The Foundation of Algebraic Geometry from Severi to André Weil

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From Max Noether to Severi

Algebraic geometry was created by MAX NOETHER. The Italian school, headed by CORRADO SEGRE, CASTELNUOVO, ENRIQUES and SEVERI, erected an admirable structure, but its logical foundation was shaky. The notions were not well-defined, the proofs were insufficient. And yet, as BERNARD SHAW puts it: "There is an Olympian ring in it. It must be true, for it is fine art."

Severi's Paper on the Conservation of Number

A first step towards a rigid foundation was taken by FRANCESCO SEVERI in 1912 in a paper "Il principio della conservazione del numero" [1]. SEVERI considers a correspondence between varieties U and V, *i.e.* a set of pairs of points (x, y), x in U, y in V, defined by homogeneous equations $H(x, y) = 0$. He supposes that U is irreducible and that to a generic point ξ of U corresponds a finite number of points $\eta^{(1)}, \ldots, \eta^{(h)}$ of V. If ξ tends to x, the points $\eta^{(\nu)}$ tend to limit points $y^{(1)}, \ldots, y^{(h)}$. If μ of the points $\eta^{(\nu)}$ tend to one and the same limit point y, this limit point is said to have *multiplicity* μ. Assuming the number of solutions of the equations $H(x, y) = 0$ for a given x to be finite, SEVERI states first, that the multiplicities are uniquely defined. While he states this, he does not prove it. Second, if the given correspondence is irreducible, every solution y of the equations $H(x, y) = 0$ occurs at least once among the $y^{(\nu)}$. Finally: the sum of the multiplicities of all solutions y for a given x is equal to the number of solutions of the set $H(\xi, \eta) = 0$ for a generic ξ. This is the *Principle of Conservation of Number*, first enunciated in a crude form by H. SCHUBERT [2]. Counterexamples to SCHUBERT'S original statement were given by G. KOHN [3] and K. ROHN [4]. KOHN'S counterexamples incited SEVERI to introduce the condition that the correspondence be irreducible.

In a later paper [29], SEVERI admitted that his first assertion, saying that the multiplicities are unique, required the additional assumption: x is a simple point of U. If this assumption is made, SEVERI'S three assertions are indeed correct, but the algebraic tools needed to prove them were not yet available in 1912, when SEVERI wrote his earlier paper. Anyhow, SEVERI was the first to give an exact definition of "multiplicity" and to formulate exact assertions.

Algebraic Tools: From Dedekind to Emmy Noether

The algebraic tools needed for a rigorous foundation of algebraic geometry were: Theory of Ideals, Elimination Theory, and Theory of Fields. They were developed in the schools of DEDEKIND, of KRONECKER, and of HILBERT. A most important paper of DEDEKIND & WEBER [5] in which DEDEKIND'S Theory of Ideals was applied to fields of algebraic functions, appeared as early as 1882. Elimination theory was developed by KRONECKER and his school [6]. Systems of resultants were studied by MERTENS [14] and HURWITZ [7]. Most important work on the theory of Polynomial Ideals was done by LASKER [8], the famous chess champion, who had got his problem from HILBERT, and by MACAULY [9], a schoolmaster who lived near Cambridge, England, but who was nearly unknown to the Cambridge mathematicians when I visited Cambridge in 1933. I guess the importance of MACAULAY'S work was known only in Göttingen. Finally, we must mention the fundamental paper of STEINITZ [10]: "Algebraische Theorie der Körper", which appeared in 1910.

When I came to Göttingen in 1924 and showed EMMY NOETHER some work I had done on generalizations of MAX NOETHER'S "fundamental theorem", she said "What you have done is all right, as far as it goes, but LASKER and MACAULAY have obtained much more general results." She told me to study the paper of STEINITZ and MACAULAY'S tract on Polynomial Ideals, and she gave me some of her papers on Ideal Theory [11] and on HENTZELT'S Elimination Theory [12].

Generic Points

Thus, armed with the powerful tools of Modern Algebra, I returned to my main problem: to give Algebraic Geometry a solid foundation. I asked myself: What do the Italian geometers mean by a generic point on a variety (punto generico), a generic plane in 3-space, *etc.*? Obviously, a generic point is supposed not to have certain undesirable properties. For instance, a generic plane is not tangent to a given curve, it does not pass through a given point, *etc.* I asked: is it possible to find, on a given variety U, a point ξ not having any special properties except those which hold for all points of U? Of course, I restricted myself to algebraic properties, which can be expressed by algebraic equations.

If U is the whole space, it is easy to construct such a point ξ. One just takes as coordinates of ξ indeterminates. If an equation $f(X)=0$ with coefficients from the ground field holds for indeterminates $X_1, \ldots, X_m$, it holds for all special values x' as well.—Now let U be any irreducible variety. Can one find a point ξ on U which is "generic" in the sense that if an equation $F=0$ with constant coefficients holds for ξ it holds for all points of U? EMMY NOETHER'S paper [11] on HENTZELT'S elimination theory showed me the way. In order to understand the fundamental ideas of this paper, we have to go back to KRONECKER'S elimination theory.

KRONECKER had developed a method to find all solutions of a set of algebraic equations by successive eliminations [6]. Using this method, he had proved: First, every variety is a union of irreducible varieties. Second, after a suitable linear transformation of coordinates, all points of an irreducible variety U of dimension d can be obtained as follows: the first d coordinates $x_1, \ldots, x_d$ are arbitrary, and the others are algebraic functions of them.

EMMY NOETHER [12] had obtained the same results anew by a more elegant elimination method due to her pupil HENTZELT, who had died in the first World War. She worked out HENTZELT's results and added new ideas of her own. One of these ideas was to replace the coordinates $x_1, \ldots, x_d$ by indeterminates $\xi_1, \ldots, \xi_d$ and to determine the other ξ_i as algebraic functions of these indeterminates in an algebraic extension field $k(\xi)$. This field she called the *Nullstellenkörper* of the prime ideal $\mathfrak{p}$ belonging to the variety.

When I studied EMMY NOETHER's paper, I saw that the point ξ with coordinates $\xi_1, \ldots, \xi_m$ was just the generic point I was looking for. I also saw that EMMY NOETHER's Nullstellenkörper was isomorphic to the quotient field of the residue class ring $\mathfrak{o}/\mathfrak{p}$, where $\mathfrak{o}$ is the polynomial ring $k[X]$. Hence it was not necessary to go through HENTZELT's elimination procedure; one could start with any prime ideal $\mathfrak{p} \neq \mathfrak{o}$, construct the residue class ring $\mathfrak{o}/\mathfrak{p}$ and its quotient field $k(\xi)$, and thus find a generic point ξ of the variety of $\mathfrak{p}$.

I wrote a paper [13] based upon this simple idea and showed it EMMY NOETHER. She at once accepted it for the *Mathematische Annalen*, without telling me that she had presented the same idea in a course of lectures just before I came to Göttingen. I heard it later from GRELL, who had attended her course.

Specializations

The next problem I studied was the definition of multiplicity. Let us recall that SEVERI had considered a correspondence between U and V, defined by a set of homogeneous equations

$$H(x, y) = 0. \tag{1}$$

SEVERI's assumptions are

1) U is irreducible.

2) If ξ is a generic point of U, the set of equations

$$H(\xi, \eta) = 0 \tag{2}$$

has a finite number of solutions $\eta^{(1)}, \ldots, \eta^{(h)}$.

3) For the given point x, the set of equations (1) has a finite number of solutions.

SEVERI now constructs specialized solutions $y^{(1)}, \ldots, y^{(h)}$ by a passage to the limit:

$$(\xi, \eta^{(1)}, \ldots, \eta^{(h)}) \to (x, y^{(1)}, \ldots, y^{(h)}). \tag{3}$$

If the ground field k is a field without topology, such a passage to the limit makes no sense. Therefore, I introduced the notion of *specialization* or, as I first called it, *relationstreue Spezialisierung*. The set $(x, y^{(1)}, \ldots)$ is called a specialization of $(\xi, \eta^{(1)}, \ldots)$, if all homogeneous equations $F(\xi, \eta^{(1)}, \ldots) = 0$ remain valid for $(x, y^{(1)}, \ldots)$.

I now had to prove the existence and uniqueness of the specialization (3) under appropriate conditions. This I did by means of elimination theory.

The Existence of Extensions of Specializations

MERTENS had already constructed, for any set of homogeneous equations $F_j(y)=0$, a set of resultants $R_1, \ldots, R_s$ depending only on the coefficients of the forms F_i, such that the equations have a non-zero solution if and only if all resultants R_j are zero. In a paper presented to the Amsterdam Academy in 1926 [15], I gave a simpler derivation, based upon HILBERT's Nullstellensatz.

No matter whether one uses MERTENS' resultants or mine, it is easy to see that every specialization $\xi \to x$ can be extended to a specialization

$$(\xi, \eta^{(1)}, \ldots, \eta^{(h)}) \to (x, y^{(1)}, \ldots, y^{(h)}).$$

Later on CHEVALLEY proved the possibility of extending specializations without making use of elimination theory. ANDRÉ WEIL included CHEVALLEY'S proof in his book [16], expressing the hope that this device "finally eliminates from algebraic geometry the last traces of elimination theory". Obviously WEIL and CHEVALLEY do not like Elimination Theory.

Uniqueness of Specialization Multiplicities

More difficult was the proof of the uniqueness of the specialization (3). In 1927, in a paper "Der Multiplizitätsbegriff der algebraischen Geometrie" [17], I made the following assumptions:

First, the coordinates $\xi_1, \ldots, \xi_m$ are independent variables.

Secondly, the equations (1) are assumed to have, for a given x, a finite number of solutions only.

From these hypotheses I proved that the specialized solutions $y^{(1)}, \ldots, y^{(h)}$ are uniquely determined but for their order. Hence the multiplicity μ of any solution y can be defined as the number of times it appears among the $y^{(\nu)}$. Obviously, the sum of the multiplicities of all solutions y is equal to h. This is the Principle of Conservation of Number. Moreover, if the correspondence (2) is irreducible, every solution of the specialized problem (1) occurs at least once among the $y^{(\nu)}$. Thus all assertions made in SEVERI's paper of 1912 [1] had been proved rigorously, under appropriate assumptions, by 1927.

In a later paper of mine [18] the assumption that the ξ_i be independent variables (which implies that the variety U is the whole affine or projective space) was replaced by the weaker assumption that x be a simple point of U. As we shall see later, ANDRÉ WEIL replaced my second assumption, which says that the specialized problem (1) has a finite number of solutions, by the weaker assumption that one solution y be isolated. The assumptions were weakened still more by NORTHCOTT [19] and LEUNG [20].

Intersection Multiplicities

My next problem was, to define *intersection multiplicities* and to generalize BÉZOUT'S theorem on the number of points of intersection of two plane curves.

Intersection of Hypersurfaces

The case of the intersection of n forms or hypersurfaces in P^n is very easy. The multiplicities can be defined as exponents in the factorization of the u-resultant of the forms, *i.e.* of the resultant of the n forms plus one generic linear form

$\sum u_k y_k$. All this is explained on one or two pages in § 7 of my paper "Multiplizitätsbegriff" [17].

Intersection of a Curve and a Hypersurface

Next consider the intersection of an irreducible curve C with a form F. (A hypersurface, *i.e.*, a cycle of dimension $n-1$, will be called a form F.) How can we define the multiplicities of the points of intersection?

The problem can be solved by several methods. First one can use the classical theory of algebraic functions. The ratios ξ_i/ξ_0 of the homogeneous coordinates of a variable point ξ of C are algebraic functions of one complex variable z, univalent on a Riemann surface. To every point on the closed Riemann surface corresponds a point on the curve C. Vice versa, to every point on the curve corresponds at least one point on the Riemann Surface. Consider the rational function

$$F(\xi_0, \ldots, \xi_n)/L(\xi)^g,$$

L being a generic linear form and g the degree of F. The poles of this meromorphic function correspond to the points of intersection of L and g, each pole having order g. The zeros correspond to the points of intersection of F and C. Now define the *multiplicity* of any point of intersection to be the sum of the orders of the corresponding zeros on the Riemann surface. The total sum of the orders of all zeros is equal to the sum of the orders of all poles, hence the sum of the multiplicities of the meeting points of C and F is equal to the product of the degrees of C and F. This generalizes BÉZOUT's theorem. The idea of this definition and proof can be found in papers of the French geometer HALPHEN.

This method of proof works just as well for an arbitrary ground field, for one can use, instead of classical function theory, the arithmetical theory of DEDEKIND & WEBER [5].

Even simpler is the following, second definition, which was systematically used in my series of papers "Zur algebraischen Geometrie", beginning in 1933 [21]. Consider the form F as a specialization of a generic form F^* with indeterminate coefficients, which meets the curve C in the points $\eta^{(1)}, \ldots, \eta^{(h)}$. The specialization $F^* \to F$ can be extended to a specialization

$$(F^*, \eta^{(1)}, \ldots, \eta^{(h)}) \to (F, y^{(1)}, \ldots, y^{(h)}).$$

The $y^{(\nu)}$ are the points of intersection of F and C, and their intersection multiplicities can be defined to be equal to their specialization multiplicities.

A third possibility is offered by the theory of polynomial ideals. Introduce non-homogeneous coordinates. Let $\mathfrak{p}$ be the prime ideal belonging to the curve. The ideal $(F, \mathfrak{p})$ is an intersection of zero-dimensional primary ideals $\mathfrak{q}$ corresponding to the single points of the meet of F and C. For every $\mathfrak{q}$, the residue class ring $\mathfrak{o}/\mathfrak{q}$ has a finite rank. This rank can be defined to be the multiplicity of the point of intersection.

All three definitions are equivalent. For a proof see, *e.g.*, LEUNG's paper [20].

Intersection of Two Varieties in P^n

Much more difficult is the case of an intersection of two varieties A and B of arbitrary dimensions r and s in a projective space P^n. The general case can

easily be reduced to the case $r+s=n$, hence we shall suppose A and B to be varieties of dimensions d and $n-d$ in P^n, for instance two surfaces in 4-space, which meet in a finite number of points.

Lefschetz' Topological Definition

LEFSCHETZ [22] gave in 1924 a topological definition of the intersection multiplicity, valid for the complex ground field. He considers the surfaces A and B as four-dimensional cycles in the complex projective space P^4. This space is an 8-dimensional orientable manifold. The space itself and the cycles A and B can be oriented in a fixed, canonical way. If y is a point of intersection and if the varieties have simple points and no common tangent at y, their topological intersection multiplicity, defined by simplicial approximation, is always $+1$. For the general case, in which there may be common tangents, I proved that the topological intersection multiplicity, or *index* of an isolated point of intersection is always positive. If this index is defined to be the intersection multiplicity, a perfectly satisfactory theory is obtained [23], and SCHUBERT'S "Calculus of Enumerative Geometry" [2] can be justified.

Applying a Projective Transformation

For an arbitrary ground field, this method does not work. I therefore proposed in 1928 [24] to bring A and B into a generic relative position by applying to one of them a projective transformation T with indeterminate coefficients. The transformed variety TA intersects B in a finite number of points. If T is specialized to the identity, the points of intersection of TA and B specialize to points of intersection of A and B. If A and B meet in a finite number of points, each of these appears with a certain specialization multiplicity, which may be defined to be the intersection multiplicity. In the complex field case, this definition is equivalent to that of LEFSCHETZ. By a method due to the Dutch geometer SCHAAKE, I proved the generalized theorem of BÉZOUT.

In my paper of 1928, the proofs were very complicated, but in a later paper [25] I gave a simple proof based on the same idea.

A and B on a Variety U: Severi's Definitions

At the Zürich International Congress in 1932 I met SEVERI, and I asked him whether he could give me a good algebraic definition of the multiplicity of a point of intersection of two varieties A and B, of dimensions d and $n-d$, on a variety U of dimension n, on which the point in question is simple. The next day he gave me the answer, and he published it in the *Hamburger Abhandlungen* in 1933 [26]. He gave several equivalent definitions; I shall explain one of them for the case of two curves A and B lying on a surface U in 3-space and meeting in a simple point y of U.

Let S be a generic point in 3-space. Connecting S with A, one obtains a cone K. The point y has, as a point of intersection of the curve B with the cone K, a certain multiplicity μ. This is defined to be the multiplicity of y as a point of intersection of A and B on U.

This definition can easily be extended to intersections of varieties A and B of arbitrary dimensions r and s on a variety U of dimension n. If $r+s$ exceeds n, the normal dimension of the intersection is $r+s-n$. If a component C of the intersection has just this dimension, it is called a *proper component*. If all components are proper, each of them has, in SEVERI's theory, a well-defined multiplicity, and one can define an *intersection cycle* $A \cdot B$ as the formal sum of the components C, counted with their multiplicities μ:

$$A \cdot B = \sum \mu C .$$

Cycles and Their Coordinates

In explaining SEVERI's results, I have used the expressions *cycle* and *intersection cycle*, defining a cycle to be a formal sum of irreducible components C multiplied by integers μ.

To be sure, these expressions do not occur in SEVERI's paper. In SEVERI's papers (and in mine too) the same word "variety" (Varietà, in German, Mannigfaltigkeit) was used for two different notions, *viz.*

1) a point set in affine or projective space, defined by algebraic equations,

2) a formal sum of irreducible varieties C of the same dimension d with integer coefficients μ.

In the present paper, the word "variety" will only be used in the sense 1), whereas the formal sums 2) will be called *cycles*, following A. WEIL [16].

If the coefficients μ are arbitrary integers, the cycle will be called a *virtual cycle* (SEVERI: Varietà virtuale). The virtual cycles of any dimension d form an additive group, the free generators of the group being the irreducible varieties. If the integers μ are non-negative, we have a *positive cycle* (Varietà effettiva). In the sequel only positive cycles will be considered.

Varieties of cycles, e.g. congruences and complexes of lines, varieties of conics, *etc.* have been objects of investigation already in the 19th century. A variety of lines in 3-space is defined by a set of homogeneous equations in the PLÜCKER coordinates, and a linear system of conics by linear equations in the coefficients of a conic.

Linear systems of plane curves and their birational transformations were studied by CASTELNUOVO in a classical paper [27]. The theory was generalized to linear systems of curves C^1 on a surface V^2, and to linear systems of cycles C^{r-1} on a variety V^r, by CASTELNUOVO, ENRIQUES, and SEVERI. An excellent exposition of these classical theories was given by ZARISKI [28].

The generalization to non-linear systems of curves on a surface was inaugurated by SEVERI [29] in 1903; for an account of this theory see Chapter V of ZARISKI's report [28].

In 1932, SEVERI opened a new field of research by considering rational and algebraic systems of cycles C^d on a variety C^r [29]. A complete exposition of this theory was given by SEVERI in three volumes [30].

In all these theories, a correct definition of the notion "variety of cycles" (sistema di varietà) was missing. It was first given by CHOW & VAN DER WAERDEN in 1936 [31]. Our idea was, to assign to every cycle C^d a form $F(u)$ in $u^{(0)}, u^{(1)},$

..., $u^{(d)}$, called the *associated form* (zugeordnete Form, or CAYLEY-form) of the cycle, and to use the coefficients of this form as coordinates (CHOW-coordinates) of the cycle. Varieties of cycles are now defined by homogeneous equations in these coordinates. The main theorem, proved in our joint paper [31], was: The totality of all cycles C^d in P^n of given dimension d an degree g is a variety of cycles, *i.e.* it can be defined by homogeneous equations in the coordinates of the cycles. The proof of this theorem is due to CHOW.

The general theory of Varieties of Cycles was further developed in my paper, "Zur algebraischen Geometrie 14" [25]. In this paper, I showed that SEVERI's definition of the intersection cycle $A \cdot B$ has all desirable properties, including the commutative and associative laws for intersection cycles. Also: If A varies in a variety of cycles and if B is fixed, the intersection cycle $A \cdot B$ varies in a variety of cycles. Just so if A and B both vary in varieties of cycles.

The intersection theory developed in "Zur algebraischen Geometrie 14" was a global theory. The first to develop a *local* algebraic theory of intersections was ANDRÉ WEIL, in his book "Foundations" [16]. To understand his line of thought, we must first return to the notion of *specialization multiplicity*.

Weil's Specialization Multiplicity

WEIL considers a specialization of a pair of points in affine spaces:

$$(\xi, \eta) \to (x, y') \tag{4}$$

and makes the following assumptions:

1) $\xi_1, \ldots, \xi_m$ are independent variables,

2) the η are algebraic functions of the ξ,

3) y' is an isolated specialization over the specialization of ξ to x. This means that if we consider all equations $H(\xi, \eta) = 0$ which hold for ξ and η, then the specialized set of equations

$$H(x, y) = 0 \tag{5}$$

is supposed to define, for given x, a variety in which y' is an isolated point. WEIL does not suppose, as I had done, that the set (5) has only a finite number of solutions.

WEIL now considers the conjugates $\eta^{(\nu)}$ of η and proves (Chapter III, § 3, Proposition 7) that there exists a unique integer μ, called the *specialization multiplicity* of y', such that y' appears exactly μ times in any specialization

$$(\xi, \eta^{(1)}, \ldots, \eta^{(h)}) \to (x, y^{(1)}, \ldots, y^{(h)}).$$

The proof is given by means of the analytic theory of specializations, which is explained in Chapter III of WEIL's book. It seems that WEIL developed this analytic theory mainly or exclusively for the purpose of proving the uniqueness of the multiplicity μ, for on p. 61 (first edition) he writes:

> The reader may now forget the whole of the contents of §§ 1–3 of this chapter, except prop. 7 of § 3, which ... will form the corner-stone for our theory of multiplicities in algebraic geometry.

Weil's Intersection Multiplicities

WEIL's theory of intersection multiplicities proceeds in two steps. First, WEIL considers two varieties A and B in affine space, one of which, say B, is a *linear* subspace. Let C be a *proper* component of the intersection, *i.e.* a component of dimension $d=r+s-n$. It is easy to reduce d to zero by intersecting all varieties with a generic set of d linear forms. Hence we may suppose d to be zero, and C to be an isolated point of the intersection of A and B. Now the linear variety B can be obtained from a generic linear variety by specialization. It follows that the intersection multiplicity can be defined as a specialization multiplicity.

The second, decisive step is taken in Chapter 6 of WEIL's book. WEIL's idea is, to reduce the general case of an intersection $A \cdot B$ to the special case in which B is linear. Let A and B be two subvarieties of an n-dimensional variety U in the affine space S^N, and let C be a proper component of their intersection. Suppose a generic point of C is simple on U. WEIL now intersects the product variety $A \times B$ with the diagonal Δ of the product space $S^N \times S^N$. This diagonal is a linear subspace, defined by N linear equations $x_i - y_i = 0$. The number of equations may be too large; therefore WEIL forms n generic linear combinations $F_j(x-y)=0$. They define a linear subspace L in $S^N \times S^N$. The diagonal Δ_C of $C \times C$ is contained in $A \times B$ and in L, hence in their intersection. WEIL now proves that Δ_C is a proper component of the intersection of $A \times B$ with L. As such, it has a definite multiplicity μ, because L is linear. This number μ is defined to be the multiplicity of C as a proper component of the intersection $A \cdot B$.

My time is up. Further important investigations on intersection multiplicities are due to SAMUEL, CHEVALLEY and SERRE. Completely new points of view were introduced by WEIL, by ZARISKI, by CHOW and in the school of GROTHENDIECK. However, in this lecture I have had to restrict myself to just one line of development, that from SEVERI to ANDRÉ WEIL.

The foregoing is an elaboration of a lecture delivered at Nice on September 5, 1970 at the International Congress of Mathematicians to the Section on the History of Science.

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