# Cells With Many Facets in a Hyperplane Mosaic

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joint work with Matthias Reitzner and Pierre Calka

Summer school Ghiffa-Oggebbio VIII.
Thursday 18th June 2015

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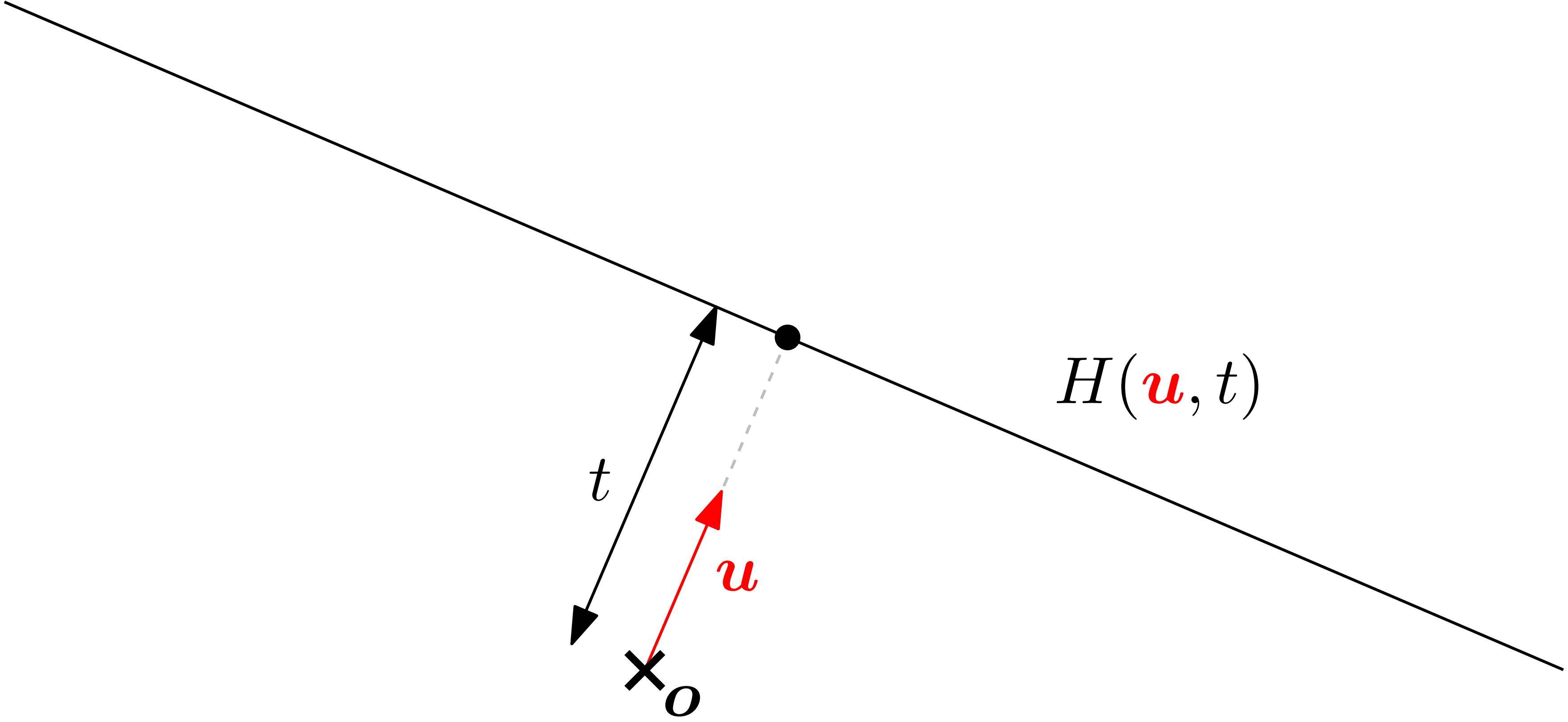
# Stationary Poisson Hyperplane Mosaic in $\mathbb{R}^d$

$\eta$ Poisson Hyperplane Process of **intensity measure** $\Theta$

$\varphi$ **directional distribution** (even measure on $\mathbb{S}^{d-1}$)

$$\Theta(\cdot) = \int_{\mathbb{S}^{d-1}} \int_0^\infty \mathbb{1}\{H(\boldsymbol{u}, t) \in \cdot\}\,\mathrm{d}t\,\varphi(\mathrm{d}\boldsymbol{u})$$



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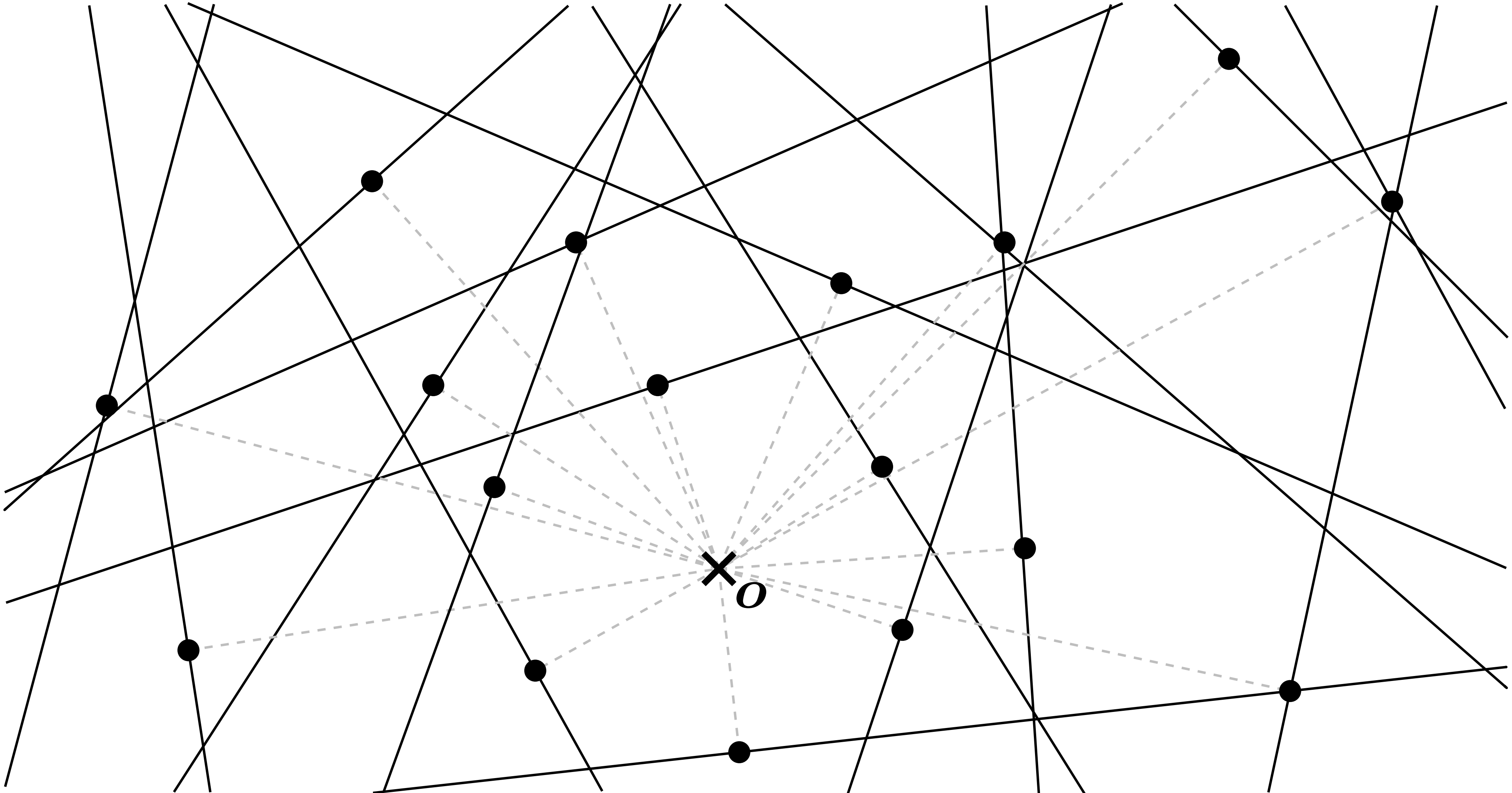

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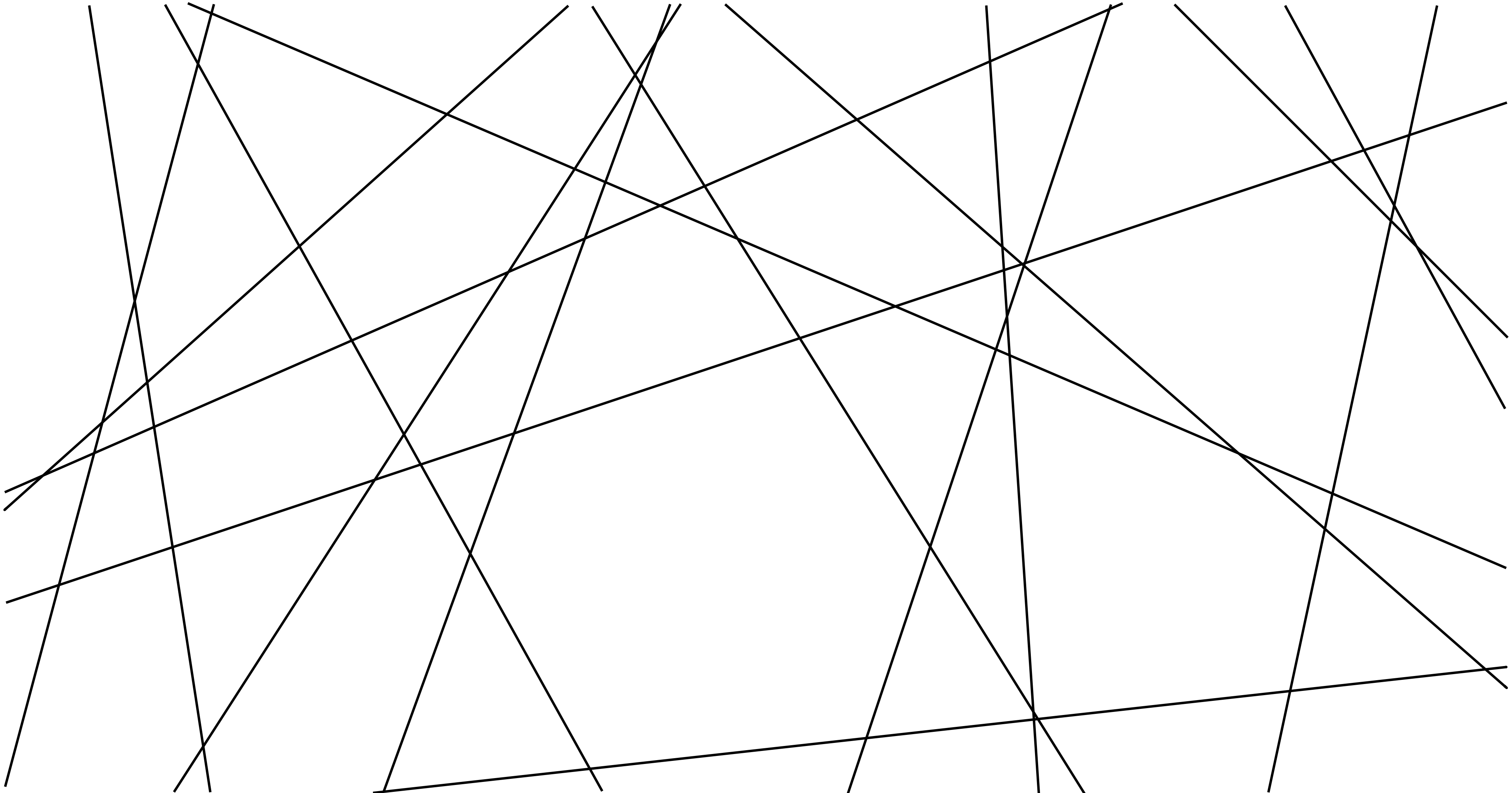

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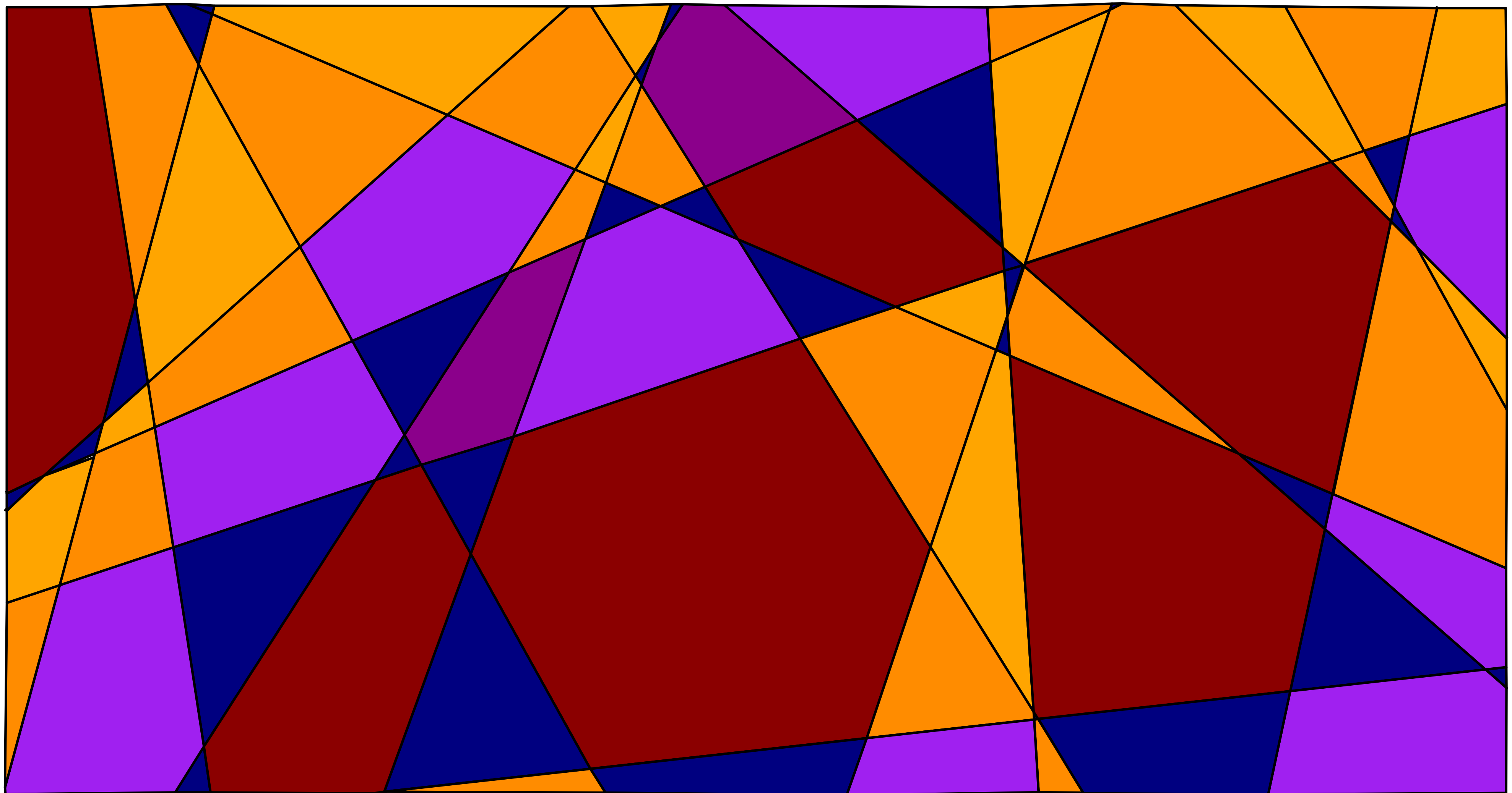

# Goal

$$\mathbb{P}(Z \text{ has } n \text{ facets})?$$

when $n \to \infty$

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In a specific case it is already known:

**Theorem [Calka and Hilhorst 2008]**

In the $2$-dimensional isotropic case we have that

$$\mathbb{P}(Z \text{ has } n \text{ facets}) \sim \alpha\beta^n n^{-2n}\sqrt{n} \text{ when } n \to \infty$$

with $\alpha = (6\pi^{5/2})^{-1}$ and $\beta = \pi^2 e^2$

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We generalize this to any dimension and nice directional distribution:

**Main Theorem**

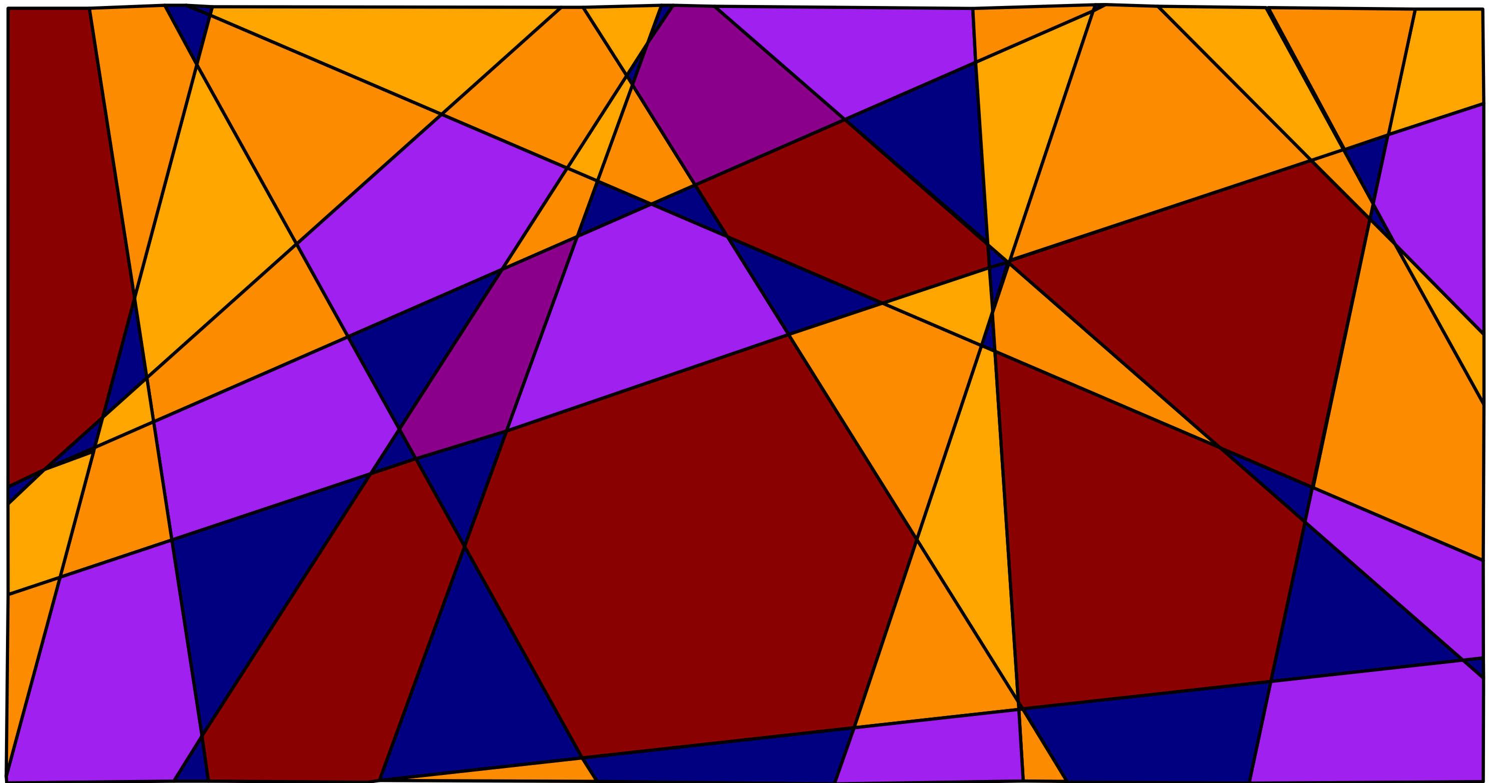
There exist constants $c_1$ and $c_2$ depending on $d$ and $\varphi$ such that for $n$ big enough we have

$$c_1^n n^{-2n/(d-1)} < \mathbb{P}(Z \text{ has } n \text{ facets}) < c_2^n n^{-2n/(d-1)}$$

# Typical Cell $Z$

$X = X_\eta$ . . . **Mosaic**: Point Process in $\mathcal{P}$

Set of polytopes



# Typical Cell $Z$

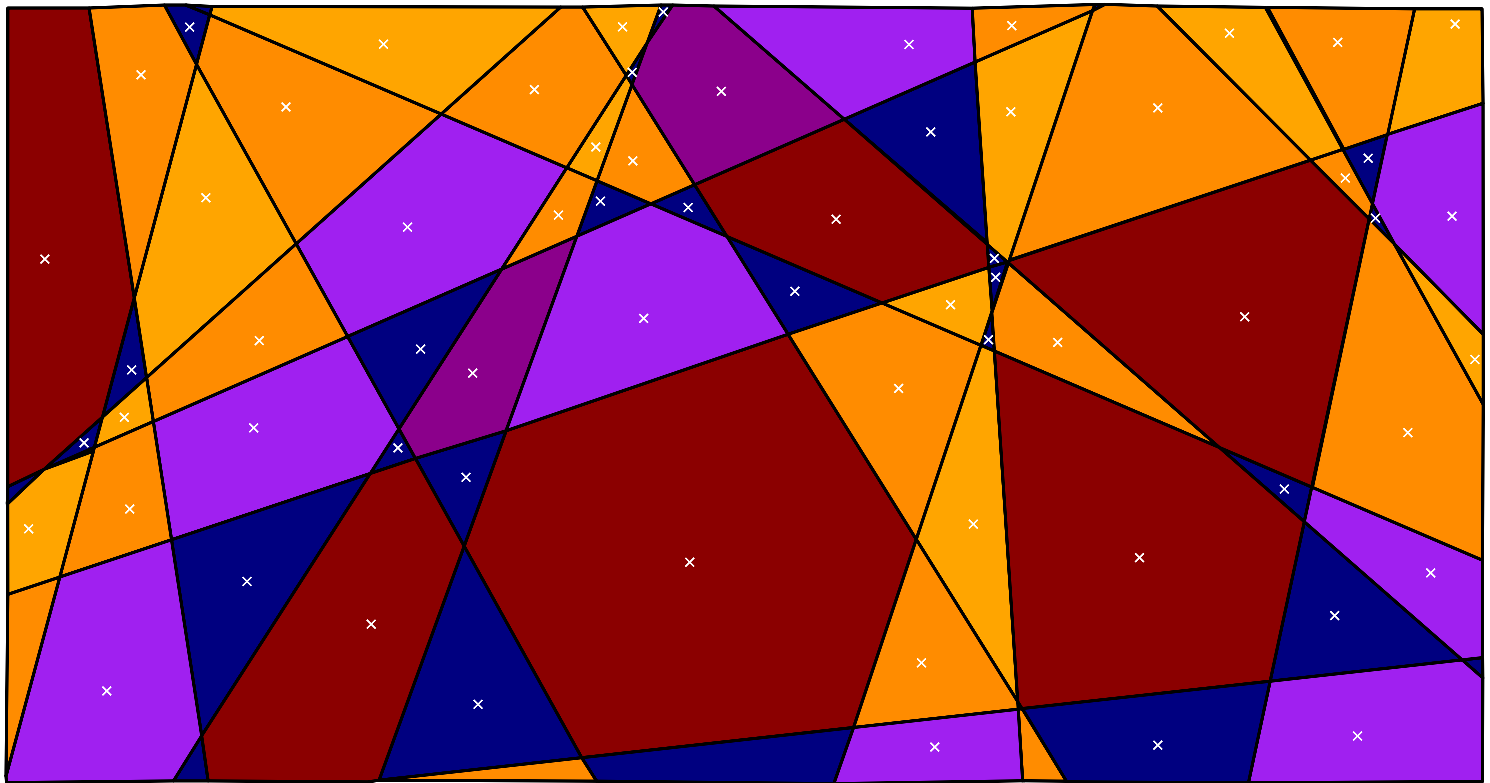
$X = X_\eta$ … **Mosaic**: Point Process in $\mathcal{P}$

Set of polytopes

$\mathfrak{c} : \mathcal{P} \to \mathbb{R}^d$ a **center function**

*e.g.* center of mass, center of the circumball…

$$\mathfrak{c}(tP + x) = t\mathfrak{c}(P) + x$$



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$X = X_\eta$ ... **Mosaic**: Point Process in $\mathbb{R}^d \times \mathcal{P}_{\mathfrak{c}}$

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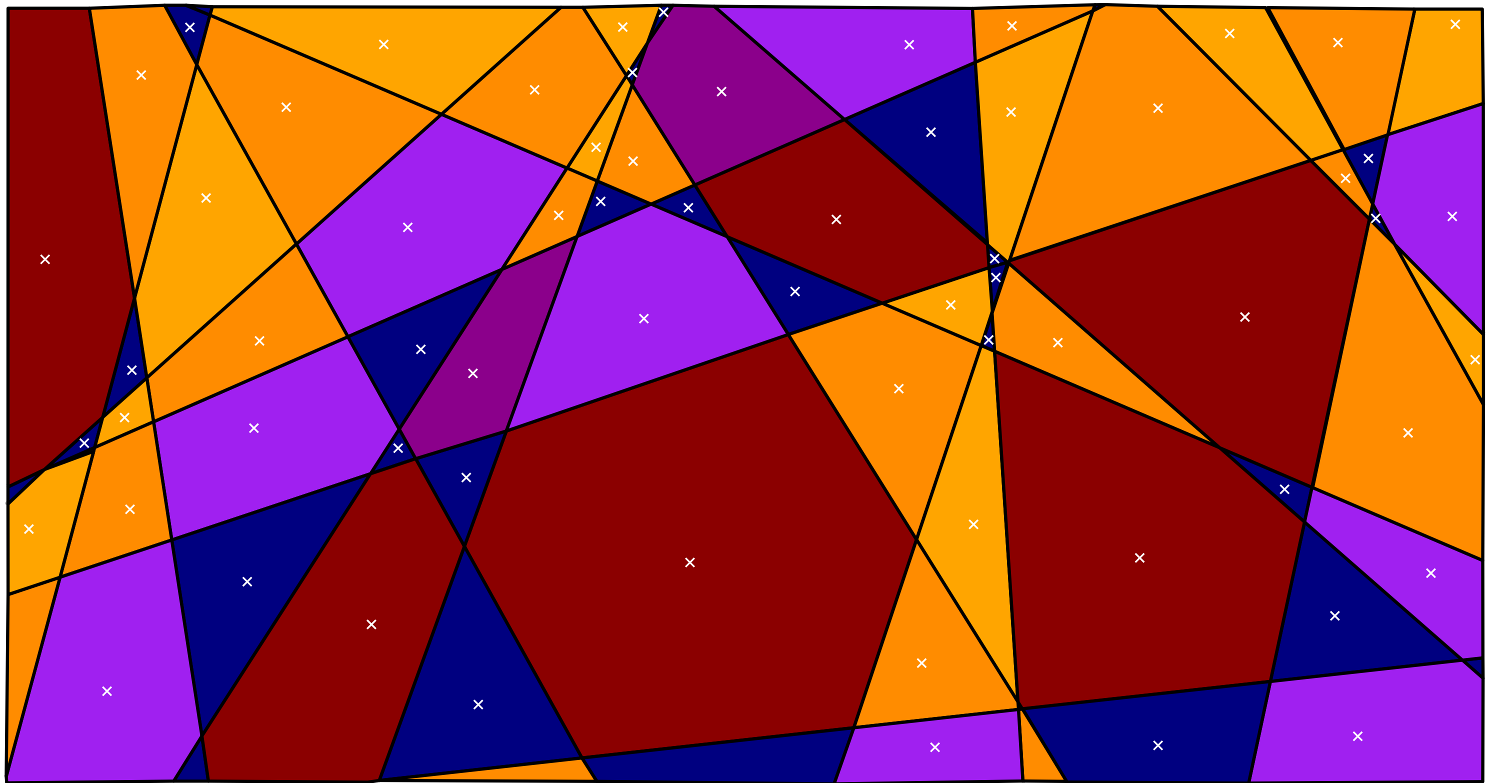
$X$ is a germ-grain process with **germs** $\mathfrak{c}(P)$ and **grains** $P - \mathfrak{c}(P)$



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grain distribution

intensity

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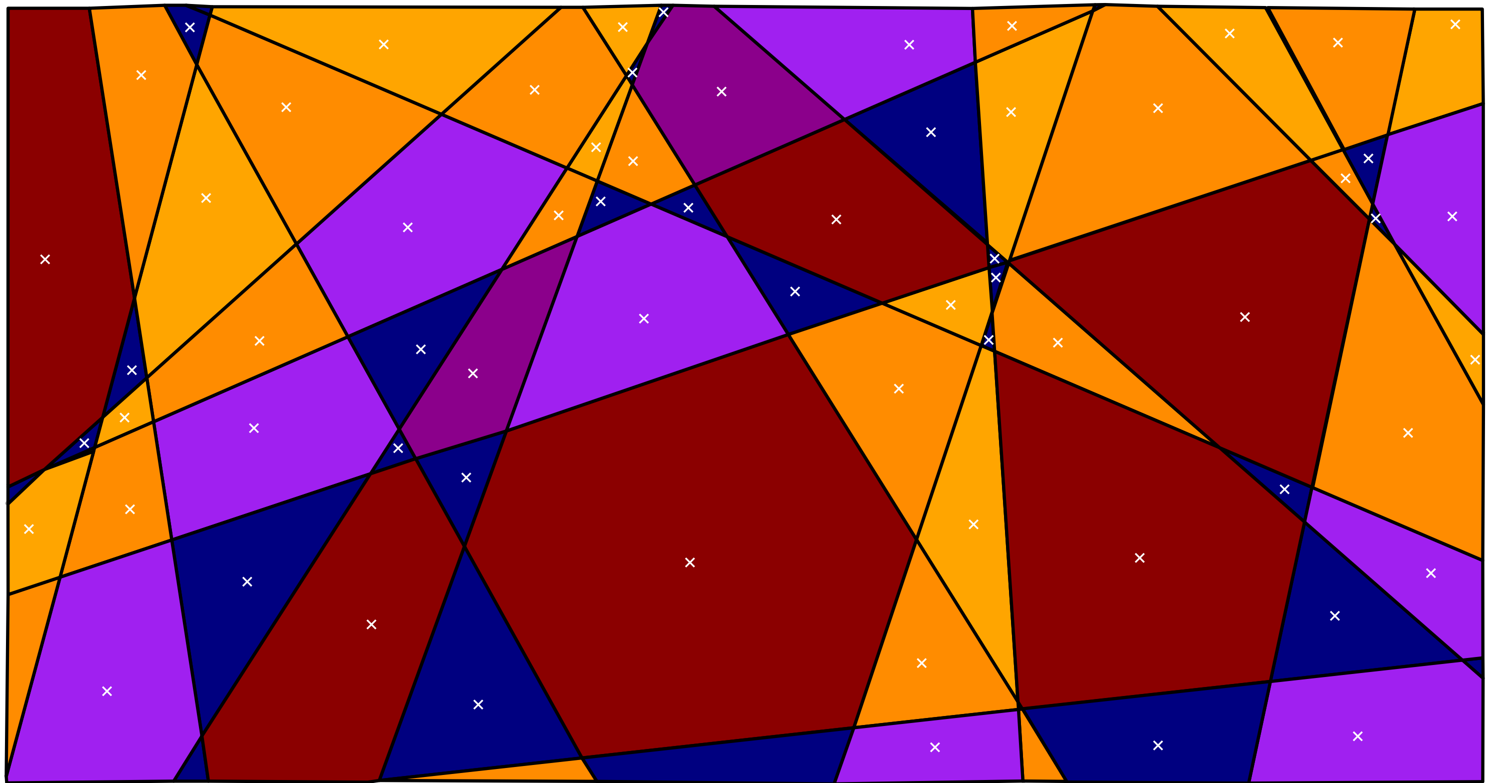

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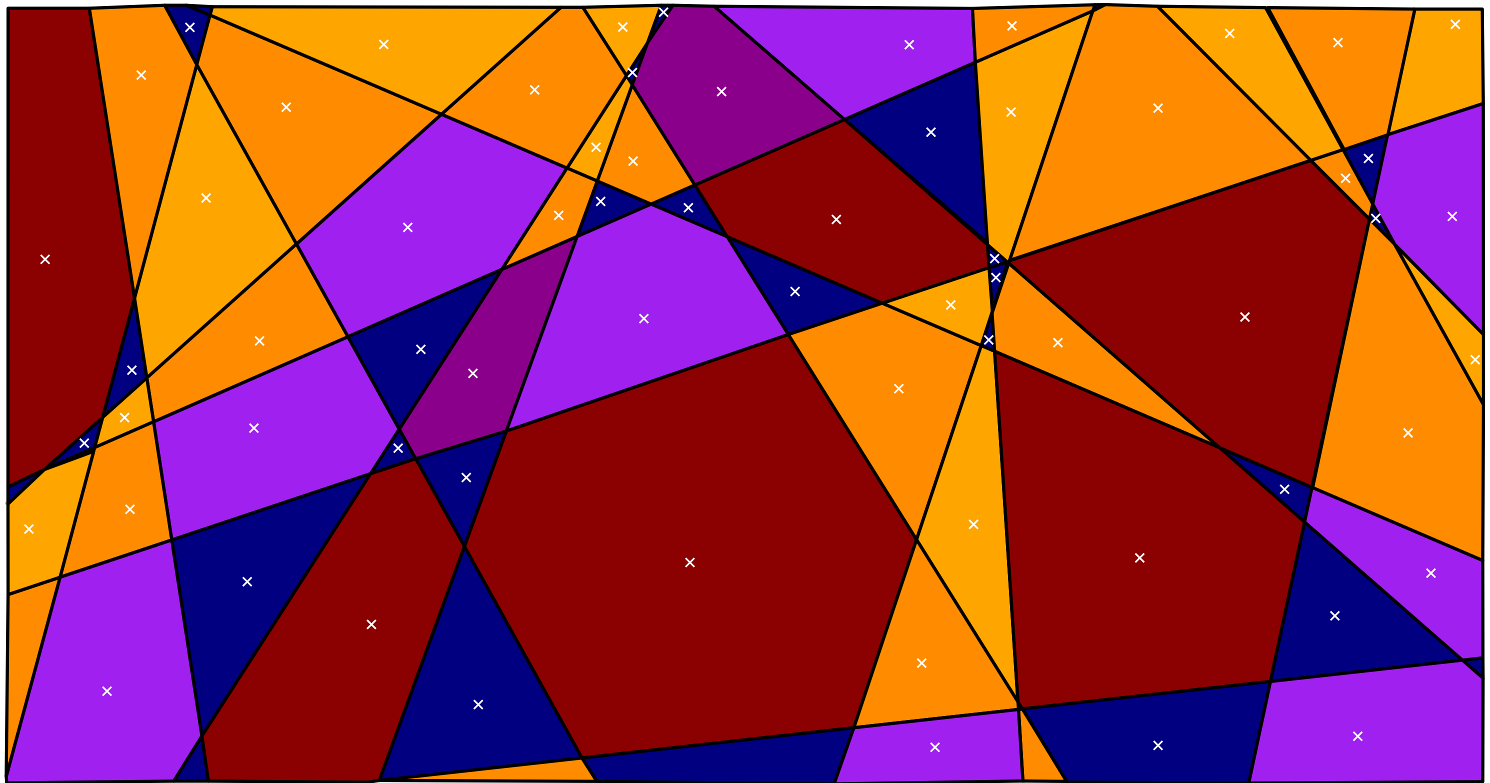
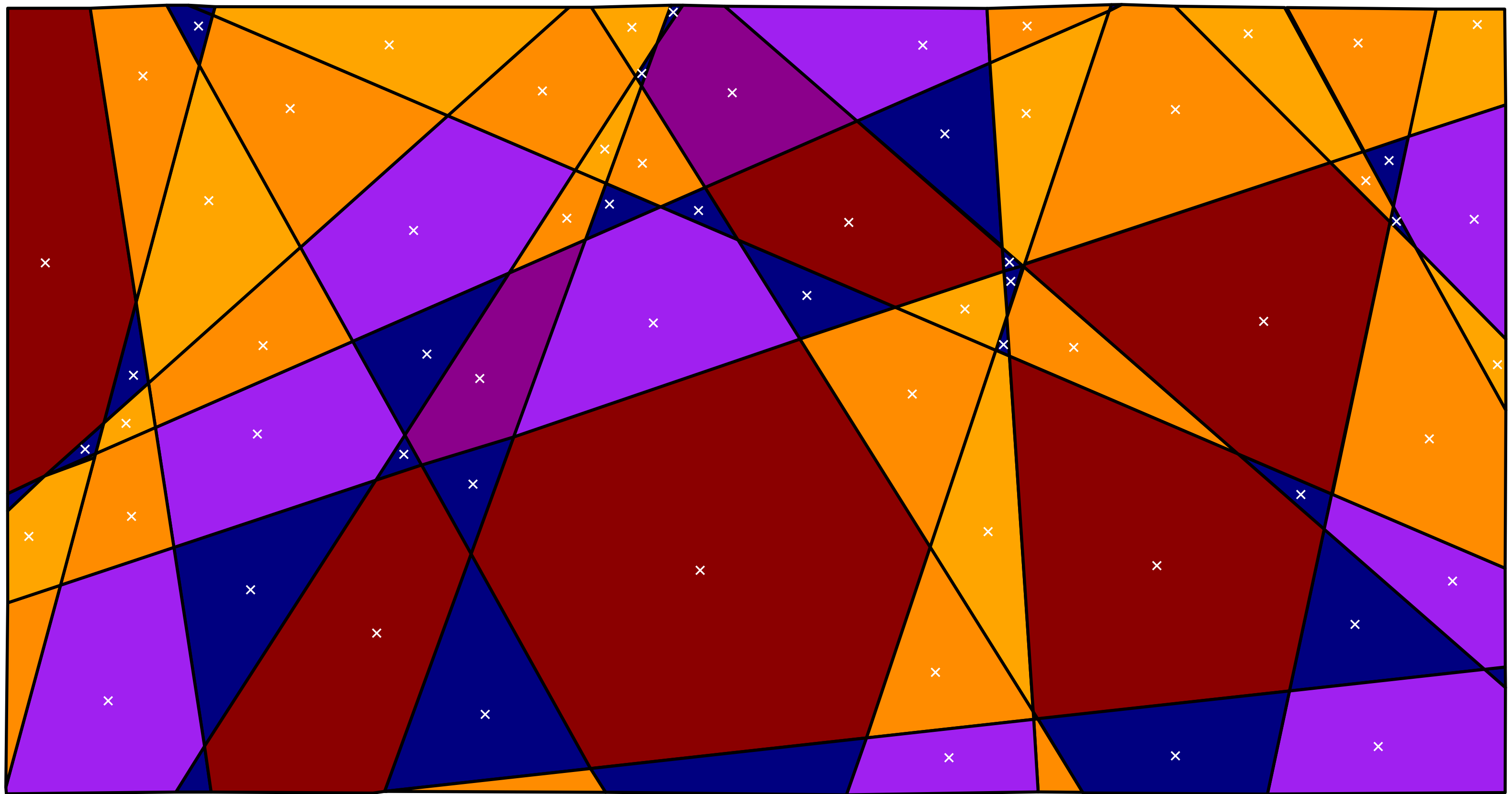
$Z$ . . . **Typical cell** $=$ random centred polytope with distribution $\mathbb{Q}$

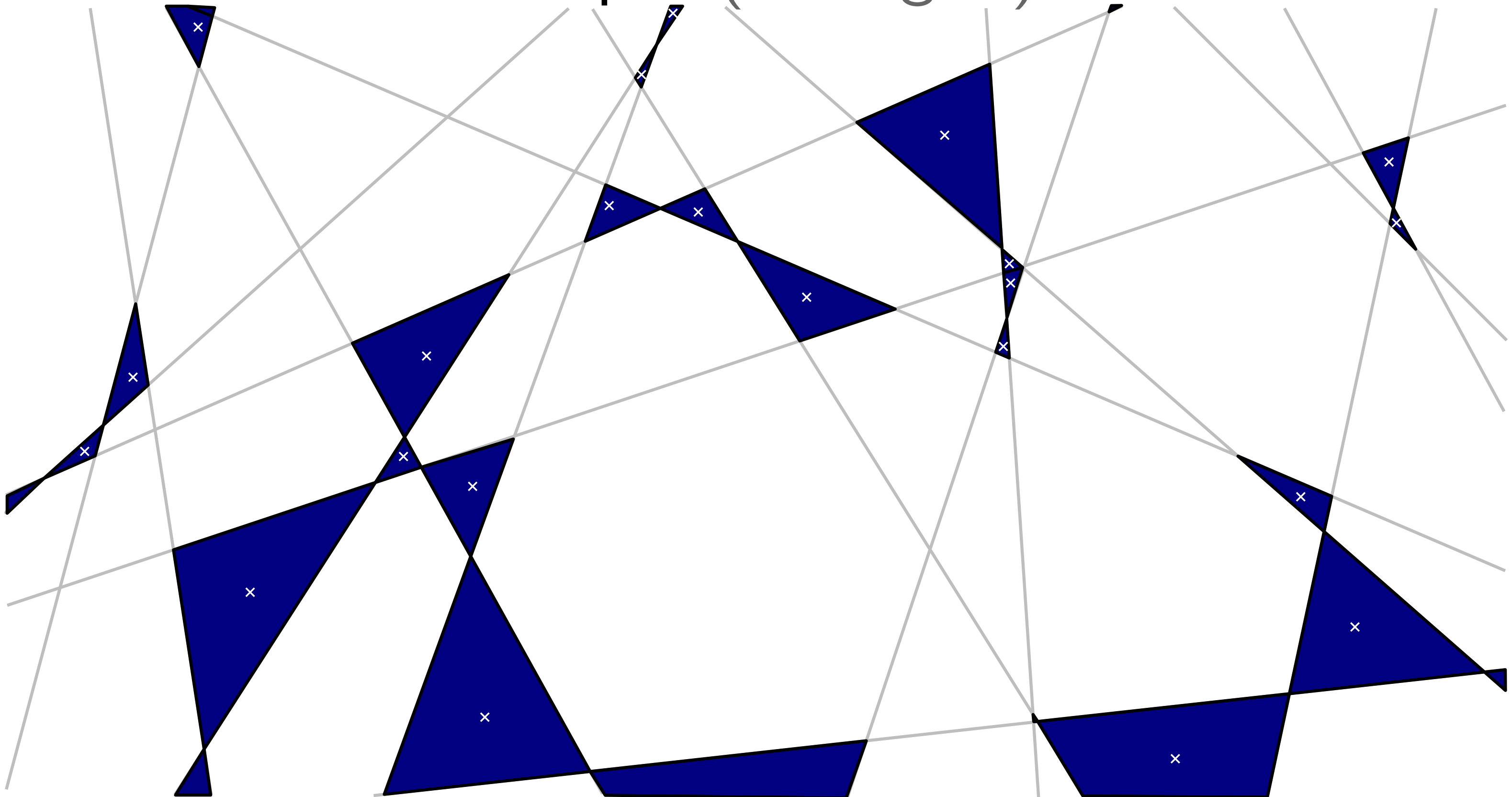


# Cells With $n$ Facets



# Cells With $n$ Facets

$3$-topes (Triangles)



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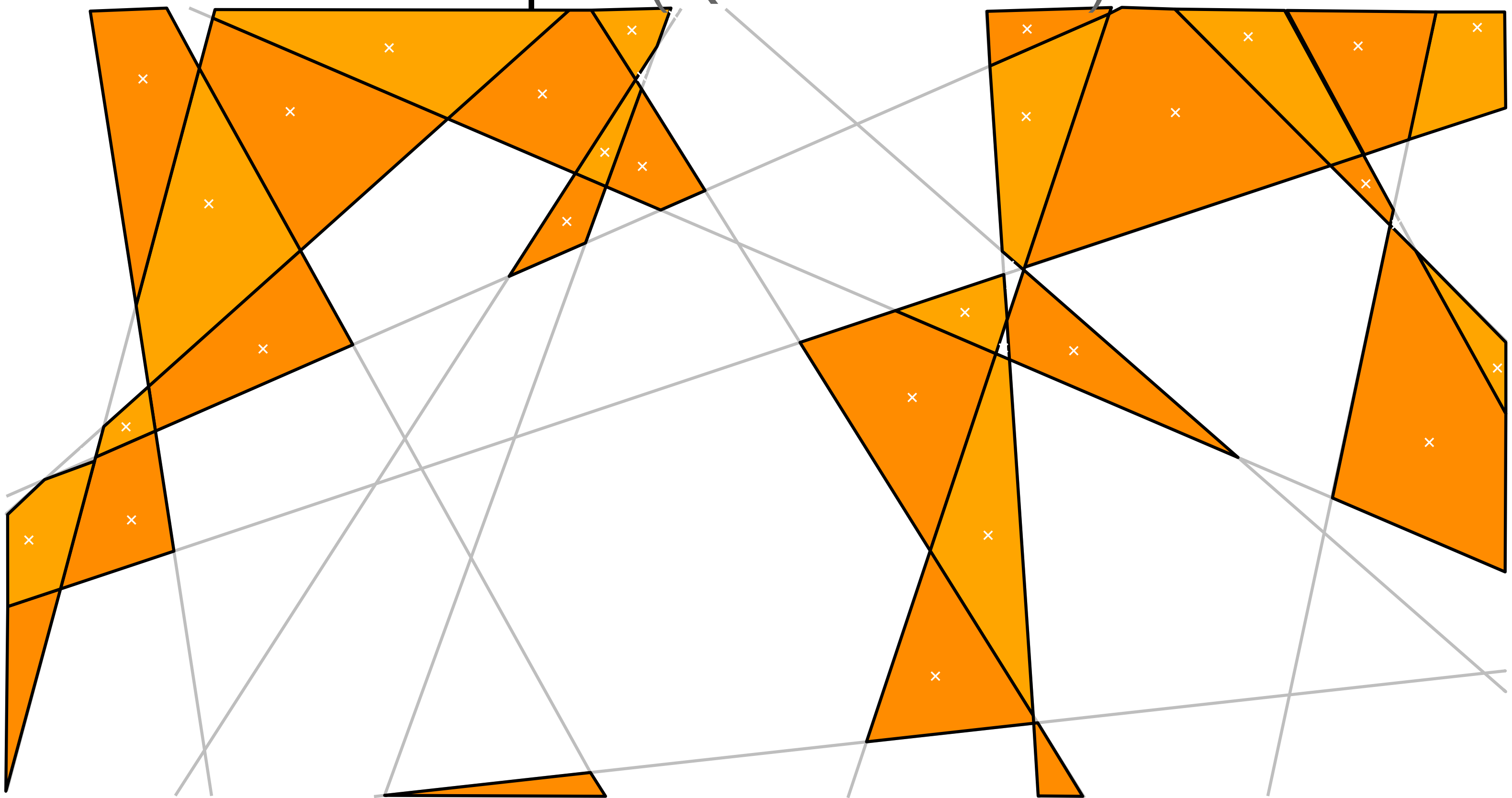
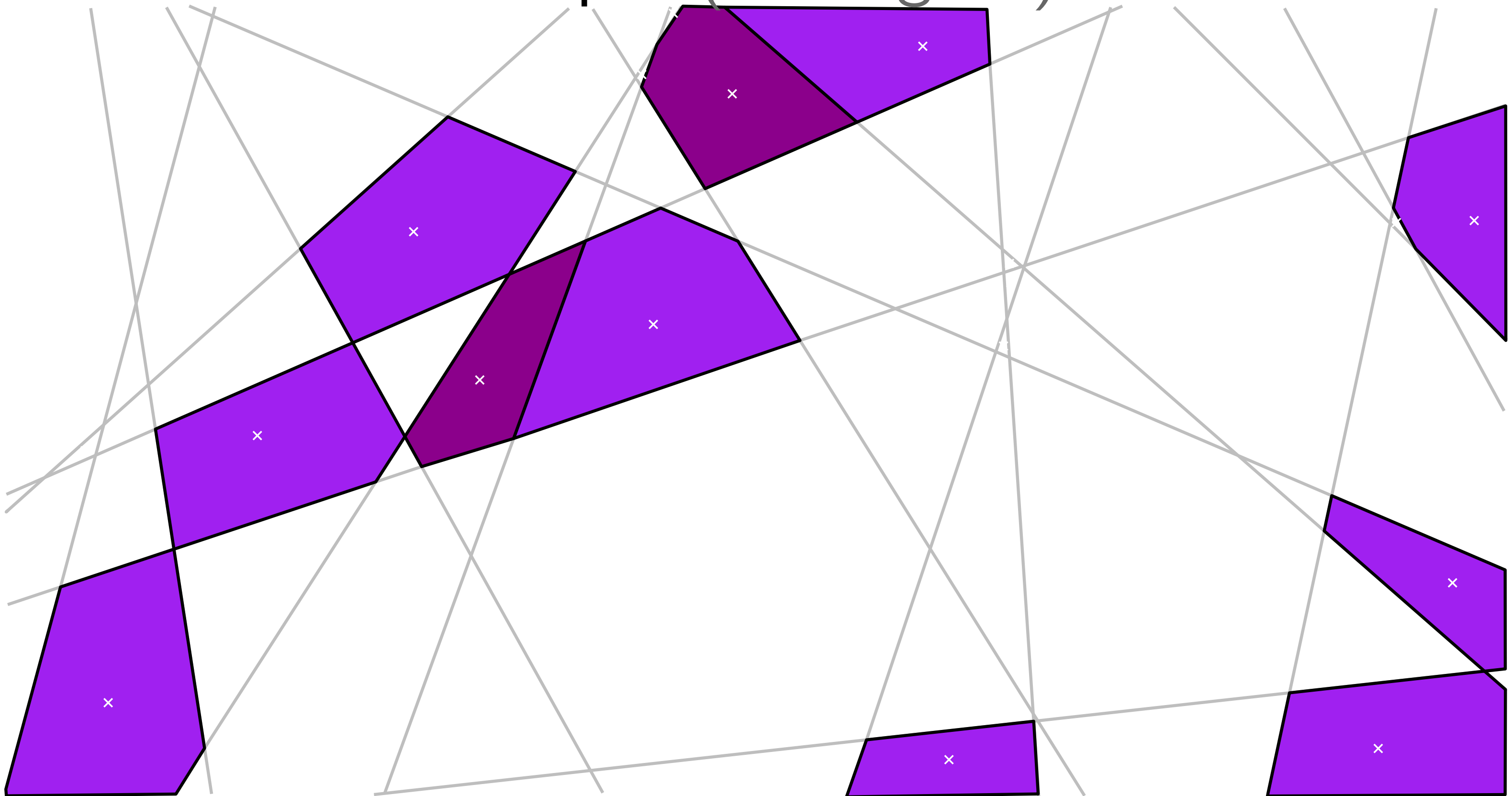
$4$-topes (Quadrilaterals)



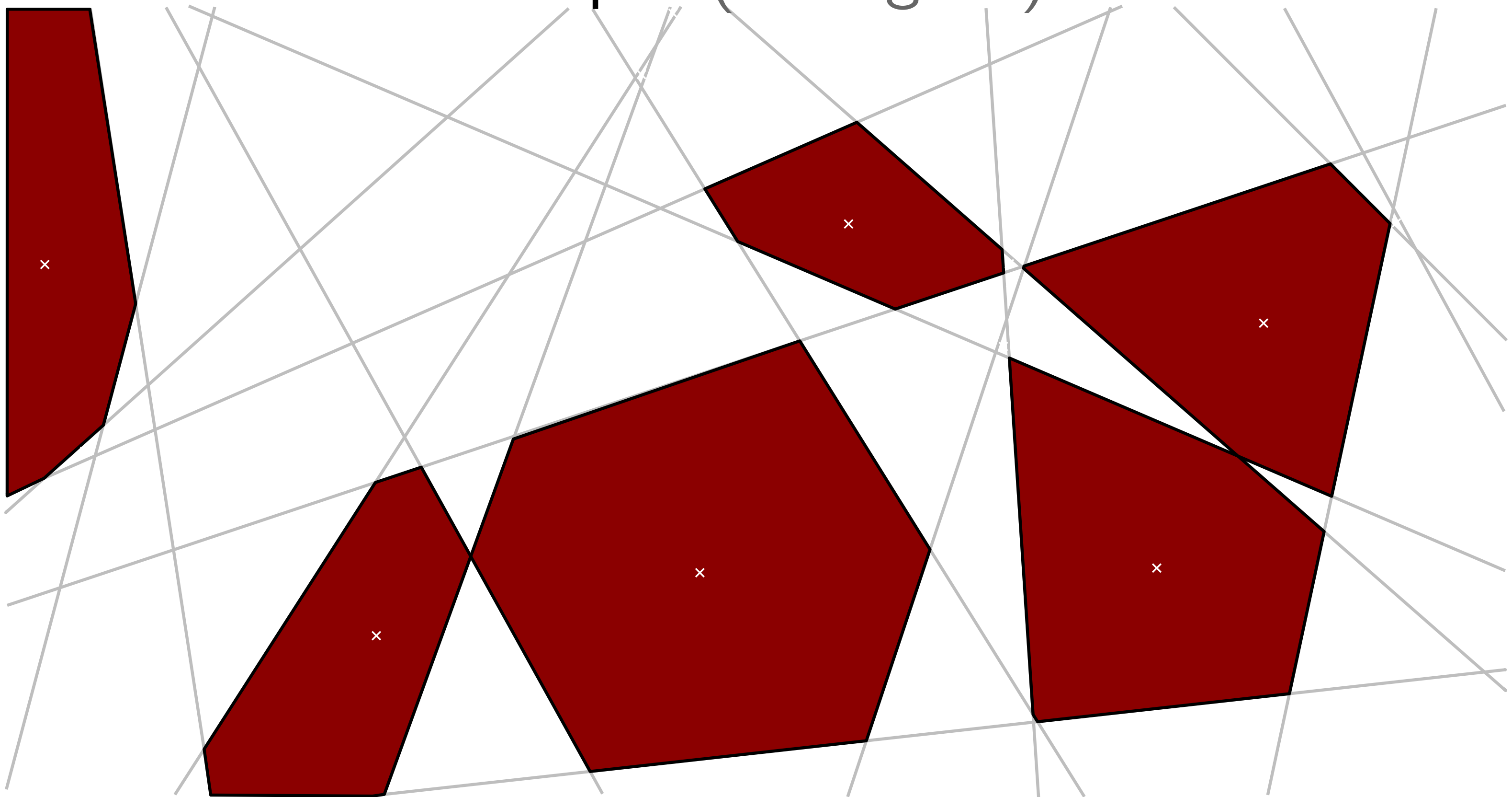
# Cells With $n$ Facets

$5$-topes (Pentagons)



# Cells With $n$ Facets

$6$-topes (Hexagons)



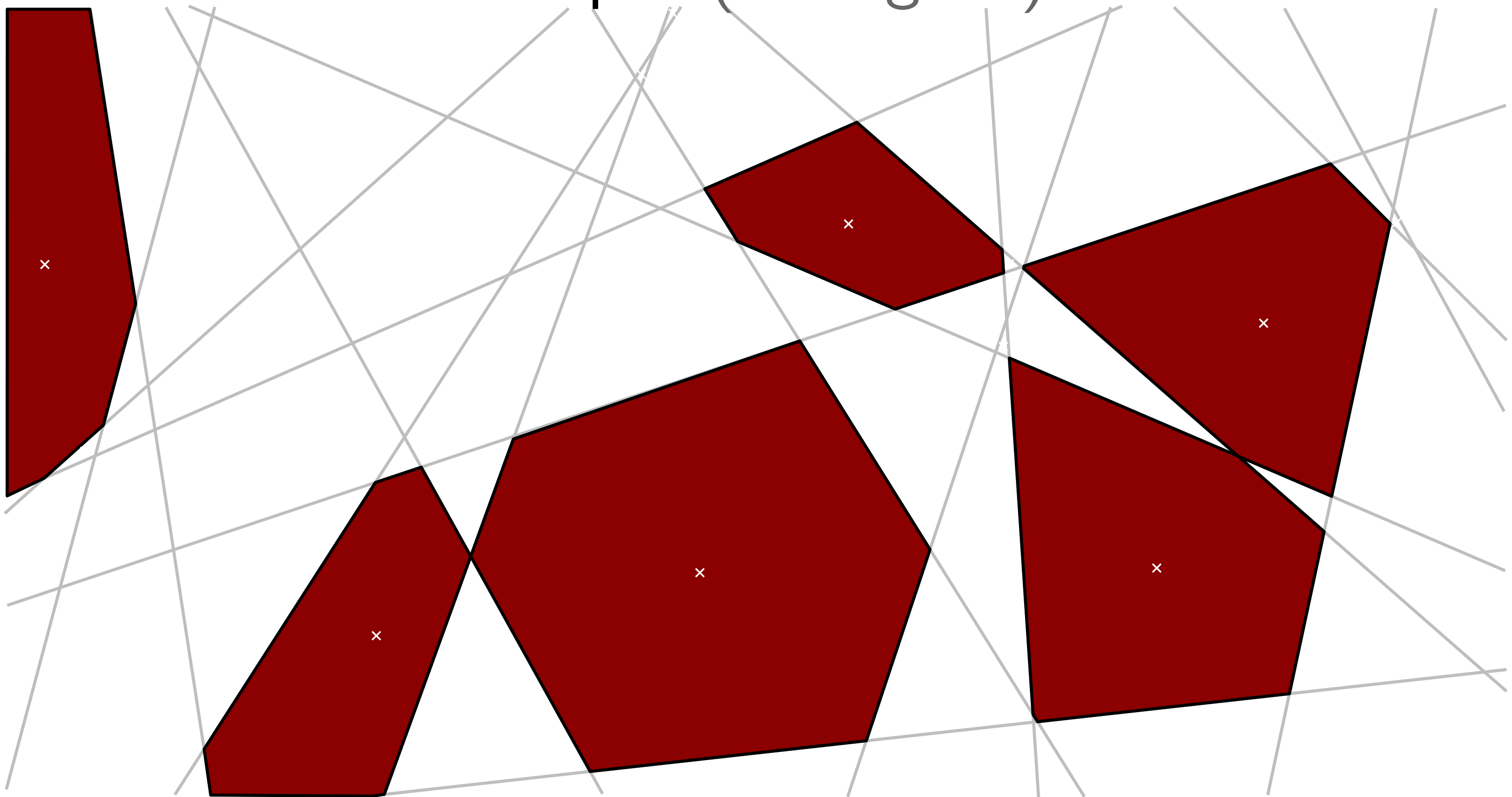
# Cells With $n$ Facets

Set of centred $n$-topes

$$\begin{aligned}\mathbb{P}(Z \text{ has } n \text{ facets}) &= \mathbb{Q}(\mathcal{P}_{n,\mathfrak{c}}) \\ &= \gamma^{-1}\gamma\lambda_d([0,1]^d)\mathbb{Q}(\mathcal{P}_{n,\mathfrak{c}}) \\ &= \gamma^{-1}\mathbb{E}\,\underbrace{X(\mathcal{P}_{n,[0,1]^d})}\end{aligned}$$

number of $n$-topes of $X$ with center in $[0,1]^d$

6-topes (Hexagons)



# Cells With $n$ Facets

Let $D \subset \mathcal{P}_n$. number of cells of $X$ in $D$ $P = \bigcap_{i=1}^{n} H_i^{\epsilon_i}$

$$X(D) = \frac{1}{n!} \sum_{P \in \eta_{\neq}^{n} \times \{\pm 1\}^{n}} \mathbf{1}(P \in D)\, \mathbf{1}(\eta \cap P = \emptyset).$$

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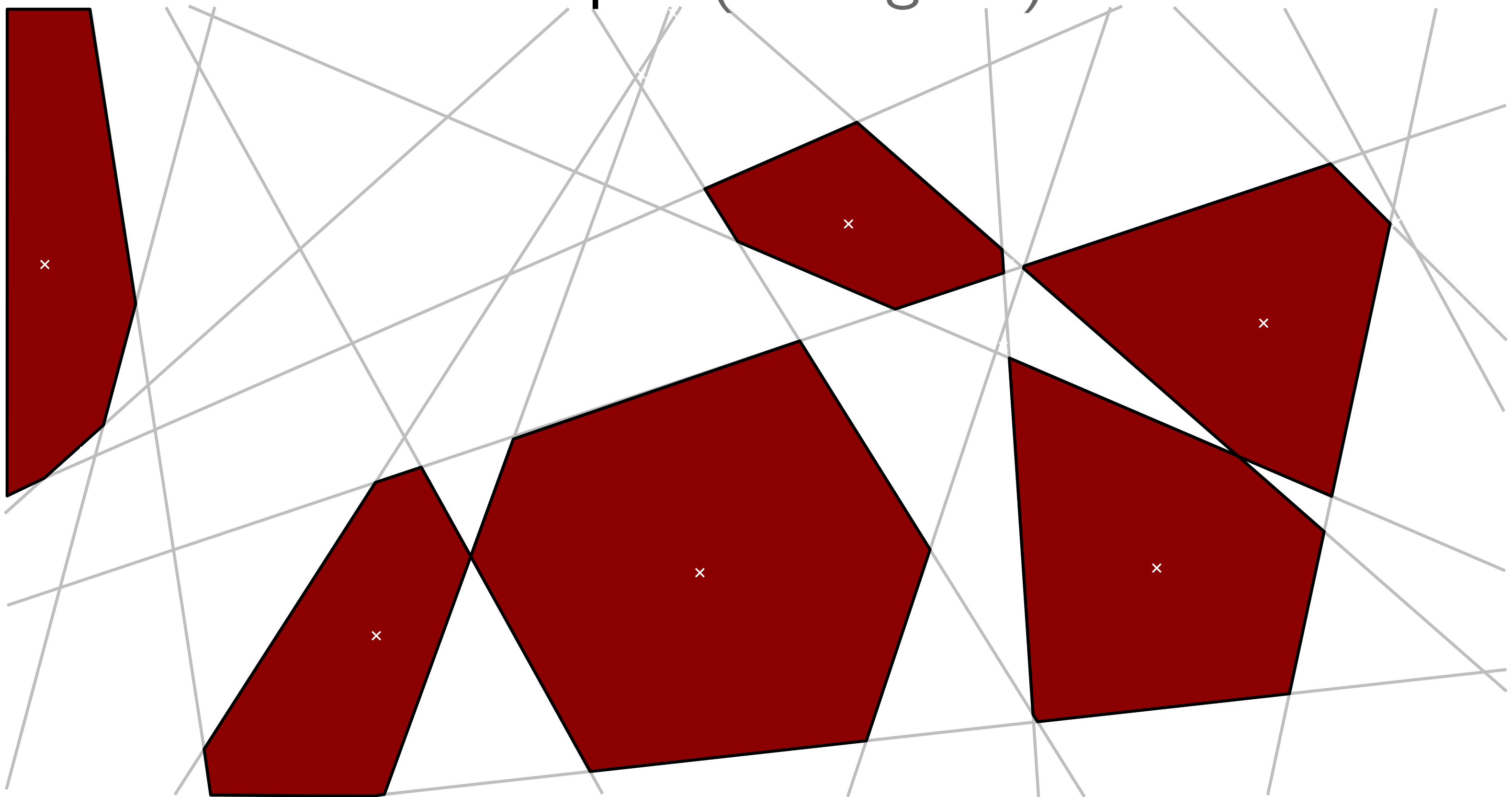

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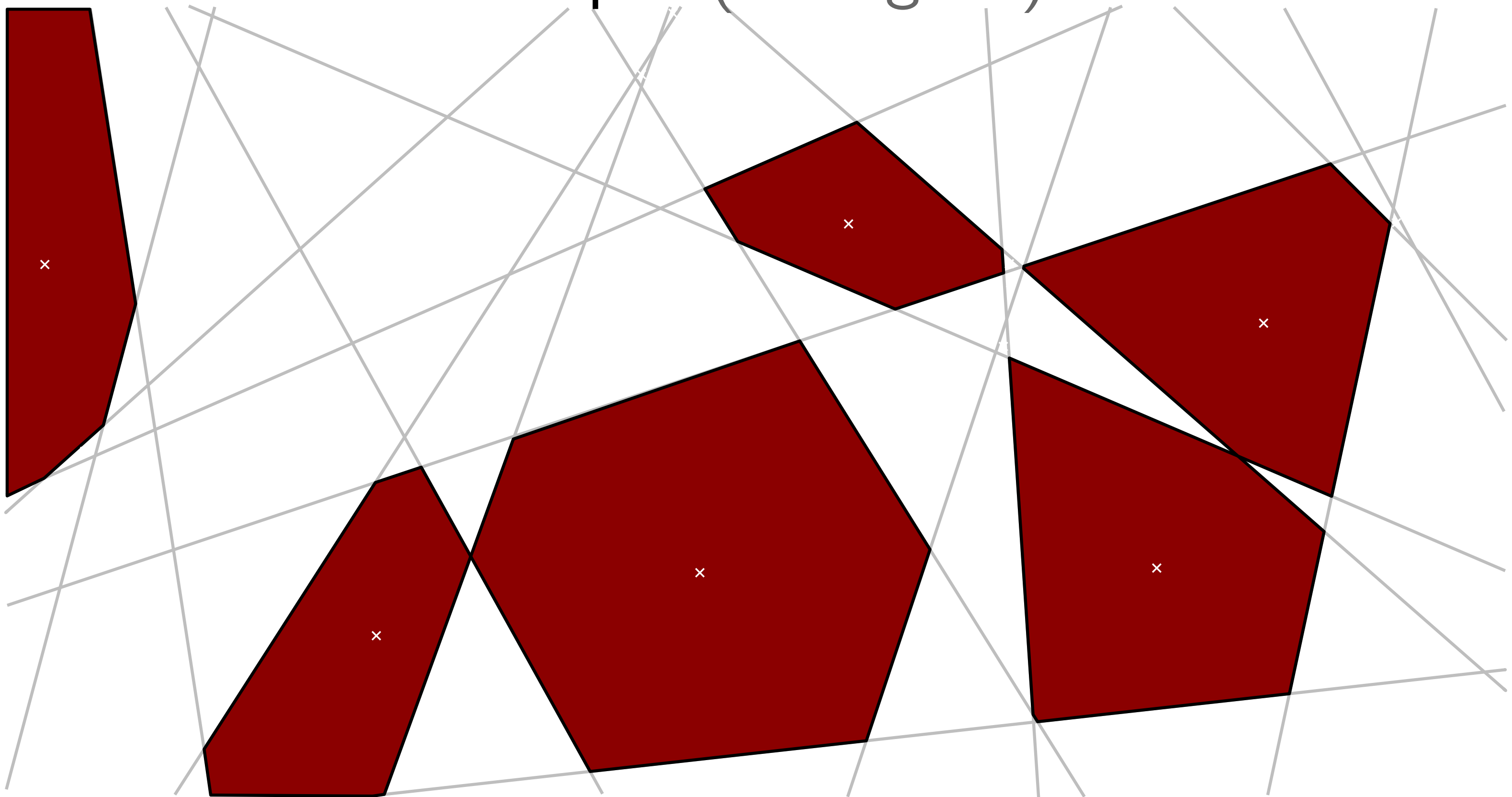

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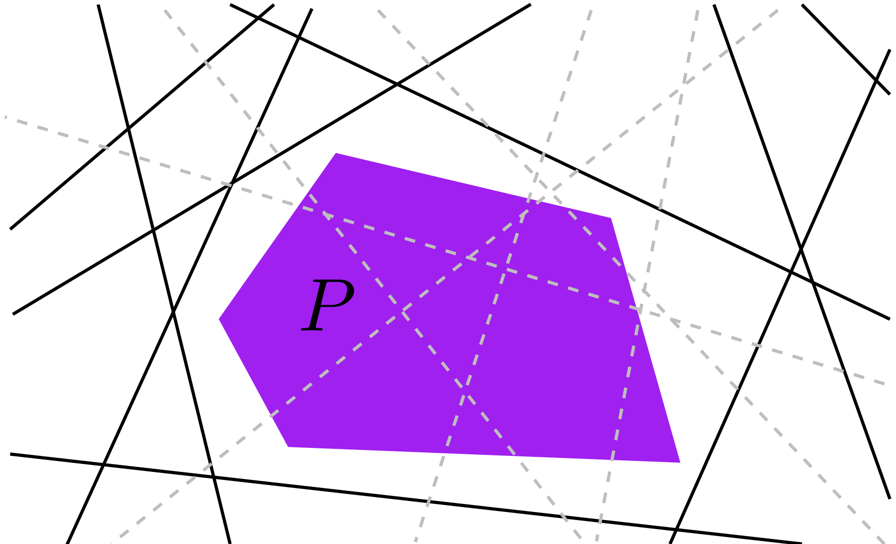

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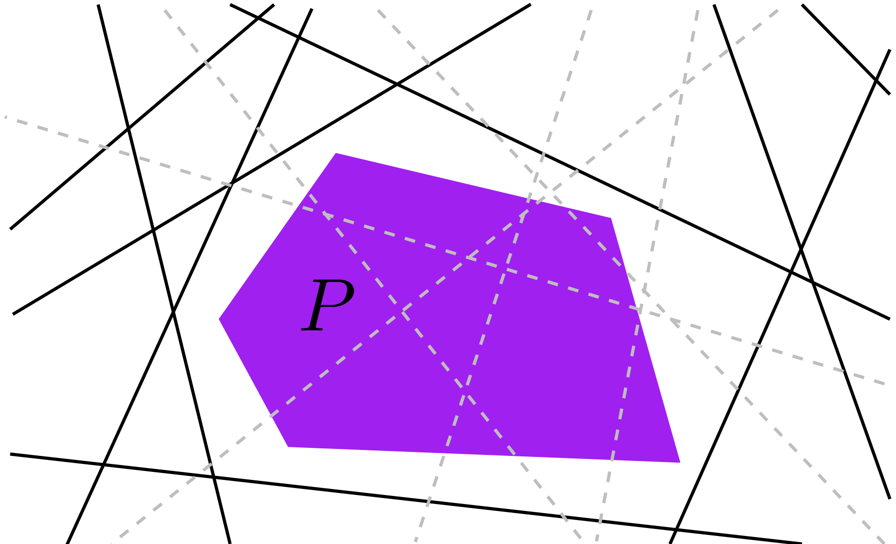

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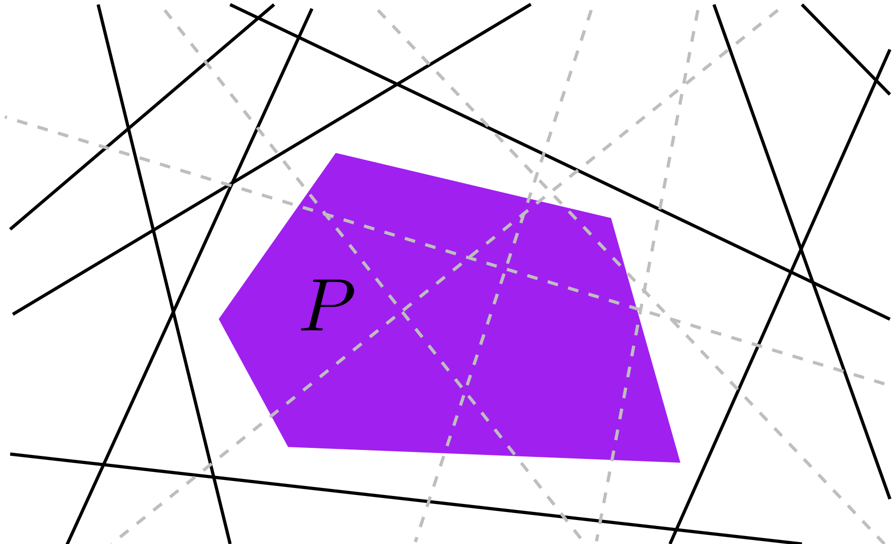

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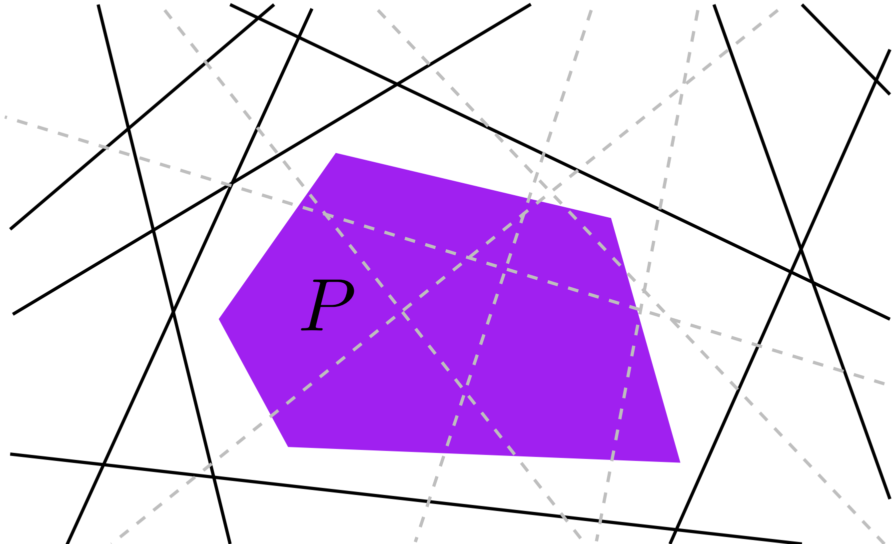

$\Phi : \mathcal{P} \to (0, \infty)$

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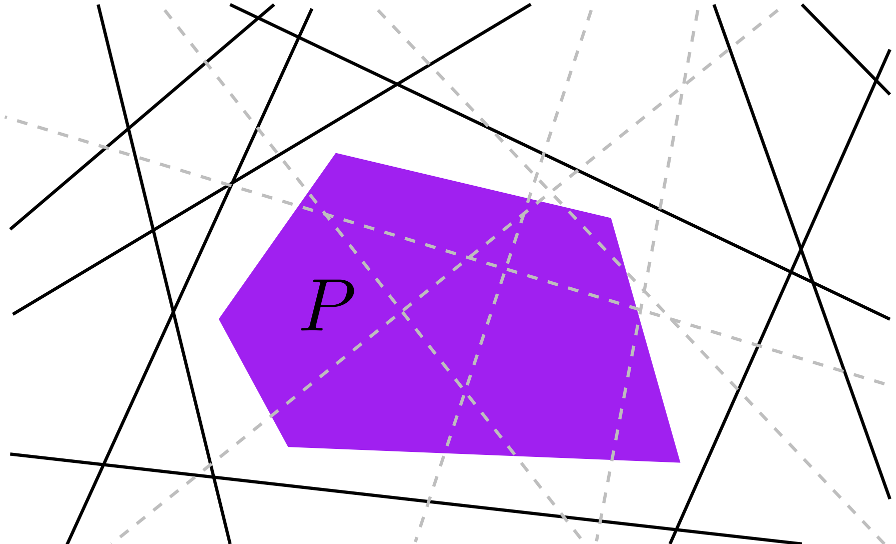

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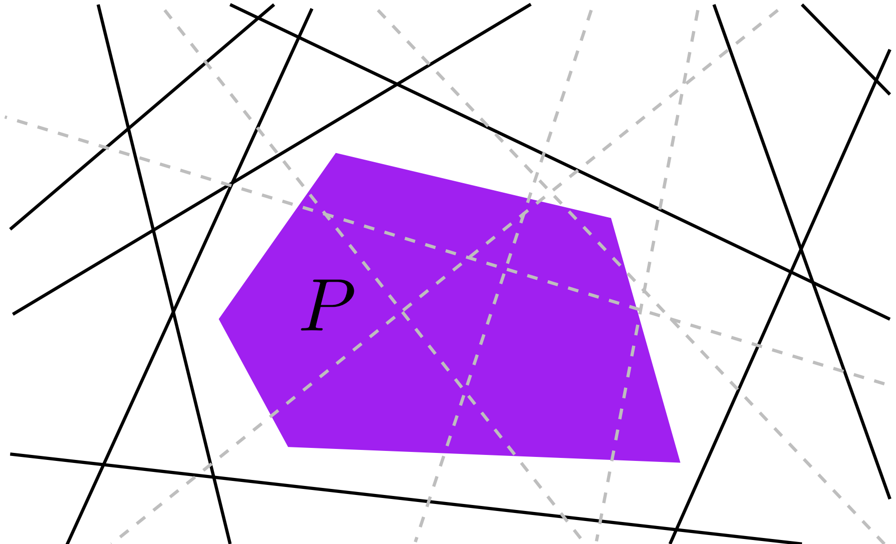

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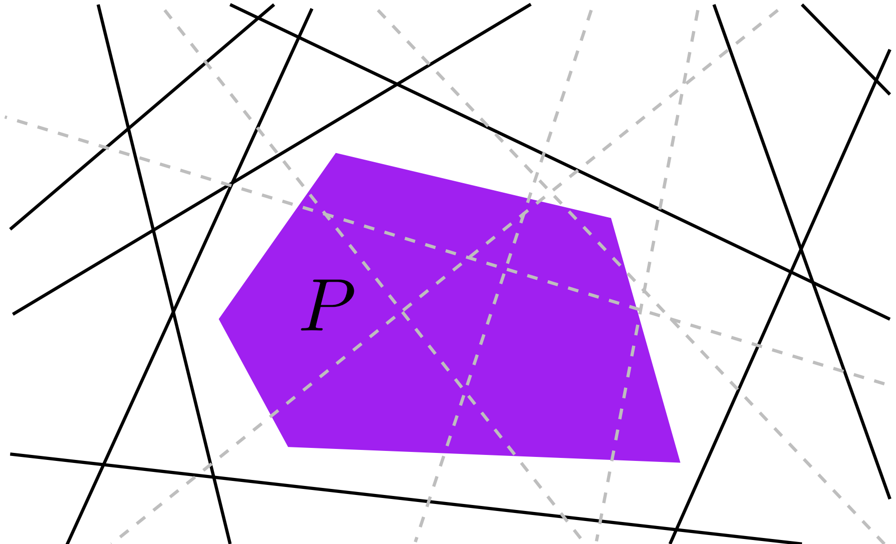
where $\Theta_n(\cdot) := \frac{1}{n!} \int \cdots \int_{\mathcal{H}^n} \sum_{\epsilon \in \{\pm 1\}^n} \mathbf{1}\left(P \in \cdot\right) \Theta(\mathrm{d}H_n) \cdots \Theta(\mathrm{d}H_1)$



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# Decomposition of the Measure $\Theta_n$

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Set of centred $n$-topes with $\Phi$-content 1

$$\begin{aligned} \mathfrak{h}_n : \mathcal{P}_n &\overset{\sim}{\to} \mathbb{R}^d \times (0,\infty) \times \mathcal{P}^1_{n,\mathfrak{c}} \\ P &\mapsto \left( \mathfrak{c}(P), \Phi(P), \underbrace{\Phi(P)^{-1}(P - \mathfrak{c}(P))}_{\mathfrak{s}(P)} \right), \end{aligned}$$

*Shape of* $P$

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*Shape of* $P$

pushforward

Lebesgue measure

$$\boxed{\mathfrak{h}_n\left(\Theta_n\right) = \lambda_d \otimes \lambda_{1,n-d} \otimes \Theta^1_{n,\mathfrak{c}}}$$

$\lambda_{1,n-d}([0,a])=a^{n-d}$

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$$\mathbb{E}X(D) = \int_{\mathcal{P}_n} \mathbf{1}\left(P \in D\right) \exp\left(-\Phi\left(P\right)\right) \Theta_n(\mathrm{d}P)$$

# Decomposition of the Measure $\Theta_n$

Set of centred $n$-topes with $\Phi$-content 1

$$\mathfrak{h}_n : \mathcal{P}_n \overset{\sim}{\to} \mathbb{R}^d \times (0,\infty) \times \mathcal{P}^1_{n,\mathfrak{c}}$$

$$P \mapsto \left(\mathfrak{c}(P), \Phi(P), \underbrace{\Phi(P)^{-1}(P-\mathfrak{c}(P))}_{\mathfrak{s}(P)}\right),$$

*Shape of* $P$

pushforward

Lebesgue measure

$$\boxed{\mathfrak{h}_n(\Theta_n) = \lambda_d \otimes \lambda_{1,n-d} \otimes \Theta^1_{n,\mathfrak{c}}}$$

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$$\mathbb{E}X(D) = \int_{\mathcal{P}^1_{n,\mathfrak{c}}}\int_{(0,\infty)}\int_{\mathbb{R}^d} \mathbf{1}\left(\mathfrak{h}_n^{-1}(c,t,s) \in D\right) e^{-t} t^{n-d-1} \mathrm{d}c\, \mathrm{d}t\, \Theta^1_{n,\mathfrak{c}}(\mathrm{d}P)$$

# Decomposition of the Measure $\Theta_n$

Set of centred $n$-topes with $\Phi$-content 1

$$\begin{aligned} \mathfrak{h}_n : \mathcal{P}_n &\overset{\sim}{\to} \mathbb{R}^d \times (0,\infty) \times \mathcal{P}^1_{n,\mathfrak{c}} \\ P &\mapsto \left(\mathfrak{c}(P), \Phi(P), \underbrace{\Phi(P)^{-1}(P-\mathfrak{c}(P))}_{\mathfrak{s}(P)}\right), \end{aligned}$$

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**Complementary Theorem (Miles 1971)**

If we condition the typical cell $Z$ to have $n$ facets, then

- $\Phi(Z)$ and $\mathfrak{s}(Z)$ are independent
- $\Phi(Z)$ is Gamma distributed with parameter $n-d$

# $\mathbb{P}(Z$ has $n$ facets$)$

$$\gamma\mathbb{P}(Z \text{ has } n \text{ facets}) = \mathbb{E}X(\mathcal{P}_{n,[0,1]^d})$$

$$= \int_{\mathcal{P}^1_{n,\mathfrak{c}}} \int_{(0,\infty)} \int_{[0,1]^d} e^{-t} t^{n-d-1} \mathrm{d}c \, \mathrm{d}t \, \Theta^1_{n,\mathfrak{c}}(\mathrm{d}P)$$

$$= (n-d-1)! \Theta^1_{n,\mathfrak{c}}(\mathcal{P}^1_{n,\mathfrak{c}})$$

$$= \frac{(n-d-1)!}{n!} \int \cdots \int_{\mathcal{H}^n} \sum_{\boldsymbol{\epsilon}\in\{\pm 1\}^n} \mathbb{1}\left(\mathfrak{c}(P) \in [0,1]^d\right) \mathbb{1}\left(\Phi(P) < 1\right) \mathbb{1}\left(P \in \mathcal{P}_n\right) \, \Theta(\mathrm{d}H_n) \cdots \Theta(\mathrm{d}H_1)$$

# Lower Bound

$$\gamma\frac{n!}{(n-d-1)!}\mathbb{P}(Z \text{ has } n \text{ facets})$$

$$=\int\cdots\int_{\mathcal{H}^n}\sum_{\epsilon\in\{\pm1\}^n}\mathbb{1}\left(\mathfrak{c}(P)\in[0,1]^d\right)\mathbb{1}\left(\Phi(P)<1\right)\mathbb{1}\left(P\in\mathcal{P}_n\right)\,\Theta(\mathrm{d}H_n)\cdots\Theta(\mathrm{d}H_1)$$

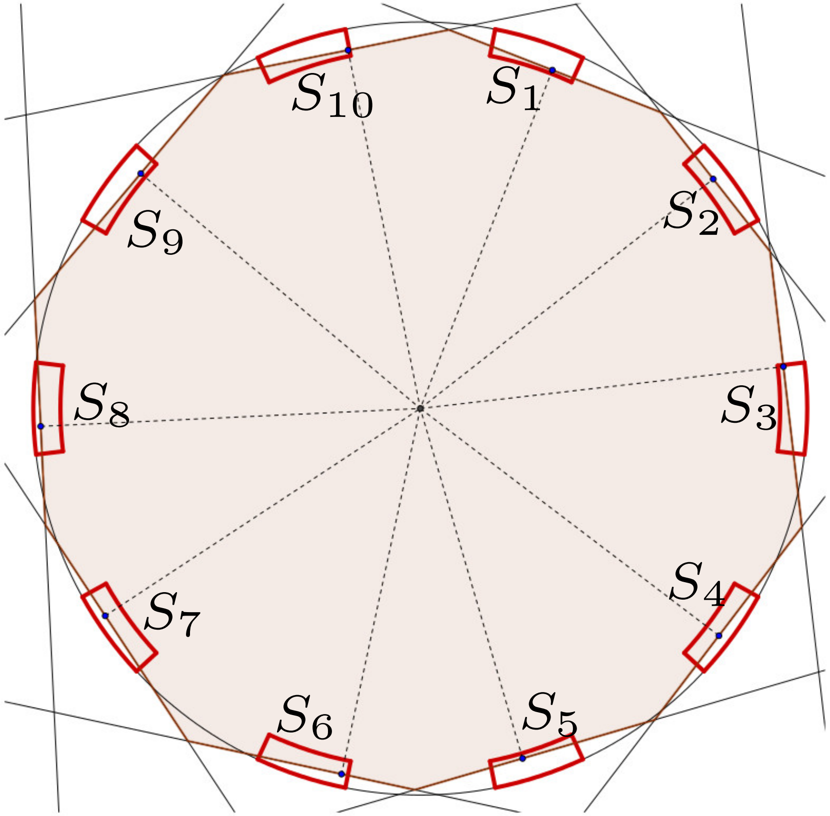
$$>n!\int\cdots\int_{\mathcal{H}^n}\mathbb{1}(H_1\in S_1)\cdots\mathbb{1}(H_n\in S_n)\,\Theta(\mathrm{d}H_n)\cdots\Theta(\mathrm{d}H_1)$$

$$=n!\Theta(S_1)^n$$

$$>n!\left(cn^{-(d-1)/(d+1)}\right)^n$$

$$\sim c^n n^{-2n/(d-1)}$$

Stirling

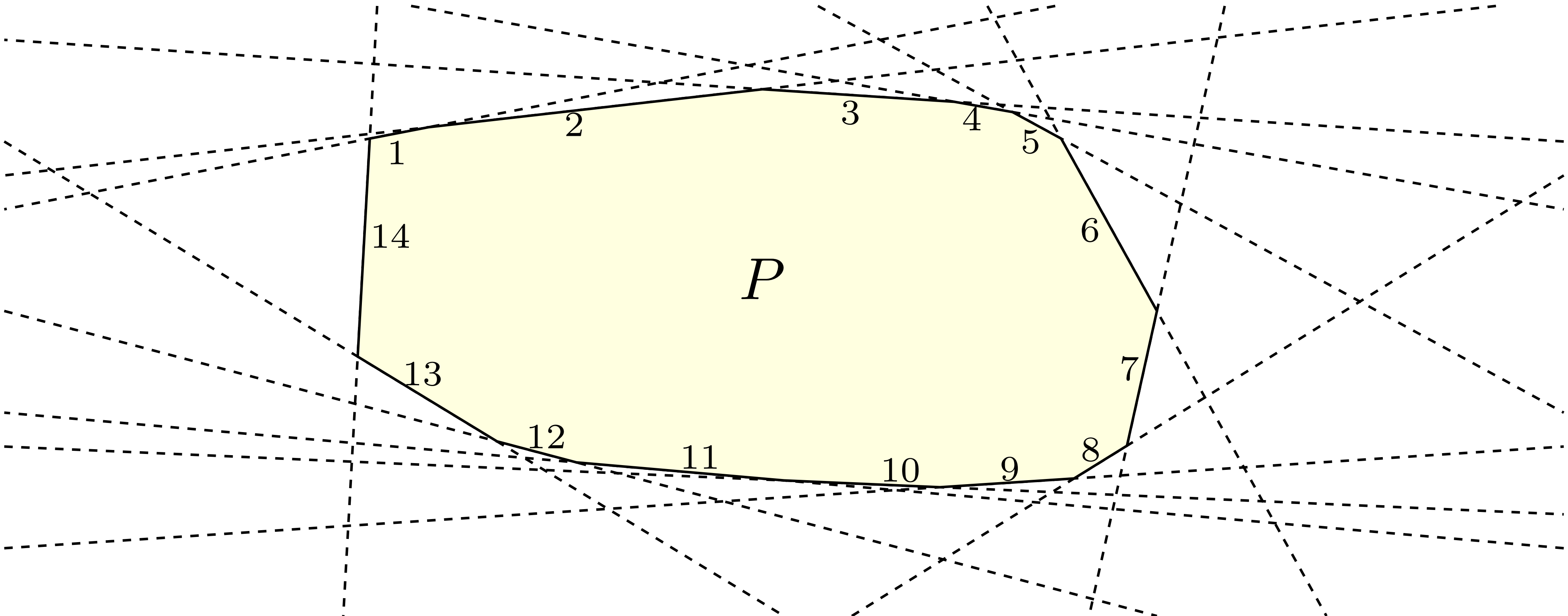


**Theorem: Lower bound**

There exists a constant $c_1$ depending on $d$ and $\varphi$ such that for $n$ big enough we have

$$\mathbb{P}(Z \text{ has } n \text{ facets}) > c_1^n n^{-2n/(d-1)}$$

# Approximation by Deleting One Facet



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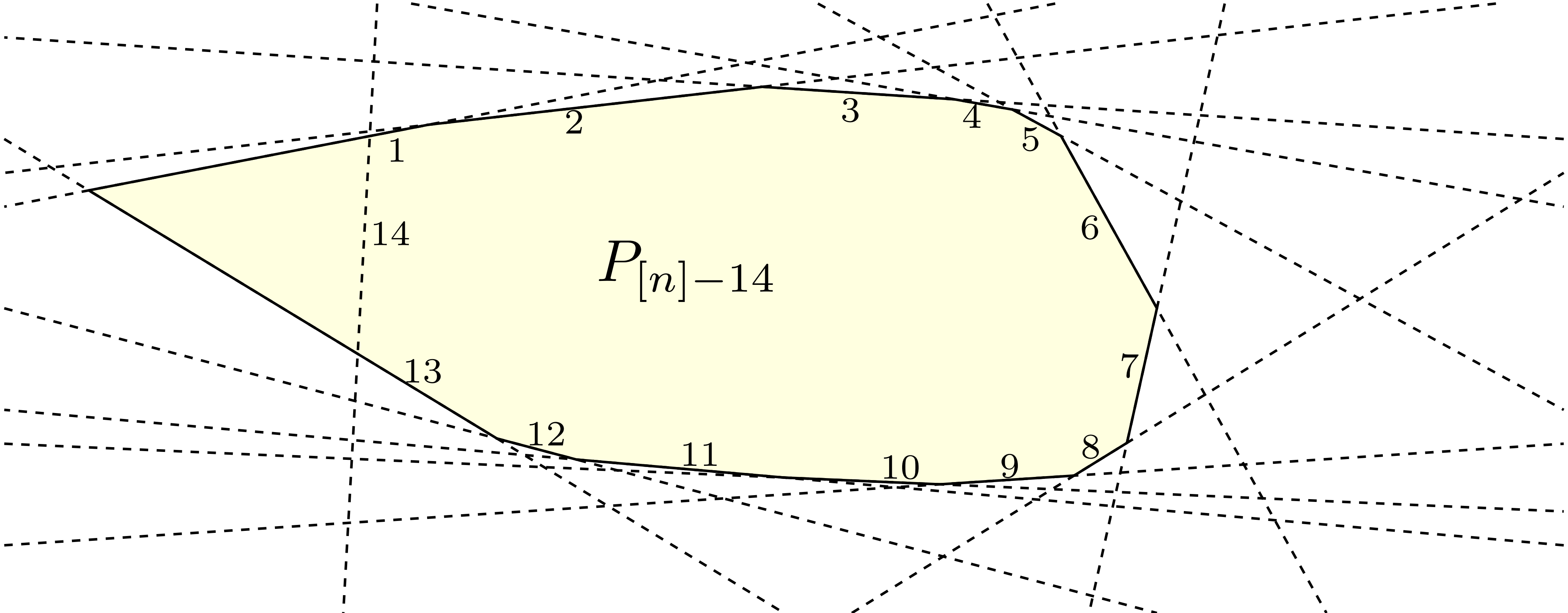

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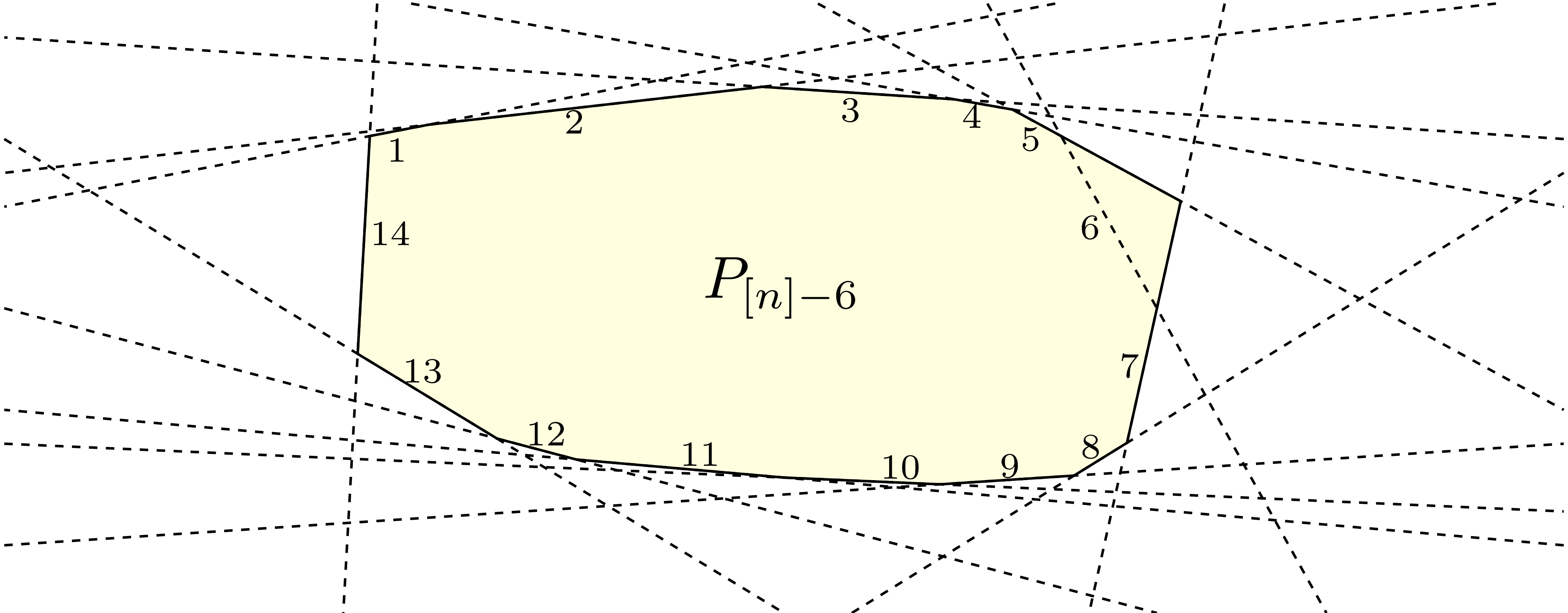

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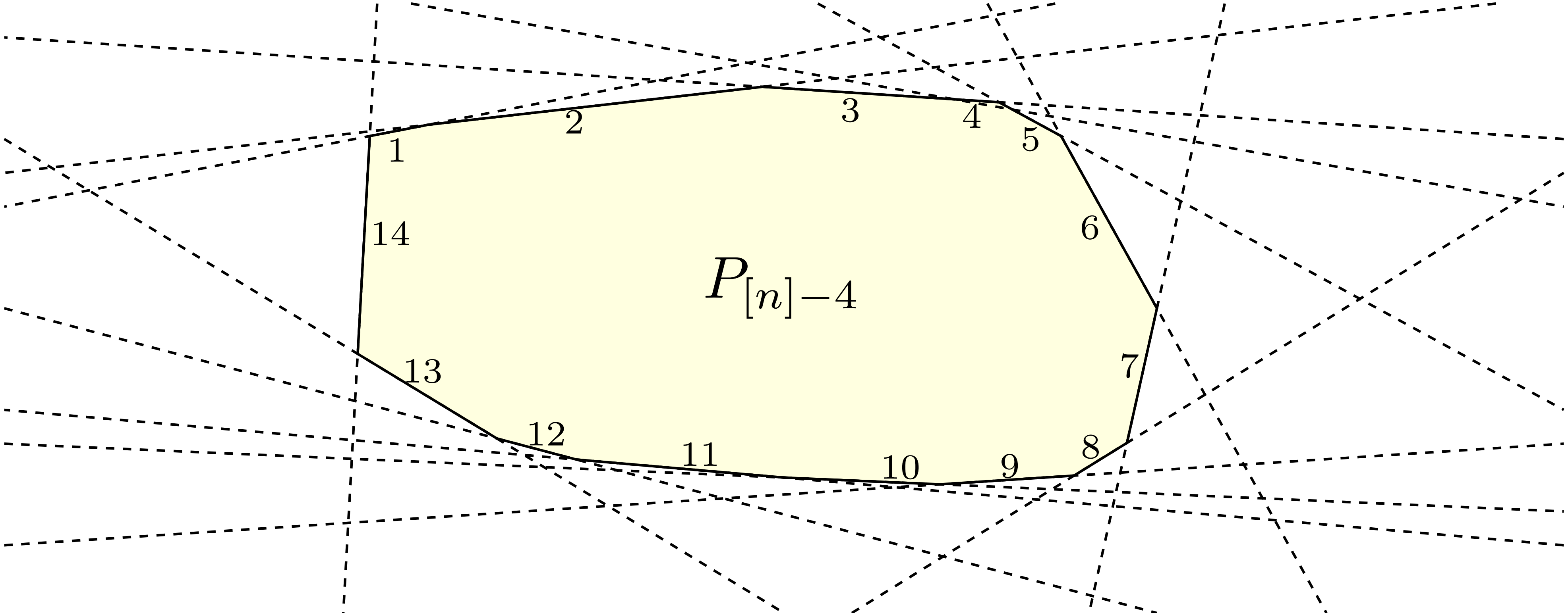


# Approximation by Deleting One Facet

There exists a constant $c_0$ such that:

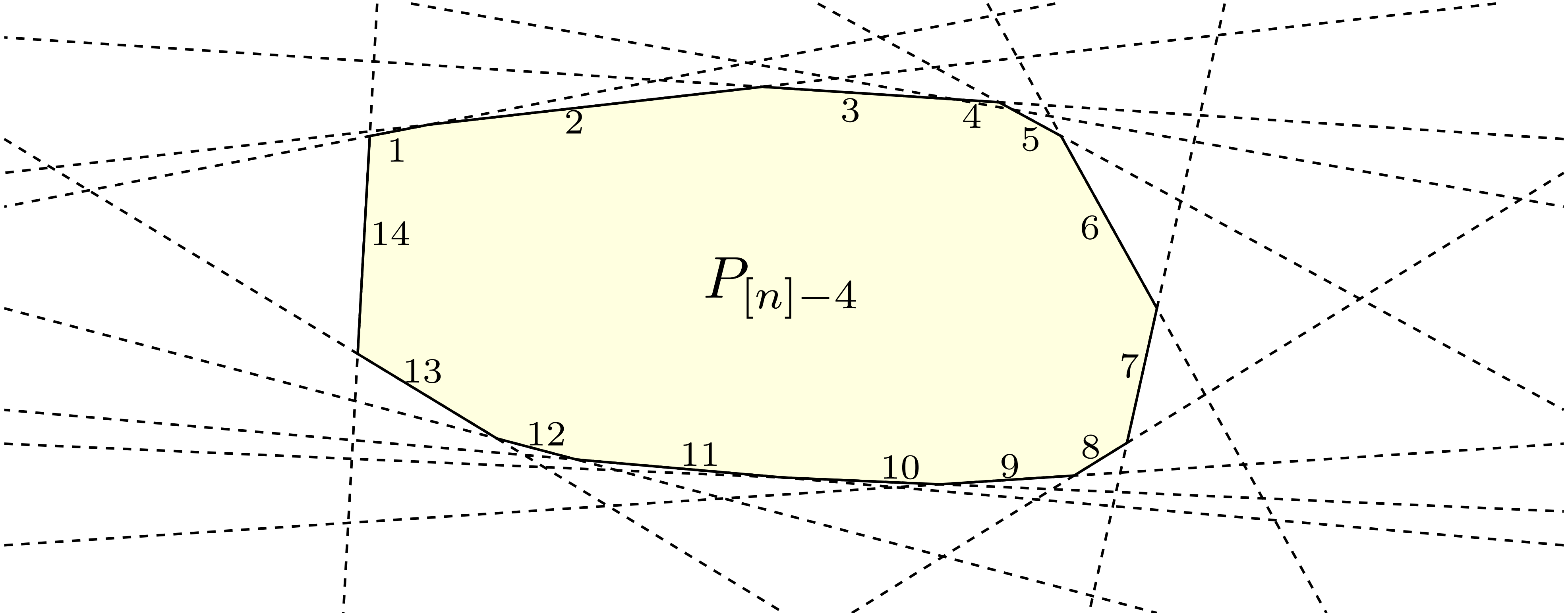
**Theorem**

Let $P = \cap_{i=1}^{n} H_i^-$ be a simple polytope with $n$ big enough.
There exists a subset $J \subset [n]$ of cardinality at least $n/4$ such that for any $j \in J$ we have

$$d_H\left(P, P_{[n]-j}\right) < c_0 n^{-2/(d-1)} \Phi\left(P\right)$$

and

$$\Phi\left(P_{[n]-j}\right) < \exp\left(c_0 n^{-1-2/(d-1)}\right)\Phi(P).$$



# Upper Bound

$$\gamma \frac{n!}{(n-d-1)!}\mathbb{P}(Z \text{ has } n \text{ facets})$$

$$=\int \cdots \int_{\mathcal{H}^n} \sum_{\boldsymbol{\epsilon}\in\{\pm 1\}^n} \mathbb{1}\left(\mathfrak{c}(P) \in [0,1]^d\right) \mathbb{1}\left(\Phi(P) < 1\right) \mathbb{1}\left(P \in \mathcal{P}_n\right) \Theta(\mathrm{d}H_n)\cdots\Theta(\mathrm{d}H_1)$$

# Upper Bound

$$\frac{\gamma}{4}\frac{n!}{(n-d-1)!}\mathbb{P}(Z \text{ has } n \text{ facets})$$

$$< \int \cdots \int_{\mathcal{H}^n} \sum_{\epsilon \in \{\pm 1\}^n} \mathbb{1}\left(\mathfrak{c}(P) \in [0,1]^d\right) \mathbb{1}\left(\Phi(P) < 1\right) \mathbb{1}\left(P \in \mathcal{P}_n\right)$$

$$\mathbb{1}\left(d_H\left(P, P_{[n-1]}\right) < c_0 n^{-2/(d-1)}\right)$$

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$$< 2^d \exp\left(c_0 n^{-1-2/(d-1)}\right)^{n-d-1} c_0 n^{-2/(d-1)}$$

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$$< 2^d \exp\left(c_0 n^{-1-2/(d-1)}\right)^{n-d-1} c_0 n^{-2/(d-1)} \gamma \frac{(n-1)!}{(n-d-2)!} \mathbb{P}(Z \text{ has } n-1 \text{ facets})$$

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# Shape of a Cell With Many Facets

Hug, Reitzner and Schneider [2004,2007] studied big cells. For example in the *isotropic case* they show that for any $j = 2, \cdots, d$ and any $\varepsilon > 0$ we have

$j$-th intrinsic volume

$$\boxed{\mathbb{P}\left(d_H\left(\mathfrak{s}(Z), \mathbb{B}^d\right) > \varepsilon \mid V_j(Z) > a\right) \to 0 \text{ when } a \to \infty}$$

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The result above remains true if you change $V_j(Z)$ by $V_1(Z)$ or $\mathrm{NumberOfFaces}(\mathrm{Z})$.

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We did a small step in the direction of this conjecture:

**Theorem**

There exists $\epsilon > 0$ such that for any $j \leq \lceil (d-1)/2 \rceil$ we have

$$\mathbb{P}\left(\frac{V_j(Z)}{V_1(Z)^j} < \epsilon \mid Z \text{ has } n \text{ facets}\right) \to 0 \text{ when } n \to \infty.$$

Cell with many facets are not '*too flat*'.

# Take Home Message and Perspectives

**Main Theorem**

There exist constants $c_1$ and $c_2$ depending on $d$ and $\varphi$ such that for $n$ big enough we have

$$c_1^n n^{-2n/(d-1)} < \mathbb{P}(Z \text{ has } n \text{ facets}) < c_2^n n^{-2n/(d-1)}$$

$\Rightarrow$ Explicit the dependence to $d$ in the constants $c_1$ and $c_2$.

$\Rightarrow$ Generalization : Other kind of mosaics, the zero cell...

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THANK YOU!