Mathematics. — *"The section of the measure-polytope M_n of space Sp_n with a central space Sp_{n-1} perpendicular to a diagonal."* By Prof. P. H. SCHOUTE.

(Communicated in the meeting of December 28, 1907).

We determine the indicated section in three different ways:

1. by means of the projection of M_n on the diagonal,
2. with the aid of the projection of M_n on a plane through two opposite edges intersecting the diagonal,
3. by regarding regular simplexes.

I. *The projection of M_n on a diagonal.*

1. We can easily prove both analytically and synthetically the following theorem:

"The vertices of the measure-polytope M_n project themselves on a "diagonal in $n+1$ points, namely in the ends of the diagonal and "in the $n-1$ points, which divide the latter into n equal parts, in "these $n+1$ points are projected successively

$$1, n, \tfrac{1}{2} n (n-1), \ldots . \tfrac{1}{2} n (n-1), n, 1$$

"points, where these numbers are the coefficients of the terms "of $(a+b)^n$".

From this general theorem ensue the results for $n = 4, 5, 6, 7, 8$ given in the diagrams added here (see the expanding plate). An explanation of the sketch belonging to $n = 4$ will sufficiently explain the others.

The horizontal lines of this figure always represent the same diagonal on which the projection takes place; on these ten lines are successively indicated the projections of vertices, of edges, of faces and of bounding bodies In order to find space for the figures indicating the numbers, the thick projection-lines have been broken off, where such was necessary.

If we designate the five points on the diagonal by $a, b, c, d, e,$ — see the bottom line of the ten horizontal ones — then in these places — see the topmost of the ten lines -- 1, 4, 6, 4, 1 vertices are projected there — bear in mind $(1+1)^4$.

On the four equal segments ab, bc, cd, de are projected successively 4, 12, 12, 4 edges — think of $4(1+1)^3$.

In like manner the three equal segments ac, bd, ce are successively the projections of 6, 12, 6 faces — think of $6(1+1)^2$.

Finally on the two equal segments ad, be are projected successively 4, 4 bounding bodies — think of $4(1+1)$.

It is easy to deduce from this the results given in the other diagrams for $n = 5, 6, 7, 8$, if we keep in mind, that the coefficients by which $(1+1)^4$, $(1+1)^3$, $(1+1)^2$, $(1+1)$ are multiplied are $1, 4, 6, 4$ and so by addition of unity at the end pass into a repetition of $(1+1)^4$.

2. More generally holds the following theorem, comprising the preceding:

"The vertices of each bounding M_p of M_n $(p \leqq n)$ are projected on "the diagonal of M_n in $p+1$ successive points of division of that "diagonal; here again the projections are distributed according to "the coefficients $1, p, \frac{1}{2}p(p-1), \ldots$ of $(a+b)^p$ over these $p+1$ "successive points."

The vertices of a bounding square are projected in three of the $n+1$ points, which naturally demands the division $1, 2, 1$. The vertices of a bounding cube are projected in four of the $n+1$ points, which of necessity must lead to the division $1, 3, 3, 1$ as by the preceding the division $2, 2, 2, 2$ is excepted.

From this ensues then directly the following theorem:

"The section of a space Sp_{n-1} perpendicular to the diagonal of M_n "forming the axis of projection, with the space Sp_p bearing a bounding "M_p of M_n is an Sp_{p-1} in Sp_p perpendicular to the diagonal of "M_p connecting the two vertices of M_p projecting themselves in the "ends of the projection of M_p." [1])

But there is more. If $p'(M_p)$ represents the section of a measure-polytope M_p with a space Sp_{p-1} of its space Sp_p perpendicular to one of its diagonals in a point of which the distance to the centre of the diagonal in the diagonal as unity amounts to $\frac{1}{2} - p'$, from which is evident that $p' \leqq \frac{1}{2}$, the two theorems hold:

"For even n a bounding measure-polytope M_p of M_n is intersected "by the central space Sp_{n-1} perpendicular to the diagonal of M_n

[1]) The indicated diagonal d_p of M_p is the projection of the axis of projection d on the space Sp_p of M_p; so we can obtain the projections of the vertices of M_p on d by projecting these vertices first in Sp_p on d_p and projecting afterwards on d the points found on d_p by the preceding means.

As d_p and d in the edge of M_n as unity are represented by $\sqrt{p}$ and $\sqrt{n}$ and d_p is projected on d as $\frac{p}{n}$ of d, the cosine of the angle between d and Sp_p is equal to $\frac{1}{n}\sqrt{np}$.

"according to an $\frac{a}{p}(M_p)$, where a according to circumstances can "assume for even p one of the $\frac{p}{2}$ values $1, 2, \ldots \frac{p}{2}$, for odd p one "of the $\frac{p-1}{2}$ values $1, 2, \ldots \frac{p-1}{2}$."

"For odd n the measure-polytope M_p is intersected under the same "circumstances according to a $\frac{2a-1}{p}(M_p)$ where a can assume for "even p one of the $\frac{p}{2}$ values $1, 2, \ldots \frac{p}{2}$, for odd p one of the $\frac{p+1}{2}$ "values $1, 2, \ldots \frac{p+1}{2}$."

We shall now, instead of losing ourselves in further generalities, give the full results of the diagrams for the cases $n = 4, 5, 6, 7, 8$ to make clear the above. In order to be able to indicate easily ratios of measure we shall suppose the edge of M_n to be unity of length.

3. Case $n = 4$. The space — see first diagram — perpendicular in the centre c of diagonal ae to this diagonal contains the six vertices of M_4 projecting themselves in c and cuts — see lines 3 and 4 — no edge; so the section has six vertices. This same space cuts twelve faces — see line 7 — according to $\frac{1}{2}(M_2)$ and eight bounding bodies — see lines 9 and 10 — according to $\frac{1}{3}(M_3)$; so the section has twelve edges with a length $\sqrt{2}$ and eight equilateral triangles as faces. So the section is a $(6, 12, 8)$ and, indeed, the regular octahedron with edges $\sqrt{2}$.

Case $n = 5$. We find — see second diagram — thirty vertices generated by intersection of edges, sixty edges, forty faces and ten bounding bodies, so a $(30, 60, 40, 10)$. The vertices are of the same kind, the edges have as $\frac{1}{4}(M_2)$ the length $\frac{1}{2}\sqrt{2}$. The forty faces consist of twenty $\frac{1}{2}(M_3)$ and two times ten $\frac{1}{6}(M_3)$, i. e. of twenty hexagons and twenty triangles, both regular[1]) with sides $\frac{1}{2}\sqrt{2}$.

[1]) Where the regularity is obvious — as e. g. with the triangles by the equal length of all edges, etc. — the additional "equilateral" or "regular" will in future be left out.

Each of the ten bounding bodies is as $\frac{3}{8}(M_4)$ — compare in the first diagram the section with a space perpendicular to ae in the point in the middle between c and d — a (12, 18, 8) bounded by four $\frac{1}{2}(M_3)$ and four $\frac{1}{6}(M_3)$, i. e. by four of the hexagons and four of the triangles, and therefore a tetrahedron truncated regularly at the vertices, i. e. the first of the equiangular semi-regular (Archimedian) bodies.

Case $n = 6$. Out of the third of the diagrams we read that the section is a (20, 90, 120, 60, 12). All the edges have a length $\sqrt{2}$, all the faces are triangles. The bounding bodies are for one half (30) as $\frac{1}{2}(M_4)$ octahedra, for the other half (15 + 15) as $\frac{1}{4}(M_4)$ tetrahedra. The twelve bounding polytopes are as $\frac{2}{5}(M_5)$ — compare now again the second diagram — polytopes (10, 30, 30, 10) bounded by five of the octahedra and five of the tetrahedra, which can be regarded as regular five-cells, regularly truncated at the vertices as far as half of the edges, so as to lose all the original edges by this truncation.

Case $n = 7$. We arrive at a (140, 420, 490, 280, 84, 14). The length of the edges is $\frac{1}{2}\sqrt{2}$. The 490 faces consist of 210 hexagons and 280 triangles, the 280 bounding bodies of 210 truncated tetrahedra and 70 tetrahedra, the 84 four-dimensional bounding polytopes of 42 polytopes $\frac{1}{2}(M_5) = (30, 60, 40, 10)$ found already under $n = 5$ and 42 polytopes $\frac{3}{10}(M_5) = (20, 40, 30, 10)$ bounded by five truncated tetrahedra and five tetrahedra — regular five-cells truncated at the vertices as far as a third of the edges. The 14 five-dimensional bounding polytopes are as $\frac{5}{12}(M_6)$ polytopes (60, 150, 140, 60, 12) bounded by six (30, 60, 40, 10) and six (20, 40, 30, 10).

Case $n = 8$. Here a (70, 560, 1120, 980, 448, 112, 16) is the result. The length of the edges is $\sqrt{2}$, all faces are triangles. The

980 bounding bodies consist of 420 octahedra and 560 tetrahedra the 448 four-dimensional bounding polytopes of 336 polytopes $\frac{2}{5}(M_5)$ and 112 polytopes $\frac{1}{5}(M_5)$, i. e. of 336 five-cells truncated as far as half of the edges, found under $n=6$, and 112 five-cells. The 112 five-dimensional bounding polytopes are as far as one half is concerned $\frac{1}{2}(M_6) = (20, 90, 120, 60, 12)$ already found above, as far as the other half is concerned $\frac{1}{3}(M_6) = (15, 60, 80, 45, 12)$ bounded by six five-cells truncated as far as half the length of the edges and six five-cells. Finally the sixteen six-dimensional bounding polytopes are as $\frac{3}{7}(M_7)$ polytopes (35, 210, 350, 245, 54, 84) bounded by seven (20, 90, 120, 60, 12) and seven (15, 60, 80,45, 12)[1]).

From this all we easily deduce the following general laws:

"The vertices of the section are vertices of M_n for even n, for odd n they are centres of edges of M_n; they are always of the same kind[2])."

"The common length of the edges is $\sqrt{2}$ for even n and $\frac{1}{2}\sqrt{2}$ for odd n; they are always of the same kind[3])."

"The faces are triangles for even n, hexagons and (smaller) triangles[4]) for odd n."

"The bounding bodies are octahedra and tetrahedra for even n, truncated tetrahedra and (smaller) tetrahedra for odd n".

"The four-dimensional bounding polyhedra are five-cells truncated as far as halfway the edges and five-cells for even n, five-cells

1) If we had set to work, when enumerating the results, in that sense inversely that with each new value of n of the bounding polytopes with the greatest number of dimensions we had descended to the vertices, we should have furnished a geometrical variation of the well known nursery-book: "the house that Jack built". However with two differences. When descending from every one round higher of the ladder we pass *every other time* again the same stadia and the ladder is a Jacob's ladder with an infinite number of rounds.

2) That is, in each vertex as many edges meet in the same way, etc.

3) The cases $n=$ odd seem to be an exception to this, as there are for the truncated tetrahedra two kinds of edges, namely: sections of two hexagonal faces and sections of an hexagonal and a triangular face. However, this is only apparently. For, for each edge we find that in the section itself always again the number of faces passing through it of each of the two sorts is steadfast, thus for $n=5$ two hexagonal faces and one triangular one.

4) We do not mention here, that for $n=3$ only an hexagon appears. Neither that of the bounding bodies the tetrahedra do not appear for $n=4$, etc.

truncated as far as a third of the edges and (smaller) five-cells for odd n."

Etc., etc. [1]).

The above results are for the greater part given in the general theorems mentioned above.

II. The projection of M_n on a plane through two opposite edges cutting the diagonal.

4. For each value of n the indicated projection — see fig. 1 for $n = 8$ and $n = 9$ — is a rectangle $PQQ'P'$ with the sides 1 and

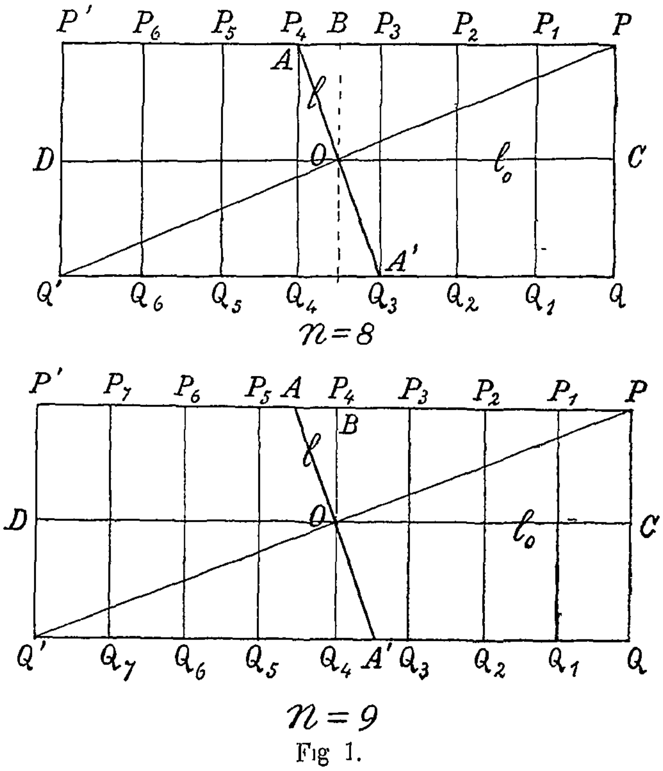


Fig 1.

$\sqrt{n-1}$, which is divided by n—2 lines P_1Q_1, P_2Q_2, ... $P_{n-2}\,Q_{n-2}$ parallel to the shorter sides PQ, $P'Q'$ into n—1 equal rectangles.[2])

[1]) We break off here because not until the third division do we indicate that everything making its appearance in the section can be regarded as simplex or truncated simplex.

The symbol which indicates the numbers of vertices, edges, faces, etc. for arbitrary n is purposely omitted as its form is rather complicated.

[2]) To compare the treatise "On the sections of a block of eightcells, etc." (Verhandelingen der K. A. v. W., vol IX, N . 7).

The diagonal on which the intersecting space Sp_{n-1} is at right angles is one of the diagonals of the rectangle, e.g, PQ'. If the normal erected in the centre O of PQ' on this line, representing the projection of the intersecting space Sp_{n-1}, cuts the side PP' in A, this point A always lies at a distance $\frac{1}{2\sqrt{n-1}}$ from the centre B of PP'. For in the right-angled triangle AOP we find that B is the foot of the normal let down out of O on AB and from this ensues $AB.BP = OB^2$ and therefore $AB = \frac{1}{4} : \frac{1}{2}\sqrt{n-1}$. So A coincides for even n with the point of division $P_{\frac{n}{2}}$ and this point lies for odd n in the middle between $P_{\frac{n-1}{2}}$ and $P_{\frac{n+1}{2}}$. From this it is again evident that the vertices of the section are vertices of M_n for even n and centres of edges of M_n for odd n.

In the paper quoted above which restricts itself to the case $n=4$ we find in a note how we can regard the section under observation as a "rhombotope" truncated at both sides; the course of thoughts is as follows. Let us imagine in the direction of the edges PQ, $P'Q'$ on either side an infinite number of measure-polytopes M_n piled on each other and let us then remove the measure-polytopes M_{n-1}, projecting themselves on PP', QQ' and lines parallel to these, with which the successive polytopes M_n bound each other; then a prism is formed with M_{n-1} as right section. If this prism is intersected by a space Sp_{n-1} which projects itself along the perpendicular l_0 let down out of O on PQ, the section is thus an M_{n-1}. What variation does this section M_{n-1} of the prism undergo when we substitute for the intersecting space projecting itself along l_0 an other one which projects itself along a line l_φ through O, enclosing with l_0 an angle φ? As is easy to see from the figure this variation consists of a regular enlargement of the perpendiculars let down out of the boundary of M_{n-1} on the space Sp_{n-2}, projecting itself in O, which enlargement means a multiplication of those perpendiculars by *sec* φ and can be regarded as a *stretching* in the direction of the diagonal CD. As for $n=4$, where M_{n-1} becomes a cube, such a stretching makes a rhombohedron of a cube, out of M_{n-1} is formed in general what we call a rhombotope.

Just as the rhombohedron regarded as a whole passes into itself when it is revolved 120° about the axis, or — in other words — just as the axis of the rhombohedron has a period three, the axis of the rhombotope under consideration has a period $n-1$. Let us now imagine this rhombotope, for the special case that the projection of the intersecting space Sp_{n-1} — so also the projection of the rhombotope itself — falls along OA and let us truncate it by the two spaces Sp_{n-2} standing normal to the plane of projection in the ends A, A' of the segment AA' of that projection lying inside the rectangle and cutting the axis of the rhombotope therefore at right angles; we then find the required section, to be indicated according to the number of its dimensions by D_{n-1}. We directly determine the length of the axis of the untruncated rhombotope and of D_{n-1}, but before this we shall deduce some general theorems easy to find.

5. The edges of M_n project themselves on the assumed plane *either* along one of the n lines $PQ, P_1Q_1, P_2Q_2, \ldots P_{n-2}Q_{n-2}, P'Q'$, *or* as parts of PP' or QQ'. Because the vertices of D_{n-1} must be vertices of M_n or points of intersection with edges of M_n, these points project themselves — compare fig. 1 for $n = 8$ and for $n = 9$ — for even n exclusively in the ends A, A', for odd n exclusively in those ends and in the centre O.

From this ensues for $n = 2n'$ the general theorem:

"The section $D_{2n'-1}$ of $M_{2n'}$ is a $2n' - 1$-dimensional prismoid with respect to each pair of opposite bounding spaces $Sp_{2n'-2}$ and so in $2n'$ ways".

Here follow two theorems holding for arbitrary n:

"Each line through the centre O normal to two opposite bounding spaces Sp_{n-2} is axis of D_{n-1} with the period $n-1$."

"Each space Sp_{n-2} through O parallel to a bounding space Sp_{n-2} divides D_{n-1} into two congruent $n-1$-dimensional prismoids."

In the demonstration of these three theorems the entire equivalence of a pair of opposite bounding spaces Sp_{n-2} with any other pair has the chief part; moreover the third causes us to inquire how the space Sp_{n-2} through the centre parallel to a bounding space intersects D_{n-1}. We prove as follows that this section is a D_{n-2}.

If the projection l of the intersecting space Sp_{n-1} revolves round O, the $Sp^{(0)}_{n-2}$ normal to the plane of projection in O remains in its place and Sp_{n-1} thus describes a pencil with this $Sp^{(0)}_{n-2}$ as axial space. Therefore then the varying section keeps going through the section of $Sp^{(0)}_{n-2}$ with M_n. We can easily know the nature of this section of $n-2$ dimensions by regarding the case in which l coincides with l_0. Then our D_{n-1} is an M_{n-1} and this measure-polytope projecting itself along l_0 is intersected according to a D_{n-2} by the space $Sp^{(0)}_{n-2}$, which is in O normal to the plane of projection and which therefore bisects the diagonal CD of this M_{n-1}. This D_{n-2} is the section of D_{n-1} with the space $Sp^{(0)}_{n-2}$, through O parallel to the spaces $Sp^{(0)}_{n-2}$, which are in A_1 and A' normal to the axis and which truncate the rhombotope. So we find:

"Each space $Sp^{(0)}_{n-2}$ through the centre O parallel to a bounding space Sp_{n-2} intersects D_{n-1} according to a D_{n-2} of which O is again the centre."

From this follows again more generally·

"Each space $Sp^{(0)}_p$ $(0 < p < n-1)$ through the centre O parallel to a bounding space Sp_p intersects D_n according to a D_{p-1}, of which O is again the centre".

Thus we find ascending from below:

"Each chord of D_{n-1} through O parallel to an edge has a length $\sqrt{2}$, each plane through O parallel to a face intersects D_{n-1} according to a regular hexagon with sides $\frac{1}{2}\sqrt{2}$, each space through O parallel to a bounding body intersects D_{n-1} according to a regular octahedron with edges $\sqrt{2}$, etc."

6. We retrace our steps and determine of the above mentioned rhombotope the length of the axis before and after the truncation. Out of the similitude of the triangles AOB and POC follows in connection with the length $\frac{1}{2}\sqrt{n-1}$, $\frac{1}{2}\sqrt{n}$, $\frac{1}{2}$ of OC, OP, OB for OA the value $\frac{1}{2(n-1)}\sqrt{n(n-1)}$ and so for half of the unmutilated axis which is $n-1$ times as large $\frac{1}{2}\sqrt{n(n-1)}$. If we represent by $Rh_p[q,r]$ a rhombotope with p dimensions of which q is the length of the axis, r are the parts of the axis removed by the truncation, the section D_{n-1} has to be represented by the symbol $Rh_{n-1}\left[\sqrt{n(n-1)}, \frac{n-2}{2(n-1)}\right]$

So the theorem holds:

"We obtain the section D_{n-1}, if we allow the measure-polytope M_{n-1} to pass in the indicated way by stretching in the direction of a diagonal as far as $\sqrt{n}$ times the original length into a rhombotope with a length of axis $\sqrt{n(n-1)}$ and if we truncate this rhombotope by two spaces Sp_{n-2}. normal to the axis to a

$$Rh_{n-1}\left[\sqrt{n(n-1)}, \frac{n-2}{2(n-1)}\right]$$"[1]

III. Explanation in details of the connection of D_{n-1} with regular and regularly truncated simplexes.

7. We consider in the space Sp_n a rectangular system of coordinates with an arbitrary point O as origin and $OX_1, OX_2, \ldots OX_n$ as axes, and we now call the $2^{n\text{th}}$ part of that space which is the locus of the point with only positive coordinates the "n-edge $O(X_1 X_2 \ldots X_n)$".

If A, A' are two opposite vertices of a measure-polytope M_n of Sp_n and if $AA_1, AA_2, \ldots AA_n$ are the edges passing through A and $A'A'_1, A'A'_2, \ldots A'A'_n$ the edges parallel to these but directed oppositely, then M_n can be regarded as the part of the space Sp_n common to the two n-edges $A(A_1 A_2 \ldots A_n)$ and $A'(A'_1 A'_2 \ldots A'_n)$.

If we intersect this figure of the two oppositely orientated n-edges and the measure-polytope M_n common to both by an arbitrary space Sp_{n-1}, the two n-edges are intersected along two oppositely orientated simplexes and the section of M_n with that space Sp_{n-1} appears as the part of that space that is enclosed at the same time by both simplexes situated in that space. If the selected space is normal to the diagonal AA', connecting the vertices of the n-edges, the simplexes are regular and they have the point of intersection P of the intersecting space Sp_{n-1} with AA' as common centre of gravity. So the general theorem holds:

"The section of M_n with a space Sp_{n-1} normal to a diagonal can always be regarded as a part of that space Sp_{n-1} enclosed by two definite concentric, oppositely orientated, regular simplexes of that space".

If we wish to make use of this theorem we must determine in a more detailed way the length of the edges of those oppositely orientated regular simplexes with common centre of gravity.

8. If we think the intersecting space Sp_{n-1} to be normal to the

[1]) This theorem shows distinctly why the sections of an octahedron parallel to two faces must be identical to those of a cube by planes normal to a diagonal in points of the middle third part of that line. The same in other words: If we truncate a cube with the unity of edge at two opposite vertices by planes normal to the connecting line in the points dividing this diagonal into three equal parts and if we compress an octahedron with edges $\sqrt{2}$ in the direction of the normal on two parallel faces as far as half the thickness, then we cause the same solid to be generated in two different ways.

diagonal AA' in the first point of division A_1, at a distance $\frac{1}{n}\sqrt{n}$ from A, the section is a simplex with edge $\sqrt{2}$. So the two simplexes, generated when an arbitrary point P of AA' is substituted for point A_1, have edges of a length of $AP\sqrt{2n}$ and $A'P\sqrt{2n}$, wherefore we indicate them, also with reference to the number of vertices, by $S_n(AP\sqrt{2n})$ and $S'_n(A'P\sqrt{2n})$. So the theorem holds:

"If we shove an M_n, of which the diagonal AA' is normal to a given space Sp_{n-1}, in the direction of that diagonal through that space Sp_{n-1}, so that the spaces Sp_{n-1} of the bounding polytopes M_{n-1} move parallel to themselves, the section of Sp_{n-1} with the moving polytope M_n is at every moment the part of that space Sp_{n-1} that is enclosed within two concentric, yet oppositely orientated, regular simplexes $S_n(p\sqrt{2n})$ and $S'_n(p'\sqrt{2n})$ where p and p' are connected in such a way that the sum $p+p'$ is equal to $\sqrt{n}$. During that movement of M_n the common centre of gravity of the two simplexes remains in its place and the spaces Sp_{n-2} of the bounding simplexes S_{n-1} and S'_{n-1} move parallel to themselves; whilst simplex S_n expands itself from this common centre of gravity to a simplex $S_n(n\sqrt{2})$, simplex S'_n inversely contracts from a simplex $S'_n(n\sqrt{2})$ to this point".

At the moment when this process has got halfway and the two simplexes are of the same size we find:

"The section D_{n-1} is the part of the intersecting space Sp_{n-1} enclosed by two definite equal concentric yet oppositely orientated regular simplexes $S_n\left(\frac{1}{2}n\sqrt{2}\right)$ and $S'_n\left(\frac{1}{2}n\sqrt{2}\right)$."

Thus for $n=3$ the regular hexagon with sides $\frac{1}{2}\sqrt{2}$ is the figure enclosed by two triangles with sides $\frac{3}{2}\sqrt{2}$ — think of the well-known trademark —, thus for $n=4$ the regular octahedron with edges $\sqrt{2}$ is the figure enclosed by two tetrahedra with edges $2\sqrt{2}$ — think of the two tetrahedra described in a cube and the octahedron common to both. So in general the problem in the space of n dimensions is reduced to another problem in space of $n-1$ dimensions and moreover the connection of the result with regular simplexes is explained.

If we think the simplex S_n to be white and the simplex S'_n to be black, the n bounding spaces Sp_{n-2} of D_{n-1} originating from S_n will be white, those originating from S'_n will be black. From this ensues that it must be possible to colour the $2n$ bounding spaces

Sp_{n-2} of D_{n-1} in such a way in turns white and black, that two opposite bounding spaces Sp_{n-2} have a different colour. The octahedron is really the only one of the regular bodies that allows this operation. [1])

9. If the simplex S_n expands from a point to an $S_n(n\sqrt{2})$ and at the same time S'_n contracts from an $S'_n(n\sqrt{2})$ to a point, then S_n lies at the beginning of the process within S'_n and at the end inversely S'_n lies within S_n. Gradually first the vertices, then the edges, then the faces, etc. of S_n have passed outward. We shall now investigate when that takes place.

From the diagrams of the expanding plate given in the first part it is evident, that the section of M_n with a space Sp_{n-1} changes its nature when the point of intersection P of that space Sp_{n-1} with the diagonal AA' passes one of the $n-1$ points of division $A_1, A_2, \ldots$ As the nature of the section of course also changes when bounding elements of S'_n lying inside S_n pass outward, the latter must take place at those moments when those points of division of the diagonal AA' of the moving M_n pass through the fixed space Sp_{n-1}. This theorem then really holds:

"In the translation of M_n in the direction of AA' through the space Sp_{n-1} in succession the vertices, the edges, the faces, bounding bodies, etc. of S_n come entirely outside S'_n at those moments that the point of intersection P of the diagonal AA' with the space Sp_{n-1} coincides successively with the points of division A_1, A_2, A_3, A_4, etc."

We regard — in order to prove this theorem — the arbitrary stadium of the simplexes $S_n(AP\sqrt{2n})$ and $S'_n(A'P\sqrt{2n})$, divide the n vertices of S_n in an arbitrary way into two groups β and γ of p and $n-p$ points, and indicate by β' and γ' the groups of the p and $n-p$ corresponding vertices of S'_n, by B, C, B', C' (fig. 2) the centres of gravity of the point-groups $\beta, \gamma, \beta', \gamma'$ — i.e. the

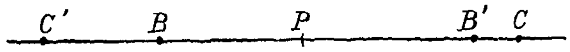


Fig. 2.

[1]) In contradiction to this seems that for $n=5$ through each edge *three* faces pass and thus *three* bounding bodies (12, 18, 8) lie around it. This contradiction however is only apparent; it is annulled by the remark that two bounding bodies (12, 18, 8) having a face in common agree or differ in colour according to the face being triangular or hexagonal. Of the three faces one is triangular, two are hexagonal; the bounding bodies to which the two hexagonal faces belong, differ in colour from the two others, these agreeing in colour.

centres of the bounding simplexes $S_p, S_{n-p}, S'_p, S'_{n-p}$ with these points as vertices. Then the five points B, C, B', C', P lie in such a way upon the same right line, that B and C' lie on one side of P and B' and C on the other side, and we have

$$\left.\begin{array}{r} p\ BP = (n-p)\,PC \\ (n-p)\,C'P = \quad p\,.\,PB' \end{array}\right\},\ \frac{AP}{PA'} = \frac{BP}{PB'} = \frac{CP}{PC'}.$$

We can now assert that the bounding simplex S_p of the vertices β of S_n lies entirely or partly inside S'_n when B is between C' and P, whilst S_p lies entirely outside S'_n when C' lies between B and P. In other words, as AP increases, the bounding simplex S_p of S_n comes entirely outside S_n when B coincides with C' and the spaces Sp_{p-1} and Sp_{n-p-1} of S_p and S'_{n-p}, crossing each other in general entirely perpendicularly, become incident because they get the point $B = C'$, then common centre of gravity, as point of intersection.

Under the condition $BP = C'P$ follows from the equations

$$\frac{BP}{PC} = \frac{n-p}{p},\ \frac{PC}{C'P} = \frac{AP}{PA'}$$

the relation

$$(n-p)\,AP = p\,.\,PA',$$

which shows that P must coincide with the p^{th} dividing point A_p of AA'.

10. If P coincides with A_p the spaces Sp_{p-1} and Sp_{n-p-1} of S_p and S'_{n-p} have, as we saw above, the common centre of S_p and S'_{n-p} in common. As this point of intersection of S_p and S'_{n-p} becomes vertex of the section, — if we call this again $\frac{p}{n}(M_n)$ in connection with preceding investigations — the theorem holds:

"The centres of the $\binom{n}{p}$ bounding simplexes S_p of a regular simplex $S_n(p\sqrt{2})$ form the vertices of a polytope congruent to $\frac{p}{n}(M_n)$ for $p = 1, 2, \ldots, n-1$."

For even $n = 2n'$ we have specially:

"The centres of the $\binom{2n'}{n'}$ bounding simplexes $S_{n'}$ of a regular simplex $S_{2n'}(n'\sqrt{2})$ form the vertices of a $D_{2n'-1}$."

11. If P lies between A_p and A_{p+1} the vertices of the section of the two simplexes S_n and S'_n are furnished by the points of intersection of each bounding simplex S_{p+1} of S_n with the $p+1$

bounding simplexes S'_{n-p} of S'_n which have the property of counting among their $n-p$ vertices only one vertex corresponding to a vertex of this S_{p+1}; in each bounding simplex S_{p+1} these $p+1$ points of intersection form the vertices of a new regular simplex $\overline{S}_{p+1}$ which is concentric to the assumed one but oppositely orientated. We determine the length of the edges of this new simplex, for the definite case that P lies just in the middle between A_p and A_{p+1}, with the aid of reflections in quite close connection with the preceding.

If B, C, B', C' (fig. 3) are successively the centres of gravity of

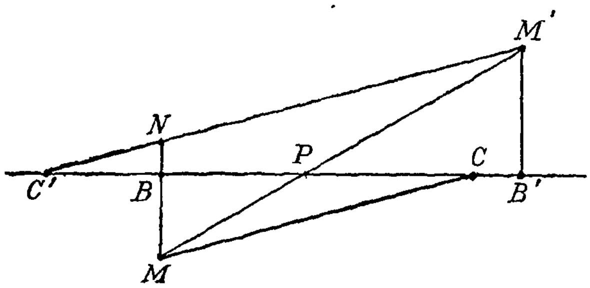


Fig. 3.

the bounding simplex S_{p+1}, of the bounding simplex S_{n-p-1} of the remaining vertices of S_n and of the bounding simplexes S'_{p+1}, and S'_{n-p-1} of the groups of vertices of S'_n corresponding with the vertices of S_{p+1} and S'_{n-p-} these points lie on a same right line through P again, viz.: B and C' on one side and C and B' on the other side of P. If furthermore M and M' are corresponding vertices of S_{p+1} and S'_{p+1} these points lie in parallel normals erected in B and B' on BB' and the line connecting M and M' passes through P. The point of intersection N of BM and $C'M'$ is the vertex of $\overline{S}_{p+1}$ corresponding to the vertex M of S_{p+1}. From CM and $C'M'$ being parallel follows

$$\frac{BN}{MB}=\frac{C'B}{BC}=\frac{C'P-BP}{BP+PC},$$

whilst the relations

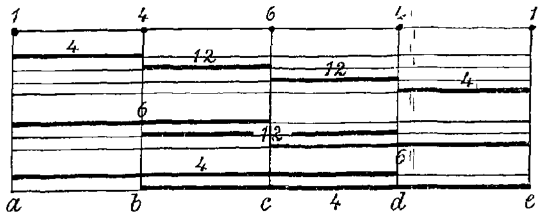
$$\frac{AP}{PA'}=\frac{BP}{PB'}=\frac{CP}{PC'}=\frac{2p+1}{2n-2p-1}$$

and

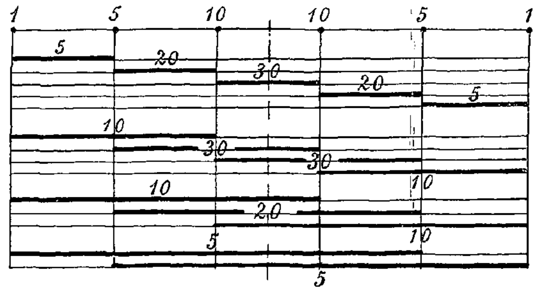
$$\frac{BP}{PC}=\frac{B'P}{PC'}=\frac{n-p-1}{p+1}$$

enable us to express $C'P$ and BP in PC. Substitution gives the result

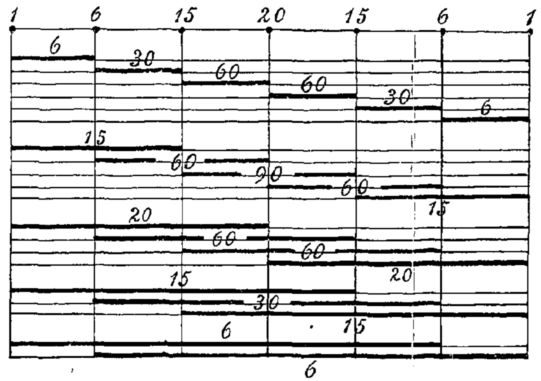
P. H. SCHOUTE. "The section of the measure-polytope M_n of space Sp_n with a central space Sp_{n-1} perpendicular to a diagonal."



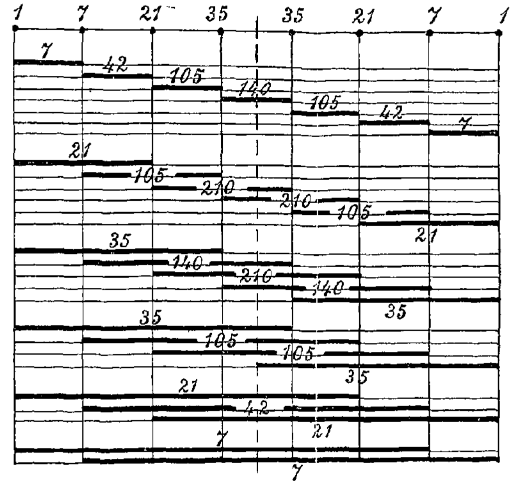
$n=4$



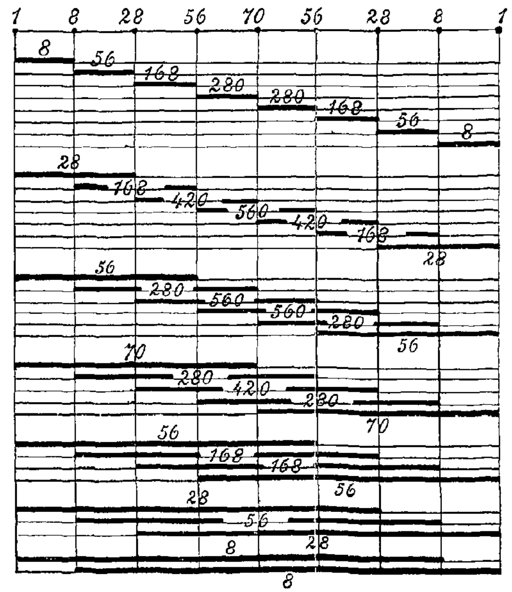
$n=5$



$n=6$



$n=7$



$n=8$

$$\frac{BN}{MB} = \frac{1}{2p+1}.$$

So the theorem holds:

"If we describe in the spaces Sp_p bearing the bounding simplexes $S_{p+1}\left(\frac{2p+1}{2}\sqrt{2}\right)$ of a regular simplex $S_n\left(\frac{2p+1}{2}\sqrt{2}\right)$ simplexes $S_{p+1}\left(\frac{1}{2}\sqrt{2}\right)$ concentric and oppositely orientated to the original ones we find the $(p+1)\binom{n}{p+1}$ vertices of a $\frac{2p+1}{2n}(M_n)$."

For odd $n = 2n' + 1$ we have in particular:

"If we describe in the spaces $Sp_{n'}$, bearing the bounding simplexes $S_{n'+1}\left(\frac{2n'+1}{2}\sqrt{2}\right)$ of a regular simplex $S_{2n'+1}\left(\frac{2n'+1}{2}\sqrt{2}\right)$ simplexes $S_{n'+1}\left(\frac{1}{2}\sqrt{2}\right)$ concentric and oppositely orientated to the original ones we find the $(n'+1)\binom{2n'+1}{n'+1}$ vertices of a $D_{2n'}$."

In connection with the results found above the length $\frac{1}{2}\sqrt{2}$ appearing here for the edges of the new simplexes contains a confirmation.

Mathematics. — "*On five pairs of four-dimensional cells derived from one and the same source.*" By Mrs. A. Boole Stott and Prof. P. H. Schoute.

(Communicated in the meeting of December 28, 1907).

Introduction.

As this paper must be regarded as a short completion of the handbook of the "Mehrdimensionale Geometrie" included in the Sammlung Schubert we keep the notation used there.

We regard in succession each of the six regular cells C_5, C_8, C_{16}, C_{24}, C_{120}, C_{600} of the space Sp_4 and derive from these two new four-dimensional cells. The first, which has the centres K_0 of the edges of the regular cell as vertices is formed by a regular truncation at the vertices as far as the centres of the edges; the second is the reciprocal polar of the first with respect to the spherical space of the points K_0.