Cells with Many Facets in a Random Hyperplane Mosaic

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SGSIA Lingen 2015

Settings

η Stationary Poisson Hyperplane Process in $\mathbb{R}^d$

Θ intensity measure of η

- intensity γ
- directional distribution φ

X hyperplane mosaic associated to η

Z_0 Zero cell of X

$Z_0 \in \mathcal{P}_0$ with distribution $\mathbb{Q}_0$

$Z_{0,n}$ Zero cell of X conditioned on the event $\{Z_0$ has n facets$\}$

$Z_{0,n} \in \mathcal{P}_{0,n}$ with distribution $\mathbb{Q}_{0,n}$

Problem

We want to give asymptotic bounds when $n \to \infty$ for

$$q_{0,n} := \mathbb{P}(Z_0 \text{ has } n \text{ facets }.)$$

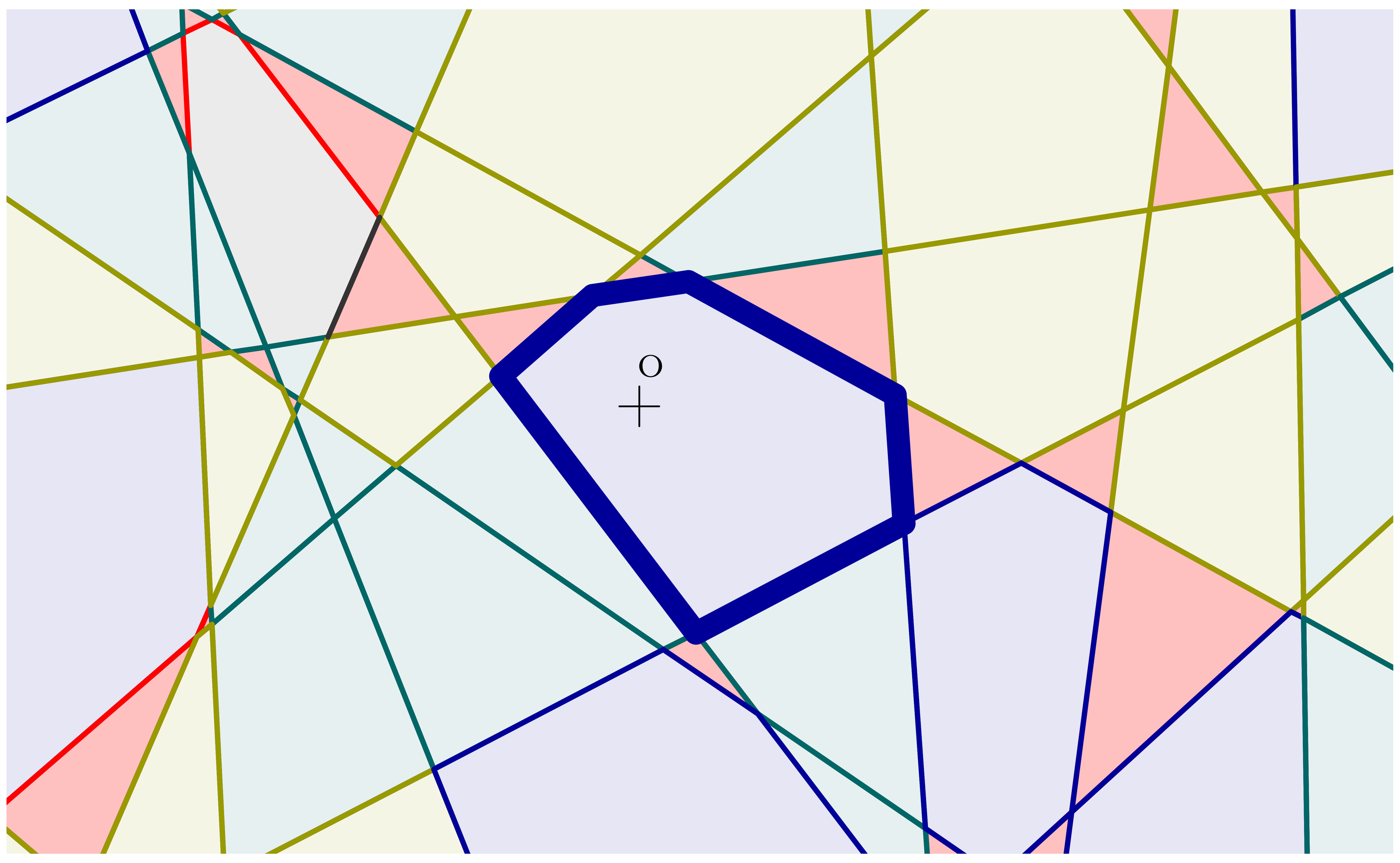


Figure: Zero cell of an isotropic mosaic

Φ-content and Shape

- Φ-content of a convex body K: $\Phi(K)$.

$$\Phi : \mathcal{C} \to (0,\infty)$$
$$K \mapsto \Phi(K) := \int_{S^{d-1}} \text{width}(u,K)\, \varphi(\mathrm{d}u)$$

Remarks: In the isotropic case Φ is the mean width. In general:

$$\mathbb{P}(K \text{ is inside a cell of } X) = e^{-\gamma\,\Phi(K)}$$

- Shape of a convex body K: $s(K)$.

$$s : \mathcal{C} \to \mathcal{C}_1 := \{K \in \mathcal{C} \mid \Phi(K) = 1\}$$
$$K \mapsto s(K) := \frac{1}{\Phi(K)} K$$

- Φ and s (restricted to $\mathcal{P}_{0,n}$) induce the following natural homeomorphism

$$h : \mathcal{P}_{0,n} \to (0,\infty) \times \mathcal{P}_{0,n,1}$$
$$P \mapsto h(P) := (\,\Phi(P)\,,\, s(P)\,)$$

Θ_n: A measure on the space of n-tope $\mathcal{P}_n$

$$\Theta_n(\cdot) := \frac{1}{n!} \int \cdots \int_{\mathcal{H}^n} \sum_{\epsilon \in \{\pm 1\}^n} \mathbb{1}\left(\bigcap_{i=1}^{n} H_i^{\epsilon_i} \in \cdot \right) \Theta(\mathrm{d}H_1) \cdots \Theta(\mathrm{d}H_n)$$

We have that

$$q_{0,n} := \mathbb{P}(Z_0 \text{ has } n \text{ facets}) = \int_{\mathcal{P}_{0,n}} e^{-\gamma\,\Phi(P)}\, \Theta_n(\mathrm{d}P)$$

and

$$\mathbb{Q}_{0,n}(\cdot) = \frac{1}{q_{0,n}} \int_{\mathcal{P}_{0,n}} \mathbb{1}(P \in \cdot)\, e^{-\gamma\,\Phi(P)}\, \Theta_n(\mathrm{d}P)$$

Complementary theorem

In the literature we find that

- $\Phi(Z_{0,n})$ and $s(Z_{0,n})$ are independent
- $\Phi(Z_{0,n})$ is gamma distributed with parameters n and γ

We get additionaly a description of the shape distribution:

- The probability distribution of $s(Z_{0,n}) \in \mathcal{P}_{0,n,1}$ is the normalisation of

$$\Theta_{n,1}(\cdot) := \Theta_n\left(h^{-1}((0,1)\,,\,\cdot)\right)$$

Notations

$\mathcal{H}$ hyperplanes in $\mathbb{R}^d$

H^- half space bounded by H containing the origin

$\mathcal{C}$ convex bodies with more then 2 points

$\mathcal{P}_0$ polytopes containing 0

$\mathcal{P}_{0,n}$ $= \{P \in \mathcal{P}_0 \mid P \text{ has } n \text{ facets}\}$

$\mathcal{C}_1$ $= \{K \in \mathcal{C} \mid \Phi(K) = 1\}$

$\mathcal{P}_{0,1}$ $= \{P \in \mathcal{P}_0 \mid \Phi(P) = 1\}$

$\mathcal{P}_{0,n,1}$ $= \{P \in \mathcal{P}_{0,n} \mid \Phi(P) = 1\}$

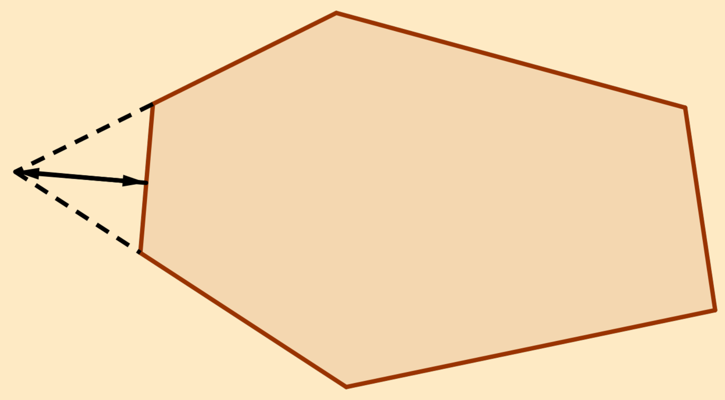
Approximation of a polytope by a polytope with less facets

There exist constants $c, n_0 > 0$ such that the following holds.
Let $P = \cap_{i=1}^{n} H_i^-$ be a simple polytope with n facets. If $n > n_0$, then there exists a subset $J \subset \{1,\dots,n\}$ of cardinality $n/4$ such that for any $j \in J$, the polytope $P_j = \cap_{i \neq j} H_i^-$ satisfies

$$d_H(P, P_j) < c\, n^{-2/(d-1)}\, \Phi(P),$$

and

$$\Phi(P_j) < \exp\left\{c\, n^{-1-2/(d-1)}\right\} \Phi(P).$$



Using the analytic expresion of q_n described on the left column, the complementary theorem and the polytope approximation result above we get the following.

Proposition : Reccurence relation

There exists a constant c such that for n big enough

$$q_{0,n+1} < c\, n^{-2/(d-1)} q_{0,n}.$$

Iterating this relation implies the next upper bound.

Theorem : Upper bound

For any directional distribution φ, there exists a constant c such that for n big enough

$$q_{0,n} < (cn)^{-2n/(d-1)}.$$

We obtain a lower bound of the q_n by construction.

Theorem : Lower bound

If φ is nice (*e.g.* isotropic), there exists a constant c such that for n big enough

$$(cn)^{-2n/(d-1)} < q_{0,n}.$$

Extensions

Our work holds as well for the typical cell with the same arguments.
We would like to extend these results to

- the zero and the typical cell of other mosaics (*e.g.* Voronoi).
- the non stationary case.
- the lower dimensional faces of the mosaic.

A big open problem:

What is the asymtotic of the shape distribution of $Z_{0,n}$ when $n \to \infty$.

References / Further reading : a really short list

Random mosaics: P. Calka, chapter *Tesselations* in *New perspectives in stochastic geometry*, 2010

Polytope approximation: S. Reisner, C. Schütt, E. Werner *Dropping a vertex of a facet from a convex polytope*, 2001

Complementary theorem: R. Cowan, *A more comprehensive complementary theorem for analysis of Poisson point processes*, 2006

Asymptotic of q_n in a special case: H. J. Hilhorst, P. Calka, *Random line tesselations of the plane: statistical properties of the many-sided cells*, 2008

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Research Training Group *Combinatorial Structures in Geometry* supported by the DFG

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