresponding to the characteristic eigenvectors (and curves).

For the shallow water equations this yields

$$\begin{bmatrix} \frac{\partial \beta_1}{\partial t} \\ \frac{\partial \beta_2}{\partial t} \end{bmatrix} = - \begin{bmatrix} v & g \\ h & v \end{bmatrix} \begin{bmatrix} \frac{\partial \beta_1}{\partial z} \\ \frac{\partial \beta_2}{\partial z} \end{bmatrix}$$

with

$$\beta_1 = \frac{1}{2} v^2 + gh, \quad \beta_2 = hv$$

being the Bernoulli function and mass flow, respectively.

This corresponds to two characteristic velocities $v \pm \sqrt{gh}$, which are, like in the transmission line case, of opposite sign (*subcritical or fluvial flow*) if

$$v^2 \leq gh$$

Because the Hamiltonian is non-quadratic, and thus the pde's are nonlinear, the characteristic curves may **intersect**, corresponding to shock waves.

Mixed lumped- and distributed-parameter port-Hamiltonian systems

Typical example: power-converter connected via a transmission line to a resistive load or an induction motor:

- The power-converter is a port-Hamiltonian system (with switching Dirac structure).

- Transmission line is distributed-parameter port-Hamiltonian system.

- Induction motor is a port-Hamiltonian system, with Hamiltonian being the electro-mechanical energy.

Power converter connected to the load via transmission line

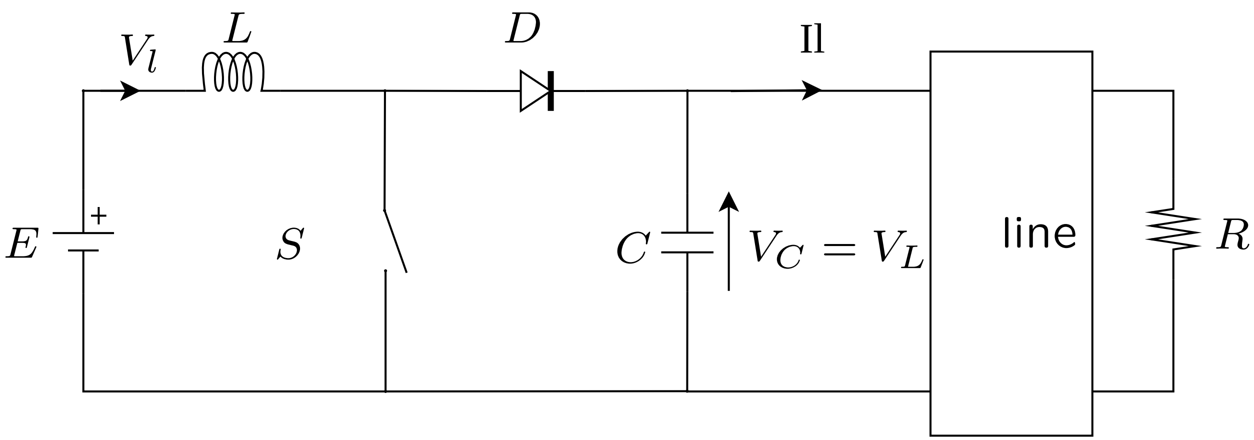


Figure 6: The Boost converter with a transmission line

Boost power converter

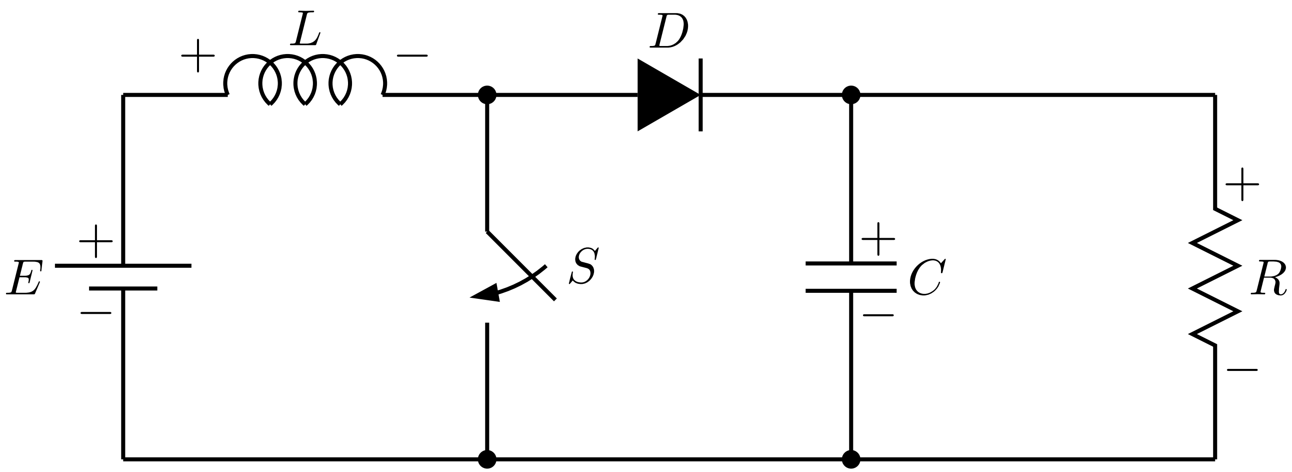


Figure 7: Boost circuit with clamping diode

The circuit consists of a capacitor C with electric charge q_C, an inductor L with magnetic flux linkage ϕ_L, and a resistive load R, together with an ideal diode and an ideal switch S, with switch positions $s = 1$ (switch closed) and $s = 0$ (switch open).

The voltage-current characteristics of the ideal diode and switch are depicted in Figure 8.

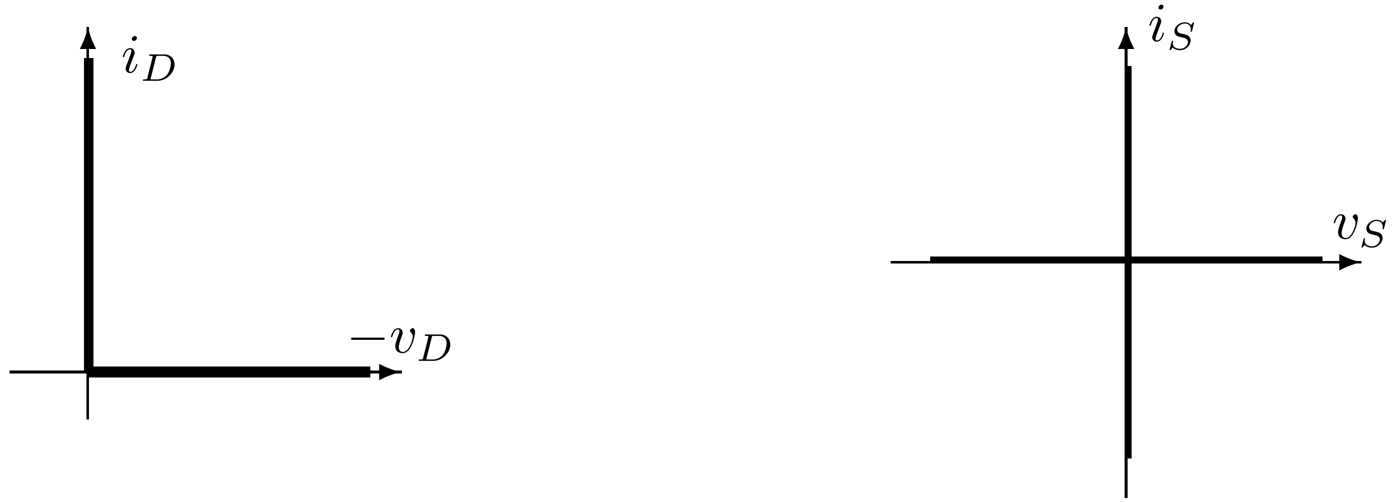


Figure 8: Voltage-current characteristic of an ideal diode and ideal switch

The ideal diode thus satisfies the complementarity conditions:

$$v_D i_D = 0, \quad v_D \leq 0, \quad i_D \geq 0.$$

This yields the port-Hamiltonian model (with $H(q_C, \phi_L) = \frac{1}{2C}q_C^2 + \frac{1}{2L}\phi_L^2$):

$$\begin{bmatrix} \dot{q}_C \\ \dot{\phi}_L \end{bmatrix} = \begin{bmatrix} -\frac{1}{R} & 1-s \\ s-1 & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial H}{\partial q_C} = \frac{q_C}{C} \\ \frac{\partial H}{\partial \phi_L} = \frac{\phi_L}{L} \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} E + \begin{bmatrix} s i_D \\ (s-1) v_D \end{bmatrix}$$

$$I = \frac{\phi_L}{L}$$

Assume that the switch and the diode are **coupled** in the following sense: if the switch is closed ($s = 1$) then the diode is open ($i_D = 0$), while if the switch is open ($s = 0$), then the diode is closed ($v_D = 0$). (This means that we disregard the so-called *discontinuous* modes.)

In this case we obtain the *switching port-Hamiltonian system*

$$\begin{bmatrix} \dot{q}_C \\ \dot{\phi}_L \end{bmatrix} = \begin{bmatrix} -\frac{1}{R} & 1-s \\ s-1 & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial H}{\partial q_C} = \frac{q_C}{C} \\ \frac{\partial H}{\partial \phi_L} = \frac{\phi_L}{L} \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} E$$

$$I = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{\partial H}{\partial q_C} = \frac{q_C}{C} \\ \frac{\partial H}{\partial \phi_L} = \frac{\phi_L}{L} \end{bmatrix} = \frac{\phi_L}{L}$$

A port-Hamiltonian system where the J-matrix depends on the switch position.

In general, a switching port-Hamiltonian system (without algebraic equality and inequality constraints) is defined as

$$\begin{aligned} \dot{x} &= F(\rho)z + g(\rho)u, \quad z = \tfrac{\partial H}{\partial x}(x) \\ y &= g^T(\rho)z \end{aligned}$$

with $\rho \in \{0,1\}^p$, and

$$F(\rho) = J(\rho) - R(\rho), \quad J(\rho) = -J^T(\rho), \quad R(\rho) = R^T(\rho) \geq 0$$

Note that the system is passive for every switching sequence:

$$\frac{d}{dt}H = -\frac{\partial^T H}{\partial x}(x)R(\rho)\frac{\partial H}{\partial x}(x) + u^T y \leq u^T y$$

Conclusions of Part I

- Port-Hamiltonian systems provide a unified framework for *modeling, analysis, and simulation* of complex lumped-parameter multi-physics systems.
- Inclusion of distributed-parameter components.
- Spatial discretization of distributed-parameter port-Hamiltonian systems to finite-dimensional port-Hamiltonian systems.

◦ Extensions to thermodynamic systems and chemical reaction networks.

◦ Further exploration of the network (graph) information.

See the forthcoming book on port-Hamiltonian systems produced by the EU-IST GeoPleX consortium,
or the website www.math.rug.nl/~arjan, for further info.

The port-Hamiltonian approach to physical system modeling and control

Part II: Control of Port-Hamiltonian systems

Contents

Use of passivity for control and beyond

- The storage function can be used as Lyapunov function, implying some sort of stability for the uncontrolled system.
- The standard feedback interconnection of two passive systems is again passive, with storage function being the **sum** of the individual storage functions.
- Passive systems can be asymptotically stabilized by adding artificial **damping**. In fact,

$$\frac{d}{dt}H \leq u^T y$$

together with the additional damping $u = -y$ yields

$$\frac{d}{dt}H \leq - \parallel y \parallel^2$$

proving **asymptotic stability** provided an observability condition is met.

Example The Euler equations for the motion of a rigid body revolving about its center of gravity with one input are

$$\begin{aligned} I_1\dot{\omega}_1 &= [I_2 - I_3]\omega_2\omega_3 + g_1 u \\ I_2\dot{\omega}_2 &= [I_3 - I_1]\omega_1\omega_3 + g_2 u \\ I_3\dot{\omega}_3 &= [I_1 - I_2]\omega_1\omega_2 + g_3 u, \end{aligned}$$

Here $\omega := (\omega_1, \omega_2, \omega_3)^T$ are the angular velocities around the principal axes of the rigid body, and $I_1, I_2, I_3 > 0$ are the principal moments of inertia. The system for $u = 0$ has the origin as an equilibrium point. Linearization yields the linear system

$$A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad B = \begin{pmatrix} I_1^{-1} g_1 \\ I_2^{-1} g_2 \\ I_3^{-1} g_3 \end{pmatrix}.$$

Hence the linearization does not say anything about stabilizability.

Stability and asymptotic stabilization by damping injection

Rewrite the system in port-Hamiltonian form by defining the angular momenta

$$p_1 = I_1\dot{\omega}_1,\, p_2 = I_2\dot{\omega}_2,\, p_3 = I_3\dot{\omega}_3$$

and defining the Hamiltonian $H(p)$ as the total kinetic energy

$$H(p) = \frac{1}{2}\left(\frac{p_1^2}{I_1} + \frac{p_2^2}{I_2} + \frac{p_3^2}{I_3}\right)$$

Then the system can be rewritten as

$$\begin{bmatrix} \dot{p}_1 \\ \dot{p}_2 \\ \dot{p}_3 \end{bmatrix} = \begin{bmatrix} 0 & -p_3 & p_2 \\ p_3 & 0 & -p_1 \\ -p_2 & p_1 & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial H}{\partial p_1} \\ \frac{\partial H}{\partial p_2} \\ \frac{\partial H}{\partial p_3} \end{bmatrix} + \begin{bmatrix} g_1 \\ g_2 \\ g_3 \end{bmatrix} u, \quad y = \begin{bmatrix} g_1 & g_2 & g_3 \end{bmatrix} \begin{bmatrix} \frac{\partial H}{\partial p_1} \\ \frac{\partial H}{\partial p_2} \\ \frac{\partial H}{\partial p_3} \end{bmatrix}$$

Since $\dot{H} = 0$ and H has a minimum at $p = 0$ the origin is **stable**. Damping injection amounts to the negative output feedback

$$u = -y = -g_1 \frac{p_1}{I_1} - g_2 \frac{p_2}{I_2} - g_3 \frac{p_3}{I_3} = -g_1 \omega_1 - g_2 \omega_2 - g_3 \omega_3,$$

yielding convergence to the largest invariant set contained in

$$\mathcal{S} := \{p \in \mathbb{R}^3 \mid \dot{H}(p) = 0\} = \{p \in \mathbb{R}^3 \mid g_1 \frac{p_1}{I_1} + g_2 \frac{p_2}{I_2} + g_3 \frac{p_3}{I_3} = 0\}$$

It can be shown that the largest invariant set contained in $\mathcal{S}$ is the origin $p = 0$ if and only if

$$g_1 \neq 0, g_2 \neq 0, g_3 \neq 0,$$

in which case the origin is rendered **asymptotically stable** (even, globally).

Beyond control via passivity: What can we do if the desired set-point is **not** a minimum of the storage function ??

Recall the proof of stability of an equilibrium $(\omega_1^*, 0, 0) \neq (0, 0, 0)$ of the Euler equations.
The total energy $H = \frac{2I_1}{p_1^2} + \frac{2I_2}{p_2^2} + \frac{2I_3}{p_3^2} = \frac{1}{2}I_1\omega_1^2 + \frac{1}{2}I_2\omega_2^2 + \frac{1}{2}I_3\omega_3^2$ has a minimum at $(0, 0, 0)$. Stability of $(\omega_1^*, 0, 0)$ is shown by taking as Lyapunov function a combination of the total energy K and another **conserved quantity**, namely the total angular momentum

$$C = p_1^2 + p_2^2 + p_3^2 = I_1^2\omega_1^2 + I_2^2\omega_2^2 + I_3^2\omega_3^2$$

This follows from

$$\begin{bmatrix} p_1 & p_2 & p_3 \end{bmatrix} \begin{bmatrix} 0 & -p_3 & p_2 \\ p_3 & 0 & -p_1 \\ -p_2 & p_1 & 0 \end{bmatrix} = 0$$

In general, for any Hamiltonian dynamics

$$\dot{x} = J(x)\frac{\partial H}{\partial x}(x)$$

one may search for conserved quantities C, called **Casimirs**, as being solutions of

$$\frac{\partial^T C}{\partial x}(x)J(x) = 0$$

Then $\frac{d}{dt}C = 0$ for *every* H, and thus also $H + C$ is a **candidate Lyapunov function**.

Note that the minimum of $H + C$ may now be **different** from the minimum of H.

Control by interconnection: set-point stabilization:

Consider first a lossless Hamiltonian plant system P

$$\begin{aligned}\dot{x} &= J(x)\frac{\partial H}{\partial x}(x) + g(x)u \\ y &= g^T(x)\frac{\partial H}{\partial x}(x)\end{aligned}$$

where the desired set-point x^* is **not** a minimum of the Hamiltonian H, while the Hamiltonian dynamics $\dot{x} = J(x)\frac{\partial H}{\partial x}(x)$ does not possess useful Casimirs.

How to (asymptotically) stabilize x^* ?

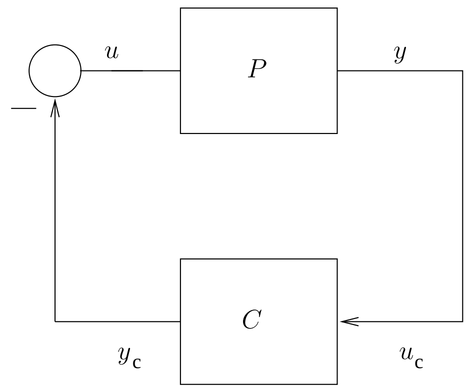
Control by interconnection:

Consider a *controller* port-Hamiltonian system

$$C: \begin{array}{rcl} \dot{\xi} & = & J_c(\xi)\frac{\partial H_c}{\partial \xi}(\xi) + g_c(\xi)u_c, \quad \xi \in \mathcal{X}_c \\ y_c & = & g^T(\xi)\frac{\partial H_c}{\partial \xi}(\xi) \end{array}$$

via the standard feedback interconnection

$$u = -y_c, \quad u_c = y$$



Then the closed-loop system is the port-Hamiltonian system

$$\begin{bmatrix} \dot{x} \\ \dot{\xi} \end{bmatrix} = \begin{bmatrix} J(x) & -g(x)g_c^T(\xi) \\ g_c(\xi)g^T(x) & J_c(\xi) \end{bmatrix} \begin{bmatrix} \frac{\partial H}{\partial x}(x) \\ \frac{\partial H_c}{\partial \xi}(\xi) \end{bmatrix}$$

with state space $\mathcal{X} \times \mathcal{X}_c$, and total Hamiltonian $H(x) + H_c(\xi)$.

Main idea: design the controller system in such a manner that the closed-loop system has useful Casimirs $C(x, \xi)$!
This may lead to a suitable candidate Lyapunov function

$$V(x, \xi) := H(x) + H_c(\xi) + C(x, \xi)$$

with H_c to-be-determined.

Thus we look for functions $C(x,\xi)$ satisfying

$$\left[\frac{\partial^T C}{\partial x}(x,\xi) \quad \frac{\partial^T C}{\partial \xi}(x,\xi)\right] \begin{bmatrix} J(x) & -g(x)g_c^T(\xi) \\ g_c(\xi)g^T(x) & J_c(\xi) \end{bmatrix} = 0$$

such that the candidate Lya[unov function

$$V(x,\xi) := H(x) + H_c(\xi) + C(x,\xi)$$

has a minimum at (x^*,ξ^*) for some (or a set of) $\xi^* \Rightarrow$ **stability**.

Remark: The set of such *achievable* closed-loop Casimirs $C(x,\xi)$ can be fully characterized.

Subsequently, one may add extra damping (directly or in the dynamics of the controller) to achieve **asymptotic** stability.

Example: the ubiquitous pendulum

Consider the mathematical pendulum with Hamiltonian

$$H(q,p) = \frac{1}{2}p^2 + (1 - \cos q)$$

actuated by a torque u, with output $y = p$ (angular velocity).
Suppose we wish to stabilize the pendulum at a **non-zero angle** q^* and $p^* = 0$.

Apply the nonlinear integral control

$$\begin{aligned} \dot{\xi} &= u_c = y \\ -u = y_c &= \tfrac{\partial H_c}{\partial \xi}(\xi) \end{aligned}$$

which is a port-Hamiltonian controller system with $J_c = 0$.

Casimirs $C(q,p,\xi)$ are found by solving

$$\begin{bmatrix} \frac{\partial C}{\partial q} & \frac{\partial C}{\partial p} & \frac{\partial C}{\partial \xi} \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} = 0$$

leading to Casimirs $C(q,p,\xi) = K(q-\xi)$, and candidate Lyapunov functions

$$V(q,p,\xi) = \frac{1}{2}p^2 + (1-\cos q) + H_c(\xi) + K(q-\xi)$$

with the functions H_c and K to be determined.

For a local minimum, determine K and H_c such that

Equilibrium assignment

$$\begin{aligned}\sin q^* + \tfrac{\partial K}{\partial z}(q^* - \xi^*) &= 0\\ -\tfrac{\partial K}{\partial z}(q^* - \xi^*) + \tfrac{\partial H_c}{\partial \xi}(\xi^*) &= 0\end{aligned}$$

Minimum condition

$$\begin{bmatrix} \cos q^* + \frac{\partial^2 K}{\partial z^2}(q^* - \xi^*) & 0 & -\frac{\partial^2 K}{\partial z^2}(q^* - \xi^*) \\ 0 & 1 & 0 \\ -\frac{\partial^2 K}{\partial z^2}(q^* - \xi^*) & 0 & \frac{\partial^2 K}{\partial z^2}(q^* - \xi^*) + \frac{\partial^2 H_c}{\partial \xi^2}(\xi^*) \end{bmatrix} > 0$$

Many possible solutions.

Example: stabilization of the shallow water equations

The dynamics of the water in a canal can be described by

$$\partial_t \begin{bmatrix} h \\ v \end{bmatrix} + \begin{bmatrix} v & h \\ g & v \end{bmatrix} \partial_z \begin{bmatrix} h \\ v \end{bmatrix} = 0$$

with $h(z,t)$ the height of the water at position z, and $v(z,t)$ its velocity (and g the gravitational constant).

Recall that by recognizing the total energy

$$H(h,v) = \int_a^b \mathcal{H} dz = \int_a^b \frac{1}{2}[hv^2 + gh^2] dz$$

this can be written (similarly to the telegrapher's equations) as the port-Hamiltonian system

$$
\begin{aligned}
\frac{\partial h}{\partial t}(z,t) &= -\frac{\partial}{\partial z}\frac{\partial \mathcal{H}}{\partial v}(h,v) \\
\frac{\partial v}{\partial t}(z,t) &= -\frac{\partial}{\partial z}\frac{\partial \mathcal{H}}{\partial h}(h,v)
\end{aligned}
$$

with the 4 boundary variables

$$
\begin{array}{l}
hv_{|a,b} \\
-(\frac{1}{2}v^2 + gh)_{|a,b}
\end{array}
$$

denoting respectively the **mass flow** and the **Bernoulli function** at the boundary points a, b.

(Note that the product $hv \cdot (\frac{1}{2}v^2 + gh)$ equals *power*.)

Suppose we want to control the water level h to a desired height h^*.

An obvious 'physical' controller is to add to one side of the canal, say the right-end b, an infinite water reservoir of height h^*, corresponding to the port-Hamiltonian 'source' system

$$\begin{aligned} \dot{\xi} &= u_c \\ y_c &= \tfrac{\partial H_c}{\partial \xi} (= gh^*) \end{aligned}$$

with Hamiltonian $H_c(\xi) = gh^* \xi$, by the feedback interconnection

$$u_c = y = h(b)v(b), \qquad y_c = -u = \frac{1}{2} v^2(b) + gh(b)$$

This yields a closed-loop port-Hamiltonian system with total Hamiltonian

$$\int_0^l \frac{1}{2}[hv^2 + gh^2] dz + gh^* \xi$$

By mass balance,

$$\int_a^b h(z,t)dz + \xi + c$$

is a Casimir for the closed-loop system. Thus we may take as Lyapunov function

$$\begin{aligned} V(h,v,\xi) &:= \tfrac{1}{2}\int_a^b [hv^2 + gh^2]dz + gh^*\xi - gh^*[\int_a^b h(z,t)dz + \xi] + \tfrac{1}{2}g(b-a)h^{*2} \\ &= \tfrac{1}{2}\int_a^b [hv^2 + g(h-h^*)^2]dz \end{aligned}$$

which has a minimum at the desired set-point $(h^*, v^* = 0, \xi^*)$ (with ξ^* arbitrary).

Remark Note that the source port-Hamiltonian system is **not** passive, since the Hamiltonian $H_c(\xi) = gh^*\xi$ is not bounded from below.

An alternative, passive, choice of the Hamiltonian controller system is to take e.g.

$$H_c(\xi) = \frac{1}{2} g h^* \xi^2$$

leading to the Lyapunov function

$$V(h, v, \xi) = \frac{1}{2} \int_a^b [hv^2 + g(h - h^*)^2] dz + \frac{1}{2} g h^* (\xi - 1)^2$$

Asymptotic stability of the equilibrium $(h^*, v^* = 0, \xi^* = 1)$ can be obtained by adding 'damping', that is, replacing $u_c = y = h(b)v(b)$ by

$$u_c := y - \frac{\partial V}{\partial \xi}(\xi) = h(b)v(b) - g h^* (\xi - 1)$$

leading to (if there is no power flow through the left-end a)

$$\frac{d}{dt} V = -g h^* (\xi - 1)^2$$

(See also the work of Bastin & co-workers for related and more refined results.)

The dissipation obstacle

Surprisingly, the presence of dissipation $R \neq 0$ may pose a problem !

$C(x)$ is a Casimir for the Hamiltonian dynamics with dissipation

$$\dot{x} = [J(x) - R(x)]\frac{\partial H}{\partial x}(x), \qquad J = J^T, R = R^T \geq 0$$

iff

$$\frac{\partial^T C}{\partial x}[J - R] = 0 \Rightarrow \frac{\partial^T C}{\partial x}[J - R]\frac{\partial C}{\partial x} = 0 \Rightarrow \frac{\partial^T C}{\partial x}R\frac{\partial C}{\partial x} = 0 \Rightarrow \frac{\partial^T C}{\partial x}R = 0$$

and thus C is a Casimir iff

$$\frac{\partial^T C}{\partial x}(x)J(x) = 0, \qquad \frac{\partial^T C}{\partial x}(x)R(x) = 0$$

The physical reason for the dissipation obstacle is that by using a passive controller only equilibria where **no** energy-dissipation takes place may be stabilized.

Similarly, if $C(x,\xi)$ is a Casimir for the closed-loop port-Hamiltonian system then it must satisfy

$$\begin{bmatrix} \frac{\partial^T C}{\partial x}(x,\xi) & \frac{\partial^T C}{\partial \xi}(x,\xi) \end{bmatrix} \begin{bmatrix} R(x) & 0 \\ 0 & R_c(\xi) \end{bmatrix} = 0$$

implying by semi-positivity of $R(x)$ and $R_c(x)$

$$\begin{aligned} \frac{\partial^T C}{\partial x}(x,\xi)R(x) = 0 \\ \frac{\partial^T C}{\partial \xi}(x,\xi)R_c(\xi) = 0 \end{aligned}$$

This is the **dissipation obstacle**, which implies that one cannot shape the Lyapunov function in the coordinates that are directly affected by energy dissipation.

Remark: For shaping the potential energy in mechanical systems this is **not** a problem since dissipation enters in the differential equations for the momenta.

To overcome the dissipation obstacle

Suppose one can find a mapping $C : \mathcal{X} \to \mathbb{R}^m$, with its (transposed) Jacobian matrix $K^T(x) := \frac{\partial C}{\partial x}(x)$ satisfying

$$[J(x) - R(x)]K(x) + g(x) = 0$$

Construct now the interconnection and dissipation matrix of an *augmented system* as

$$J_{aug} := \begin{bmatrix} J & JK \\ K^T J & K^T JK \end{bmatrix}, \quad R_{aug} := \begin{bmatrix} R & RK \\ K^T R & K^T RK \end{bmatrix}$$

By construction

$$[K^T(x) \mid -I]J_{aug} = [K^T(x) \mid -I]R_{aug} = 0$$

implying that the components of C are Casimirs for the Hamiltonian dynamics

$$\begin{bmatrix} \dot{x} \\ \dot{\xi} \end{bmatrix} = [J_{aug} - R_{aug}] \begin{bmatrix} \frac{\partial H}{\partial x}(x) \\ \frac{\partial H_c}{\partial \xi}(\xi) \end{bmatrix}$$

Furthermore, since $[J(x) - R(x)]K(x) + g(x) = 0$

$$\begin{aligned} J_{aug} - R_{aug} &= \begin{bmatrix} J - R & [J-R]K \\ K^T[J-R] & K^TJK - K^TRK \end{bmatrix} \\ &= \begin{bmatrix} J - R & -g \\ [g - 2RK]^T & K^TJK - K^TRK \end{bmatrix} \end{aligned}$$

Thus the augmented system is a closed-loop system for a **different output** !

Port-Hamiltonian systems with **feedthrough term** take the form

$$\begin{aligned}\dot{x} &= [J(x) - R(x)]\frac{\partial H}{\partial x}(x) + g(x)u \\ y &= (g(x) + 2P(x))^T\frac{\partial H}{\partial x}(x) + [M(x) + S(x)]u,\end{aligned}$$

with M skew-symmetric and S symmetric, while

$$\begin{bmatrix} R(x) & P(x) \\ P^T(x) & S(x) \end{bmatrix} \geq 0$$

The augmented system is thus the feedback interconnection of the nonlinear integral controller

$$\begin{aligned} \dot{\xi} &= u_c \\ y_c &= \frac{\partial H_c}{\partial \xi}(\xi) \end{aligned}$$

with the plant port-Hamiltonian system with **modified** output with feedthrough term

$$\begin{aligned} \dot{x} &= [J(x) - R(x)]\frac{\partial H}{\partial x}(x) + g(x)u \\ y_{mod} &= [g(x) - 2R(x)K(x)]^T \frac{\partial H}{\partial x}(x) + [-K^T(x)J(x)K(x) + K^T(x)R(x)K(x)]u \end{aligned}$$

Remark: See Jeltsema, Ortega and Scherpen for further explorations.

Generalization to feedback interconnection with state-modulation.

Recall that $K^T(x) := \frac{\partial C}{\partial x}(x)$ is a solution to $[J(x) - R(x)]K(x) + g(x) = 0$. This can be generalized to

$$[J(x) - R(x)]K(x) + g(x)\beta(x) = 0$$

with $\beta(x)$ an $m \times m$ design matrix.

The same scheme as above works if we extend the standard feedback interconnection $u = -y_c, u_c = y$ to the state-modulated feedback

$$u = -\beta(x)y_c, \quad u_c = \beta^T(x)y$$

Note that $K(x)$ **is a solution for some** $\beta(x)$ iff

$$g^{\perp}(x)[J(x) - R(x)]K(x) = 0$$

(In fact, $\beta(x) := -(g^T(x)g(x))^{-1}g^T(x)[J(x) - R(x)]K(x)$ does the job.)

A state feedback perspective: shaping the Hamiltonian

Restrict (without much loss of generality) to Casimirs of the form

$$C(x,\xi) = \xi_j - G_j(x)$$

It follows that for all time instants

$$\xi_j = G_j(x) + c_j, \quad c_j \in \mathbb{R}$$

Suppose that in this way **all** control state components ξ_i can be expressed as function

$$\xi = G(x)$$

of the plant state x. Then the dynamic feedback reduces to a **state feedback**, and the Lyapunov function $H(x) + H_c(\xi) + C(x,\xi)$ reduces to the **shaped** Hamiltonian

$$H(x) + H_c(G(x))$$

A direct state feedback perspective:
Interconnection-Damping Assignment (IDA)-PBC control

A direct way to generate candidate Lyapunov functions H_d is to look for state feedbacks $u = \hat{u}_{IDA}(x)$ such that

$$[J(x) - R(x)]\frac{\partial H}{\partial x}(x) + g(x)\hat{u}_{IDA}(x) = [J_d(x) - R_d(x)]\frac{\partial H_d}{\partial x}(x)$$

where J_d and R_d are newly assigned interconnection and damping structures.

Remark: For mechanical systems IDA-PBC control is equivalent to the theory of Controlled Lagrangians (Bloch, Leonard, Marsden, .).

For $J_d = J$ and $R_d = R$ (*Basic IDA-PBC*) this reduces to

$$[J(x) - R(x)]\frac{\partial(H_d - H)}{\partial x}(x) = g(x)\hat{u}_{BIDA}(x)$$

and thus in this case, there exists an $\hat{u}_{BIDA}(x)$ if and only if

$$g^{\perp}(x)[J(x) - R(x)]\frac{\partial(H_d - H)}{\partial x}(x) = 0$$

which is the same equation as obtained for stabilization by Casimir generation with a state-modulated nonlinear integral controller !

Conclusion: Basic IDA-PBC $\Leftrightarrow$ State-modulated Control by Interconnection.

Shifted passivity w.r.t. a controlled equilibrium

(see Jayawardhana, Ortega). Consider a port-Hamiltonian system

$$\begin{aligned} \dot{x} &= Fz + gu, \qquad z = \tfrac{\partial H}{\partial x}(x) \\ y &= g^T z \end{aligned}$$

where $F = J - R, g$ are constant, and a controlled equilibrium x_0:

$$Fz_0 + gu_0 = 0, \qquad z_0 = \frac{\partial H}{\partial x}(x_0)$$

Define the shifted storage function

$$V(x) := H_p(x) - (x - x_0)^T \frac{\partial H_p}{\partial x}(x_0) - H_p(x_0)$$

Note that $\frac{\partial V}{\partial x} = z - z_0$. It follows that

$$\tfrac{d}{dt}V = (z - z_0)^T \dot{x} = (z - z_0)^T (Fz + gu) =$$
$$(z - z_0)^T F(z - z_0) + (z - z_0)^T g(u - u_0) + (z - z_0)^T (Fz_0 + gu_0) \leq (y - y_0)^T (u - u_0)$$

implying passivity w.r.t. the shifted inputs $u - u_0$ and outputs $y - y_0$.

Application to switching control

Consider the port-Hamiltonian model of a power-converter

$$\dot{x} = F(\rho)z + g(\rho)E + g_l u, \quad z = \frac{\partial H_p}{\partial x}(x),\ F(\rho) := J(\rho) - R(\rho)$$

with vector of Boolean variables $\rho \in \{0,1\}^k$, $H_p(x)$ the total stored electromagnetic energy, and output vector $y = g_l^T z$.

Let x_0 be an equilibrium of the *averaged* model, that is

$$F(\rho_0)z_0 + g(\rho_0)E + g_l u_0 = 0, \quad z_0 = \frac{\partial H}{\partial x}(x_0)$$

for some $\rho_0 \in [0,1]^k$ and u_0. Then

$$\begin{aligned}
\dot{x} &= F(\rho)(z - z_0) + F(\rho)z_0 + g(\rho)E + g_l u \\
&= F(\rho)(z - z_0) + [F(\rho) - F(\rho_0)]z_0 + [g(\rho) - g(\rho_0)]E + g_l(u - u_0) \\
&\quad + F(\rho_0)z_0 + g(\rho_0)E + g_l u_0 \\
&= F(\rho)(z - z_0) + [F(\rho) - F(\rho_0)]z_0 + [g(\rho) - g(\rho_0)]E + g_l(u - u_0)
\end{aligned}$$

For many power converters we know that

$$\begin{aligned} F(\rho) - F(\rho_0) &= \textstyle\sum_{i=1}^p F_i(\rho_i - \rho_{0i}) \\ g(\rho) - g(\rho_0) &= \textstyle\sum_{i=1}^p g_i(\rho_i - \rho_{0i}) \end{aligned}$$

and thus

$$\dot{x} = F(\rho)(z - z_0) + \sum_{i=1}^{p} [F_i z_0 + g_i E](\rho_i - \rho_{0i}) + g_l(u - u_0)$$

Take as Lyapunov/storage function

$$V(x) := H_p(x) - (x - x_0)^T \frac{\partial H_p}{\partial x}(x_0) - H_p(x_0)$$

Then

$$\tfrac{d}{dt}V(x) = [\tfrac{\partial H_p}{\partial x}(x) - \tfrac{\partial H_p}{\partial x}(x_0)]^T \dot{x} = (z - z_0)^T \dot{x} =$$
$$(z - z_0)^T F(\rho)(z - z_0) + \textstyle\sum_{i=1}^p (z - z_0)^T [F_i z_0 + g_i E](\rho_i - \rho_{0i}) + (z - z_0)^T g_l(u - u_0)$$

with $(z - z_0)^T F(\rho)(z - z_0) \leq 0$.

Thus at any time we can choose $\rho_i \in \{0,1\}$ such that

$$\frac{d}{dt}V(x) \leq (z - z_0)^T g_l(u - u_0)$$

implying passivity of the switched system with respect to the input vector $u - u_0$ and output vector $y - y_0 = g_l^T(z - z_0)$. As a consequence, if the converter is terminated on a static resistive load then the switched converter is (asymptotically) stable around x_0. Thus the voltage over the resistive load can be stabilized around any set-point.

This can be immediately generalized to converters **connected to a load via a transmission line** (see Part I).

Note that for linear capacitors and inductors we have

$$H_p(x) = \frac{1}{2}x^T Q x, \quad V(x) = \frac{1}{2}(x - x_0)^T Q(x - x_0)$$

(cf. Buisson & co-workers)

New control paradigms

Example: Energy transfer control

Consider two port-Hamiltonian systems Σ_i

$$\begin{aligned}\dot{x}_i &= J_i(x_i)\frac{\partial H_i}{\partial x_i}(x_i) + g_i(x_i)u_i \\ y_i &= g_i^T(x_i)\frac{\partial H_i}{\partial x_i}(x_i), \qquad i = 1,2\end{aligned}$$

Suppose we want to transfer the energy from the port-Hamiltonian system Σ_1 to the port-Hamiltonian system Σ_2, while keeping the total energy $H_1 + H_2$ constant.

This can be done by using the output feedback

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0 & -y_1 y_2^T \\ y_2 y_1^T & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

It follows that the closed-loop system is energy-preserving. However, for the individual energies

$$\frac{d}{dt} H_1 = -y_1^T y_1 y_2^T y_2 = -||y_1||^2 ||y_2||^2 \leq 0$$

implying that H_1 is decreasing as long as $||y_1||$ and $||y_2||$ are different from 0. On the other hand,

$$\frac{d}{dt} H_2 = y_2^T y_2 y_1^T y_1 = ||y_2||^2 ||y_1||^2 \geq 0$$

implying that H_2 is increasing at the same rate. Has been successfully applied to **energy-efficient path-following control** of mechanical systems (cf. Duindam & Stramigioli).

Impedance control

Consider a system with two (not necessarily distinct) ports

$$\begin{aligned} \dot{x} &= [J(x) - R(x)]\frac{\partial H}{\partial x}(x) + g(x)u + k(x)f, \quad x \in \mathcal{X},\, u \in \mathbb{R}^m \\ y &= g^T(x)\frac{\partial H}{\partial x}(x) \quad u, y \in \mathbb{R}^m \\ e &= k^T(x)\frac{\partial H}{\partial x}(x) \quad f, e \in \mathbb{R}^m \end{aligned} \tag{1}$$

The relation between the f and e variables is called the 'impedance' of the (f, e)-port. In **Impedance Control** (Hogan) one tries to *shape* this impedance by using the control port corresponding to u, y.
Typical application: the (f, e)-port corresponds to the end-point of a robotic manipulator, while the (u, y)-port corresponds to actuation.

Basic question: what are achievable impedances of the (f, e)-port ?

Model reduction of port-Hamiltonian systems

(Ongoing joint work with Polyuga, Scherpen.)

- Network modeling of complex lumped-parameter systems (circuits, multi-body systems) often leads to high-dimensional models.
- Structure-preserving spatial discretization of distributed-parameter port-Hamiltonian systems yields high-dimensional port-Hamiltonian models.
- Lumped-parameter modeling of systems like MEMS gives high-dimensional port-Hamiltonian systems.
- Controller systems may be in first instance *distributed-parameter*, and need to be discretized to low-order controllers.

In many cases we want the reduced-order system to be **again port-Hamiltonian**:

- Port-Hamiltonian model reduction preserves passivity.
- Port-Hamiltonian model reduction may (approximately) preserve other balance laws /conservation laws.
- Physical interpretation of reduced-order model.
- Reduced-order system can replace the high-order port-Hamiltonian system in a larger context.

Thus there is a need for **structure-preserving** *model reduction of high-dimensional port-Hamiltonian systems.*

Controllability analysis (see talk by Polyuga)

Consider a linear port-Hamiltonian system, written as

$$\begin{aligned} \dot{x} &= FQx + Bu, \quad F := J - R, \quad J = -J^T, \; R = R^T \geq 0 \\ y &= B^T Q x, \quad Q = Q^T \geq 0 \end{aligned}$$

Take linear coordinates $x = (x_1, x_2)$ such that the upper part of

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix} \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u$$

$$y = \begin{bmatrix} B_1^T & B_2^T \end{bmatrix} \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

is the reachability subspace R.

By invariance of R this implies

$$\begin{aligned} F_{21}Q_{11} + F_{22}Q_{21} &= 0 \\ B_2 &= 0 \end{aligned}$$

If follows that the dynamics restricted to R is given as

$$\begin{aligned} \dot{x}_1 &= (F_{11}Q_{11} + F_{12}Q_{21})x_1 + B_1 u \\ y &= B_1^T Q_{11} x_1 \end{aligned}$$

Now solve for Q_{21} as $Q_{21} = -F_{22}^{-1}F_{21}Q_{11}$. This yields

$$\begin{aligned} \dot{x}_1 &= (F_{11} - F_{12}F_{22}^{-1}F_{21})Q_{11}x_1 + B_1 u \\ y &= B_1^T Q_{11} x_1 \end{aligned}$$

which is **again a port-Hamiltonian system**.

Observability analysis

Suppose the system is not observable. Then there exist coordinates $x = (x_1, x_2)$ such that the *lower* part is the unobservability subspace N. By invariance of N it follows that

$$\begin{aligned} F_{11}Q_{12} + F_{12}Q_{22} &= 0 \\ B_1^T Q_{12} + B_2^T Q_{22} &= 0 \end{aligned}$$

Then the dynamics on the quotient space $\mathcal{X}/N$ is

$$\begin{aligned} \dot{x}_1 &= (F_{11}Q_{11} + F_{12}Q_{21})x_1 + B_1 u \\ y &= B_1^T Q_{11} x_1 + B_2^T Q_{21} x_1 \end{aligned}$$

It follow from that $F_{12} = -F_{11}Q_{12}Q_{22}^{-1}$ and $B_2^T = -B_1^T Q_{12}Q_{22}^{-1}$. Substitution yields

$$\begin{aligned} \dot{x}_1 &= F_{11}(Q_{11} - Q_{12}Q_{22}^{-1}Q_{21})x_1 + B_1 u \\ y &= B_1^T(Q_{11} - Q_{12}Q_{22}^{-1}Q_{21})x_1 \end{aligned}$$

which is again a port-Hamiltonian system with Hamiltonian $\bar{H} = \frac{1}{2}x_1^T(Q_{11} - Q_{12}Q_{22}^{-1}Q_{21})x_1$.

Remark Note that the Schur complement $(Q_{11} - Q_{12}Q_{22}^{-1}Q_{21}) \geq 0$ if $Q \geq 0$.

This suggests two canonical ways for structure-preserving model reduction.

Conclusions of Part II

- Beyond passivity by port-Hamiltonian systems theory.
- Control by interconnection and Casimir generation, IDA-PBC control.
- Allows for 'physical' interpretation of control strategies. Suggests new control paradigms for nonlinear systems.

◦ Use of passivity generally yields good robustness, but performance theory is yet lacking.

◦ Observer theory for port-Hamiltonian systems; see talk Venkatraman.

◦ Structure-preserving model reduction of port-Hamiltonian systems, see talk Polyuga.

THANK YOU !