Mathematics. — *An analogue of the nine-point circle in the space of n-dimensions.* By J. C. H. GERRETSEN. (Communicated by Prof. J. G. VAN DER CORPUT.)

(Communicated at the meeting of October 27, 1945.)

1. Let the vertices of a simplex in the space of n dimensions be A_0, A_1, A_2, ..., A_n. It can be assumed that each edge $A_i A_k$ is perpendicular to the opposite $(n-2)$-dimensional boundary face, so the altitude lines through the vertices are concurrent in a point called the orthocentre of the simplex. Moreover the same peculiarity holds for every $(n-1)$-face of the simplex and the orthogonal projection of the orthocentre on any $(n-1)$-face is just the orthocentre of that face. In the trivial case $n = 2$ every point of the line $A_i A_k$ must be regarded as an orthocentre of the 1-simplex with edges A_i and A_k.

In the case of an orthocentric simplex a generalisation of the nine-point circle of a triangle has been given by R. MEHMKE, Arch. f. Math. u. Phys. **70** (1884), p. 210. It is our aim to give an analogue in the general case. In that case the altitude lines have not a point in common, but — as we shall see — there will be a point that has analogous properties as the orthocentre in the special case mentioned above.

2. Let M_{ik} be the middle point of the edge $A_i A_k$ and G_{ik} the centre of gravity of the opposite $(n-2)$-face A_{ik}^{n-2}. If G be the centre of gravity of the n-simplex, it is easily proved that the points M_{ik}, G and G_{ik} are collinear, the segment $M_{ik} G_{ik}$ being divided by G in the ratio $(n-1):2$.

The $n+1$ hyperplanes each going through the points M_{ik} and perpendicular to the lines $A_i A_k$ are concurrent in a point M, the centre of the circumscribed hypersphere of the n-simplex. By a similitude with centre G and with ratio $-2:(n-1)$ these hyperplanes are transformed into the hyperplanes through the points G_{ik}, each being perpendicular to the opposite edge $A_i A_k$. *Hence these hyperplanes are concurrent in a point H.* In the case $n = 3$ this is the well-known point of MONGE; we will denote it as the point of MONGE in the general case also. The result can be formulated in the following way: *the centre of gravity G is collinear with the centre of the circumscribed hypersphere M and the point of* MONGE *H and divides the segments $M H$ in the ratio* $(n-1):2$. It is easily seen that in the case of the orthocentric simplex the point of MONGE and the orthocentre are coincident.

3. Let H_k be the point of MONGE of the $(n-1)$-face A_k^{n-1} opposite to the vertex A_k, $(k = 0, 1, \ldots, n)$. Let $'A_k$ be the orthogonal projection of

A_k and $'H_k$ the orthogonal projection of H on that face. *We will prove that the points $'A_k$, $'H_k$ and H_k are collinear and that $'H_k$ divides the segment $'A_k H_k$ in the ratio* $(n—2) : 1$. It is to be noted that the theorem is trivial if $n \leqq 2$.

To prove the theorem we regard at first the edge $A_0 A_1$. The point H_n is laying in the $(n—2)$-space perpendicular to $A_0 A_1$ which is passing through the centre of gravity of the $(n—3)$-dimensional boundary simplex of the simplex $A_0 \dots A_{n-1}$ opposite to $A_0 A_1$ and contained in that simplex. A similitude with centre $'A_n$ and ratio $(n—2) : (n—1)$ transforms this space into an $(n—2)$-dimensional space perpendicular to $A_0 A_1$ going through the centre of gravity $'G_{01}$ of the simplex $A_2 \dots A_{n-1} {}'A_n$. But this point is just the orthogonal projection of G_{01} on the space through the points $A_0 \dots A_{n-1}$ and therefore the $(n—2)$-space just mentioned is the intersection of the $(n—1)$-space through $A_0, \dots, A_{n-1}$ and the hyperplane through G_{01} perpendicular to $A_0 A_1$. This space cuts the line $'A_n H_n$ in the point $'H_n$ and the segment $'A_n H_n$ is divided by $'H_n$ in the ratio $(n—2) : 1$. If we take instead of $A_0 A_1$ any other edge of the $(n—1)$-simplex $A_0 \dots A_{n-1}$, we always find the same point $'H_n$. Therefore the point of MONGE from the n-simplex must be situated on the line through $'H_n$ perpendicular to the hyperplane through $A_0, A_1, \dots, A_{n-1}$, q.e.d.

4. Now we are able to give an analogue of the nine-point circle in the following manner: Let G_k denote the centre of gravity of the $(n—1)$-dimensional face opposite to the vertex A_k, P_k the point on the segment $A_k H$ on a distance $\frac{1}{n} A_k H$ from H, $'P_k$ the harmonic conjugate of H_k with regard to $'A_k$ and $'H_k$. *Then the* $2(n+1)$ *points* G_k, P_k, $'P_k$ *are on one hypersphere. The centre N of this hypersphere is the harmonic conjugate of M with regard to G and H.*

In proving this theorem we see that a similitude with centre G and ratio $— 1 : n$ transforms the circumscribed hypersphere of the n-simplex into the same hypersphere as is obtained by a similitude with centre H and ratio $1 : n$. The points P_k and G_k are diametrically situated on this sphere and — if $''P_k$ denotes the orthogonal projection of P_k on the face opposite to A_k — $\angle P_k {}''P_k G_k$ is a right angle; hence $''P_k$ is also on that sphere. An easy computation shows that $''P_k$ is the harmonic conjugate of H_k with regard to $'A_k$ and $'H_k$, so the points $''P_k$ and $'P_k$ coalesce.