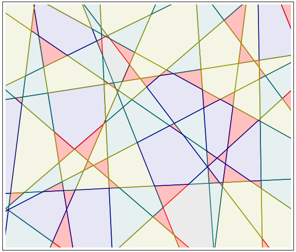
Small cells in a random hyperplane mosaic

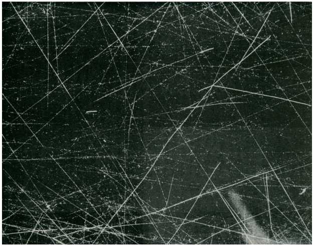
Gilles Bonnet, Matthias Reitzner

Monday 22nd September 2014

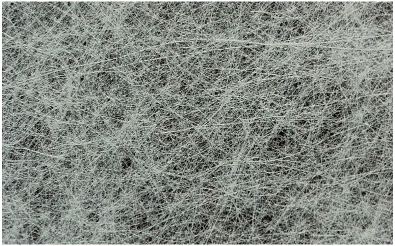
Motivations



(a) Line mosaic



(b) Cloud chamber, S.Goudsmit, 1945

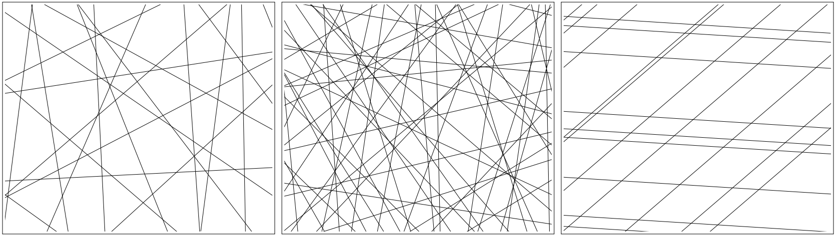


(c) Paper fibres, Miles, 1964

Typical cell of a Stationary Hyperplane Mosaic

η Stationary Poisson Hyperplane Process in $\mathbb{R}^d$.

- **intensity** γ
- **directional distribution** φ (even measure on $\mathcal{S}^{d-1}$)



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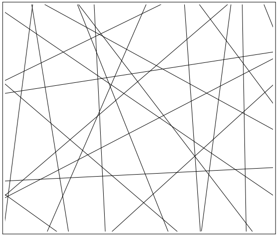
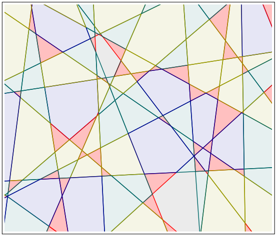
X Stationary Random Hyperplane Mosaic





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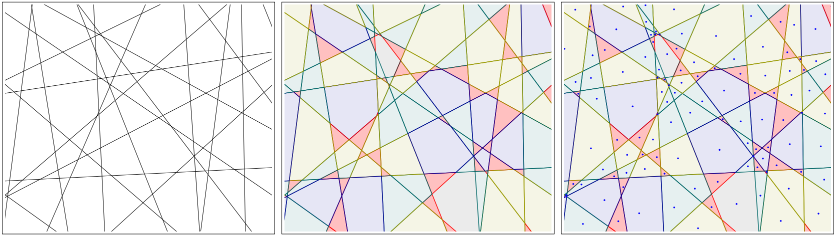
germs in $\mathbb{R}^d$ with intensity $\tilde{\gamma}$

grains in $\mathcal{P}_c$ with distribution $\mathbb{Q}$



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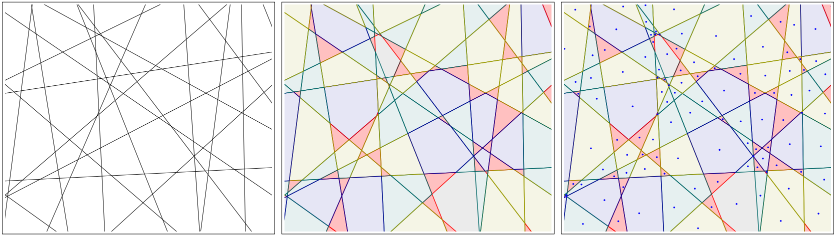

A random polytope with distribution $\mathbb{Q}$ is called **typical cell** Z.

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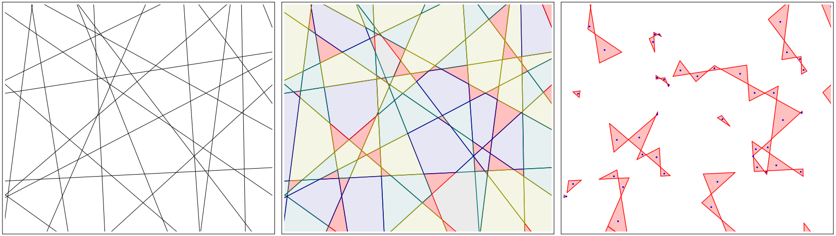


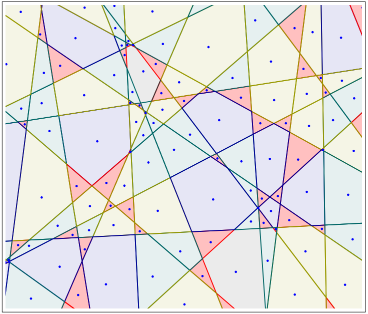
A random polytope with distribution $\mathbb{Q}$ is called **typical cell** Z.

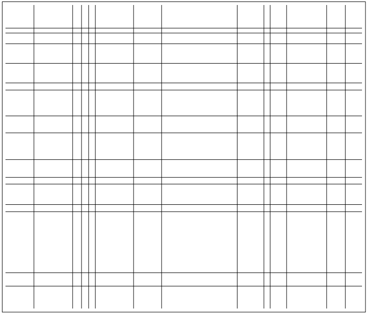
Typical cell with a fixed number of facets $\rightsquigarrow$ **typical n-tope** Z_n.

How does the typical cell Z look like?

Distribution of:

- the number of facets
- the size (mean-width, Φ-content, volume, surface area ...)
- the shape (=scale and translation invariant characteristics)





Let us consider a simpler question: How does the typical n-tope Z_n look like?

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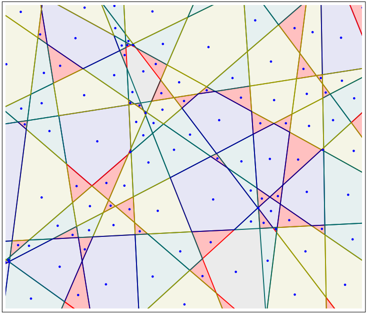

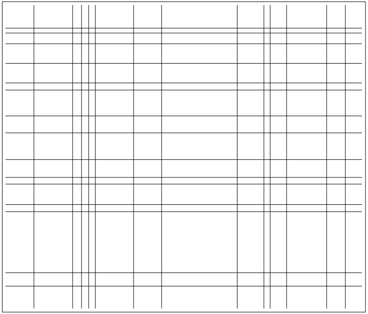

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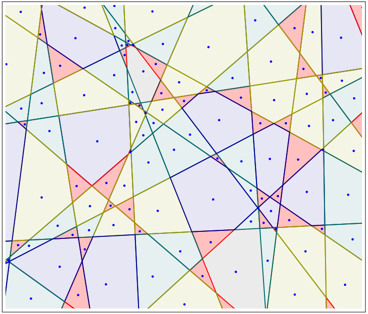
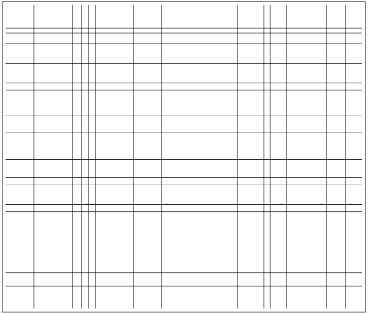
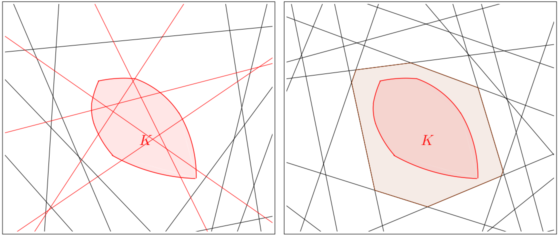



Φ-content

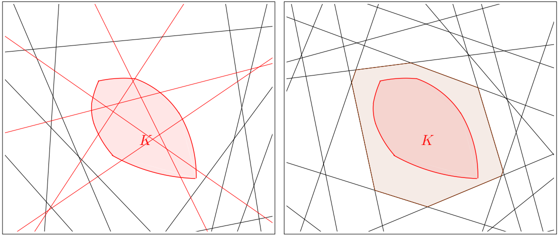
$\mathbb{P}(K \text{ is inside a cell of } X) = \textbf{?}$



Φ-content

$$\mathbb{P}(K \text{ is inside a cell of } X) = \mathbb{P}(\eta(\mathcal{H}_K) = 0) = e^{-\Theta(\mathcal{H}_K)}$$

$$\mathcal{H}_K = \{H \text{ hyperplane} \mid H \cap K \neq \emptyset\}$$



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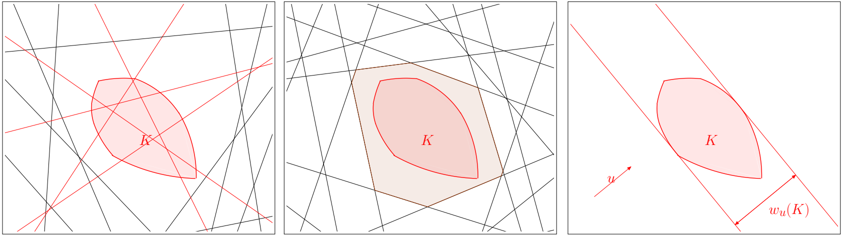
$$\Theta(\mathcal{H}_K) = \gamma \int_{\mathcal{S}^{d-1}} w_u(K)\, \varphi(\mathrm{d}u)$$



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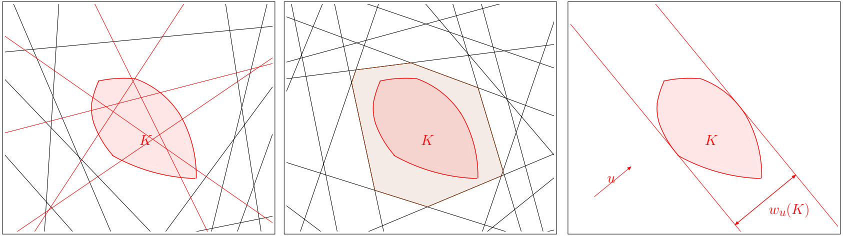

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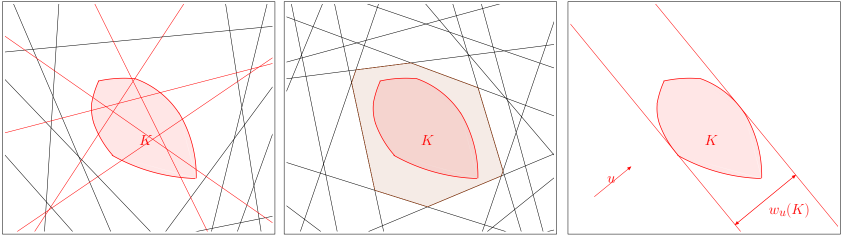
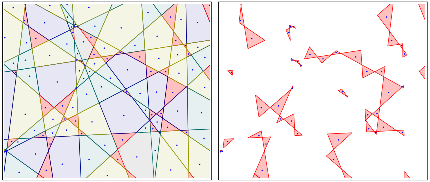


Distribution of the Φ-content and the shape of Z_n

$$\mathcal{S}(K) = \frac{1}{\Phi(K)}\,(K - c(K))\ldots\text{shape of } K$$



Distribution of the Φ-content and the shape of Z_n

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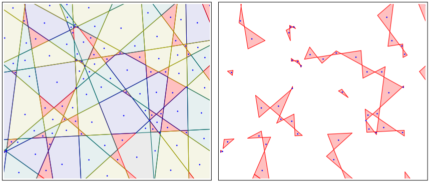
Theorem 1 (Møller,Zuyev 1996)

Given a fixed number of facets $n \geq d+1$,

- $\Phi(Z_n)$ *and* $\mathcal{S}(Z_n)$ *are* ***independent***
- $\Phi(Z_n)$ *is* ***gamma distributed*** *with parameters γ and $n-d$:*

$$\mathbb{P}(\Phi(Z_n) \in A) = \text{cst} \int_A e^{-\gamma t}\, t^{n-d-1} \mathrm{d}t \qquad \forall A \subset [0,\infty)$$



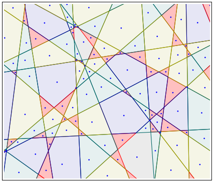
Distribution of the Φ-content and the shape of Z_n

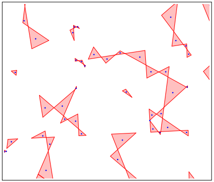
Corollary 1

For $C \subset \mathcal{P}_{n,c}$ we have

$$\mathbb{P}(Z_n \in C) = \text{cst} \int_{\mathcal{S}_n} \int_0^\infty \mathbb{1}(P(s,t) \in C)\, e^{-\gamma t}\, t^{n-d-1} \, \mathrm{d}t \; \mu_{\mathcal{S}_n}(\mathrm{d}s)$$

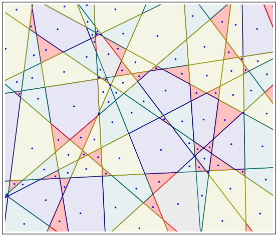
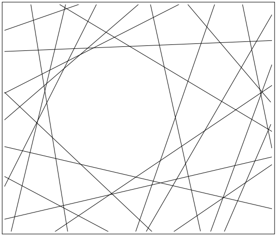
$\mathcal{P}_{n,c}$ ..... centred n-topes
$P(s,t)$... centred n-tope of shape s and Φ-content t
$\mu_{\mathcal{S}_n}(\cdot)$ ... shape distribution of Z_n





Small and big cells

- What is the shape of the *'big'* cells?
  Minimize a certain isoperimeter [Hug, Reitzner, Schneider: 2004,2007]

Small and big cells

- What is the shape of the *'big'* cells?
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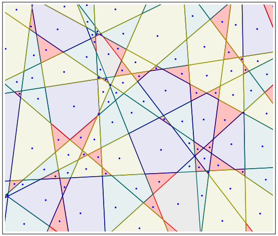
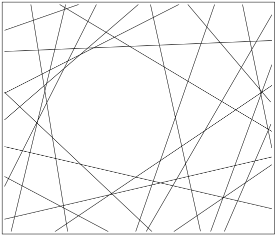
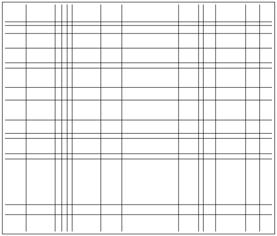
- What is the shape of the *'small'* cells?
  - 2-dimensional case, 2 directions [Beermann, Redenbach, Thäle: 2014]:
    - small area: degenerated shape, needle
    - small perimeter: random shape

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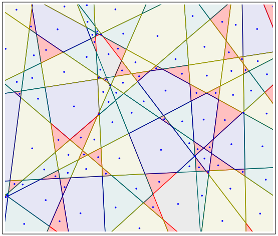
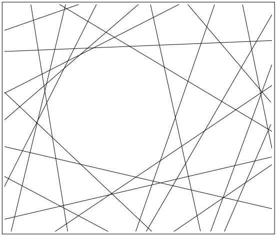
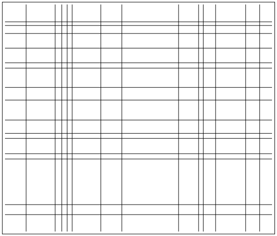
  - More general case: We will talk about it now!



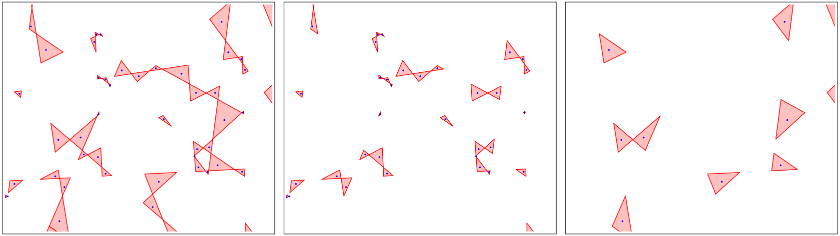




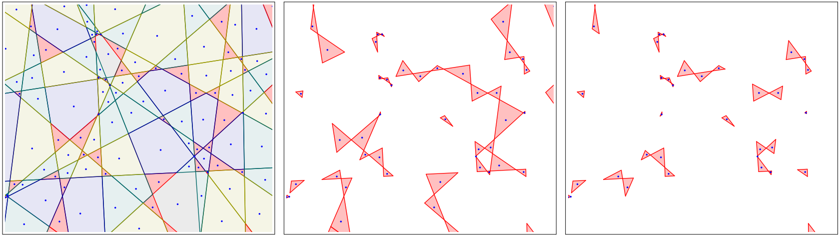
Shape of Z_n with small/big Φ-content

Corollary 2

$$\begin{aligned}\mu_{S_n}(\cdot) &:= \mathbb{P}(\mathcal{S}(Z_n) \in \cdot)\\ &= \mathbb{P}(\mathcal{S}(Z_n) \in \cdot \mid \Phi < a)\\ &= \mathbb{P}(\mathcal{S}(Z_n) \in \cdot \mid \Phi > a)\end{aligned}$$



Shape of Z_n with small volume $\lambda_d(Z_n)$



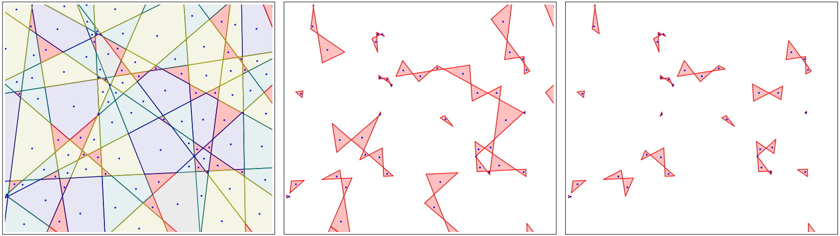
Shape of Z_n with small volume $\lambda_d(Z_n)$

Corollary 3

If $B \subset \mathcal{S}_n$ and $v > 0$ then

$$\mathbb{P}(\mathcal{S}(Z_n) \in B, \lambda_d(Z_n) < v) = \mathrm{cst} \int_B \int_0^{f(v,s)} e^{-\gamma t} t^{n-d-1} \, \mathrm{d}t \, \mu_{\mathcal{S}_n}(\mathrm{d}s)$$

$f(v, s) \ldots \Phi$-content of a n-tope of shape s and volume v.



Shape of Z_n with small volume $\lambda_d(Z_n)$

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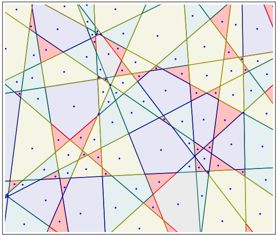
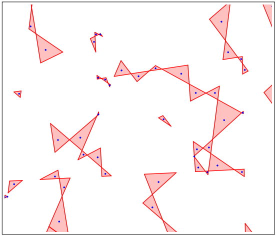
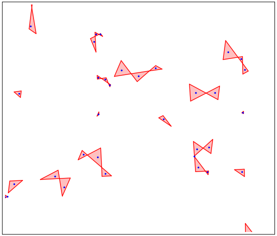
$f(v, s) \ldots$ Φ-content of a n-tope of shape s and volume v.

$$I_n(B) = \int_B f(1, s)^{n-d} \, \mu_{\mathcal{S}_n}(\mathrm{d}s)$$

Shape of Z_n with small volume $\lambda_d(Z_n)$

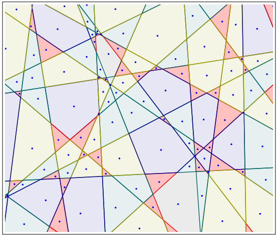
Corollary 3

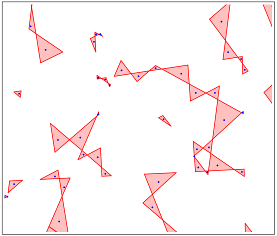
If $B \subset \mathcal{S}_n$, $v \to 0$, *and* $I_n(B) < \infty$ *then*

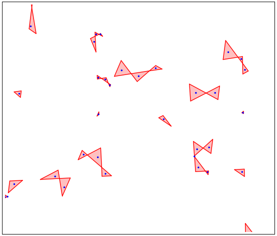
$$\mathbb{P}(\mathcal{S}(Z_n) \in B, \lambda_d(Z_n) < v) \sim \text{cst}\, v^{(n-d)/d} I_n(B)$$

$f(v,s)\ldots$ Φ-content of a n-tope of shape s and volume v.

$$I_n(B) = \int_B f(1,s)^{n-d}\, \mu_{\mathcal{S}_n}(\mathrm{d}s)$$







Shape of Z_n with small volume $\lambda_d(Z_n)$

Theorem 2

1. *if* $I_n(\mathcal{S}_n) < \infty$ *then*

$$\mathbb{P}(\mathcal{S}(Z_n) \in \cdot \mid \lambda_d(Z_n) < v) \xrightarrow{v \to 0} \frac{I_n(\cdot)}{I_n(\mathcal{S}_n)}$$

random shape

$f(v, s) \ldots \Phi$-content of a n-tope of shape s and volume v.

$I_n(B) = \int_B f(1, s)^{n-d} \, \mu_{\mathcal{S}_n}(\mathrm{d}s)$

Shape of Z_n with small volume $\lambda_d(Z_n)$

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random shape

2. *if* $I_n(\mathcal{S}_n) = \infty$ *then for any* $C > 0$,

$$\mathbb{P}\left(\frac{\Phi(Z_n)^d}{\lambda_d(Z_n)} < C \;\middle|\; \lambda_d(Z_n) < v \right) \xrightarrow{v \to 0} 0$$

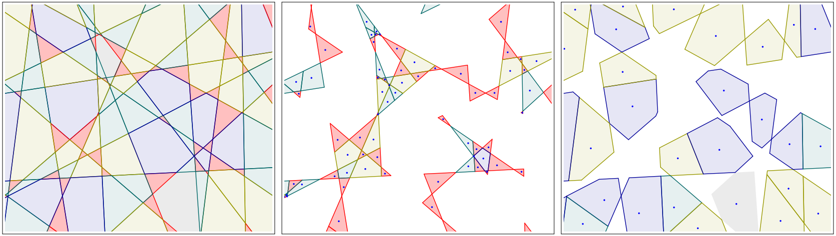
degenerated shape, high isoperimeter

$f(v, s) \ldots \Phi$-content of a n-tope of shape s and volume v.

$I_n(B) = \int_B f(1, s)^{n-d} \, \mu_{\mathcal{S}_n}(\mathrm{d}s)$

Number of facets of cells with small/big Φ-content

$n \in [n_{\min}, n_{\max}] \ldots$ admissible number of facets ($\subset [d+1, \infty]$)



Number of facets of cells with small/big Φ-content

$n \in [n_{\min}, n_{\max}]$... admissible number of facets ($\subset [d+1, \infty]$)

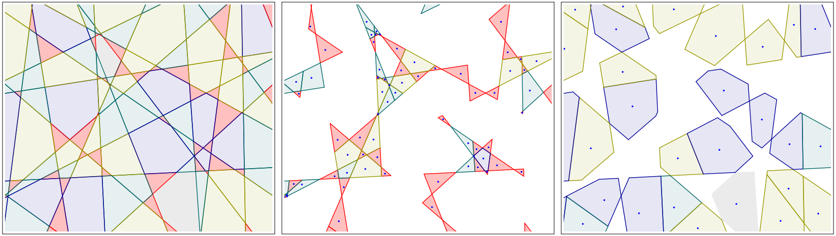
Corollary 4

When $a \to 0$, we have

(1) $$\mathbb{E}(f_{d-1}(Z) \mid \Phi(Z) < a) = n_{\min} + O(a)$$

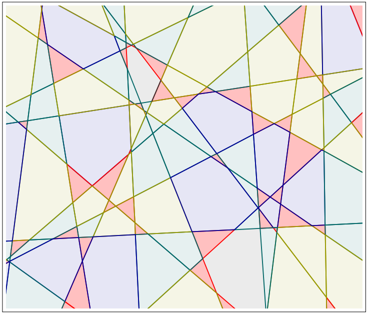
When $a \to \infty$ we have,

(2) $$\mathbb{E}(f_{d-1}(Z) \mid \Phi(Z) > a) \to n_{\max}$$



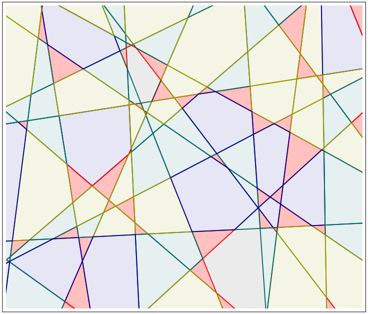
Perspectives

- Distribution of the number of facets of cells with small/big volume

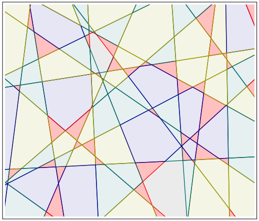
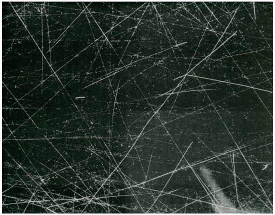
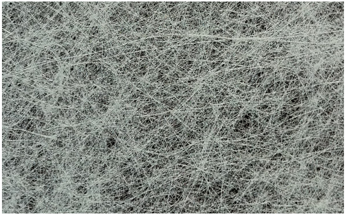


Perspectives

- Distribution of the number of facets of cells with small/big volume
- Distribution of the shape (isoperimeter...) of the cells with many facets

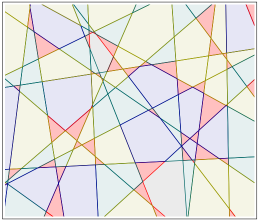
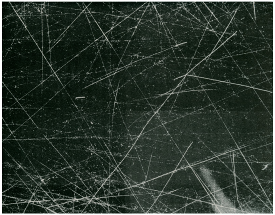
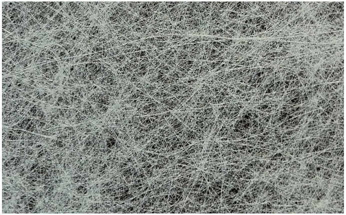


3 things you should you remember

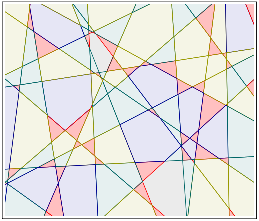
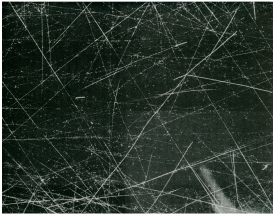
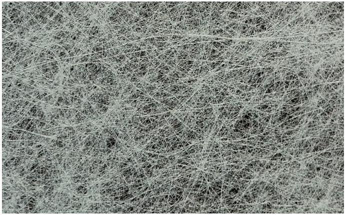
3 things you should you remember

- The shape and the Φ-content of Z_n are independent

3 things you should you remember

- The shape and the Φ-content of Z_n are independent
- The evaluation of a certain integral $\int_{\mathcal{S}_n} f(1,s)^{n-d}\,\mu_{\mathcal{S}_n}(\mathrm{d}s)$ tell us the behaviour of the *'small'* cell

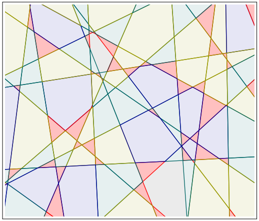
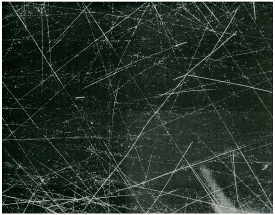
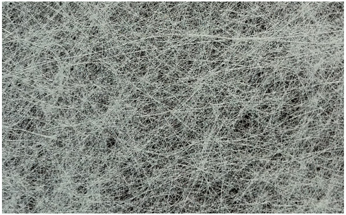




3 things you should you remember

- The shape and the Φ-content of Z_n are independent

- The evaluation of a certain integral $\int_{\mathcal{S}_n} f(1,s)^{n-d}\,\mu_{\mathcal{S}_n}(\mathrm{d}s)$ tell us the behaviour of the *'small'* cell

- Random mosaics are beautiful!

Thank you!