Cells With Many Facets in a Hyperplane Mosaic

Gilles Bonnet,
joint work with Matthias Reitzner and Pierre Calka

4th Stochastic Geometry Days

Poitiers, Friday August 28, 2015

DFG GK 1916

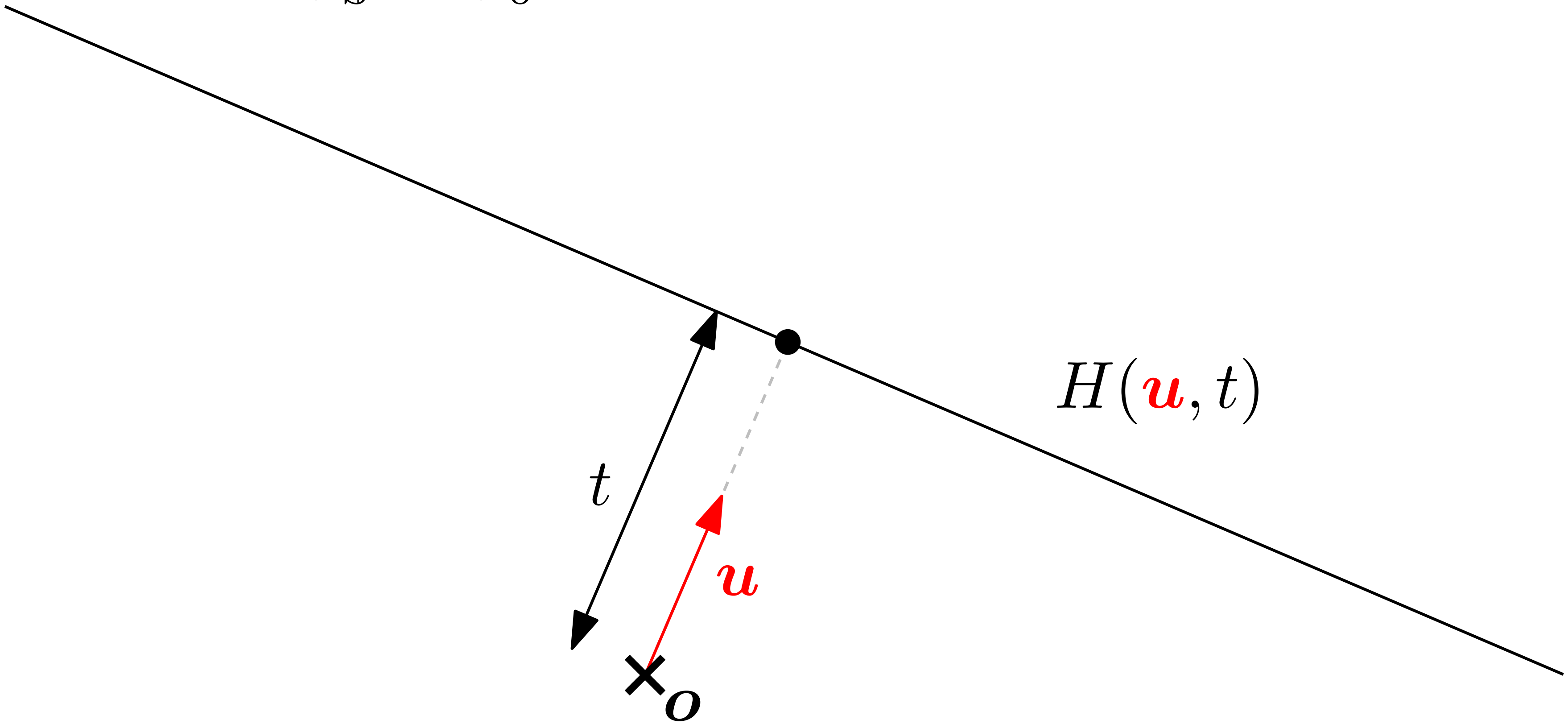
UNIVERSITÄT OSNABRÜCK

Stationary Poisson Hyperplane Mosaic in $\mathbb{R}^d$

η Poisson Hyperplane Process of **intensity measure** Θ

φ **directional distribution** (even measure on $\mathbb{S}^{d-1}$)

$$\Theta(\cdot) = \int_{\mathbb{S}^{d-1}} \int_0^\infty \mathbb{1}\{H(\boldsymbol{u}, t) \in \cdot\}\, \mathrm{d}t\, \varphi(\mathrm{d}\boldsymbol{u})$$

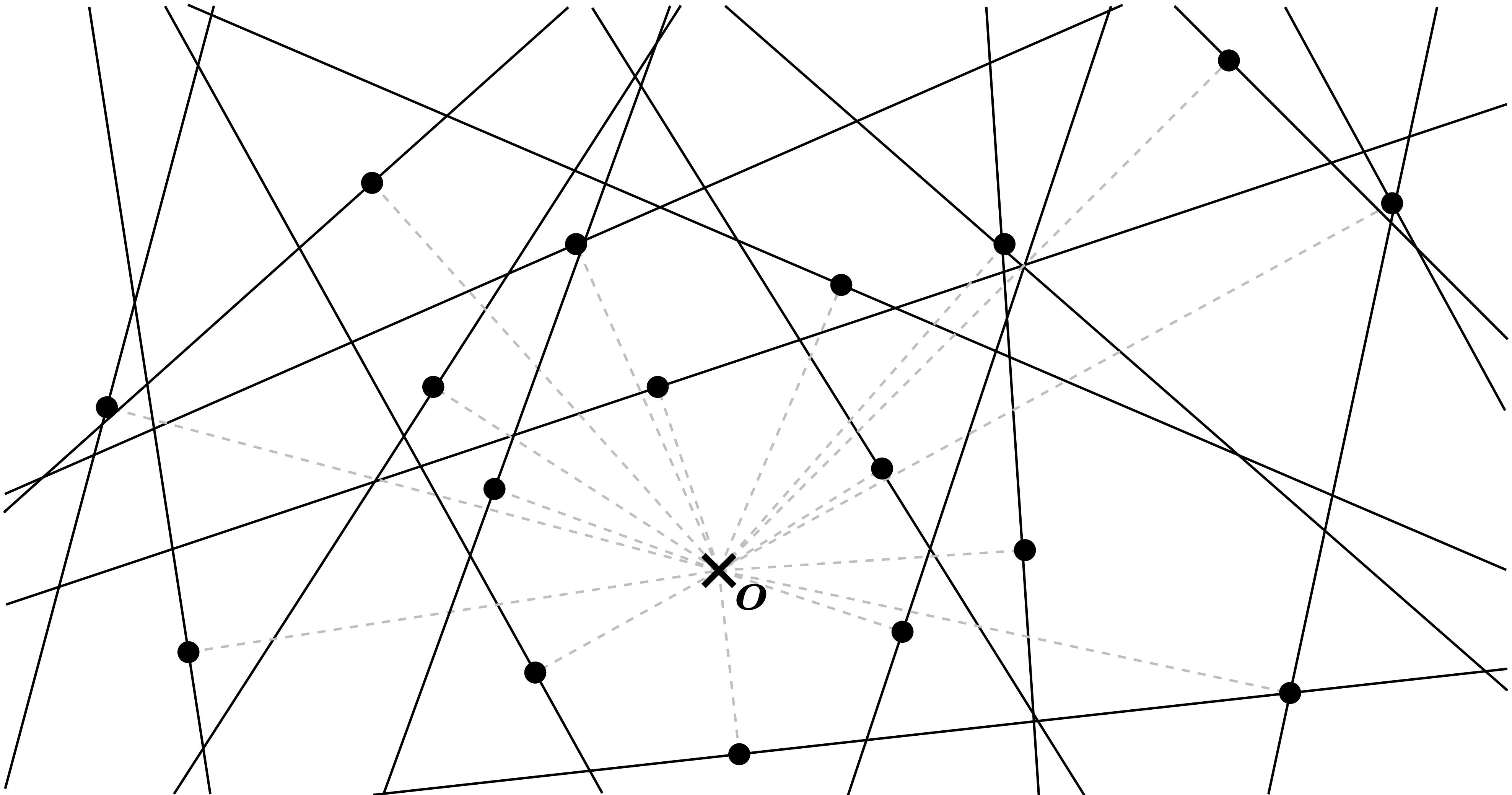


Stationary Poisson Hyperplane Mosaic in $\mathbb{R}^d$

η Poisson Hyperplane Process of **intensity measure** Θ

φ **directional distribution** (even measure on $\mathbb{S}^{d-1}$)

$$\Theta(\cdot) = \int_{\mathbb{S}^{d-1}} \int_0^\infty \mathbb{1}\{H(\boldsymbol{u}, t) \in \cdot\}\, \mathrm{d}t\, \varphi(\mathrm{d}\boldsymbol{u})$$

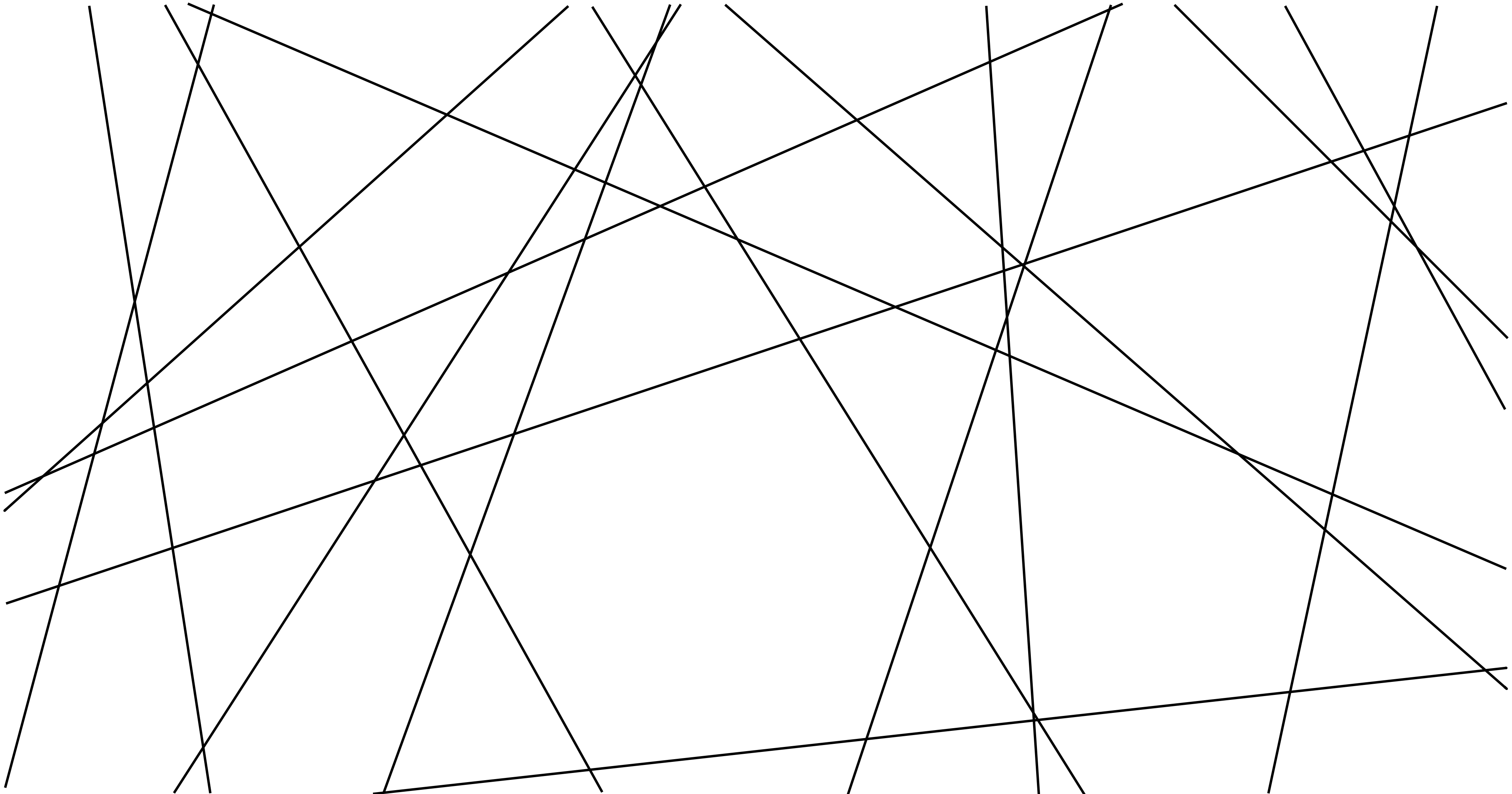


Stationary Poisson Hyperplane Mosaic in $\mathbb{R}^d$

η Poisson Hyperplane Process of **intensity measure** Θ

φ **directional distribution** (even measure on $\mathbb{S}^{d-1}$)

$$\Theta(\cdot) = \int_{\mathbb{S}^{d-1}} \int_0^\infty \mathbb{1}\{H(\boldsymbol{u}, t) \in \cdot\}\, \mathrm{d}t\, \varphi(\mathrm{d}\boldsymbol{u})$$

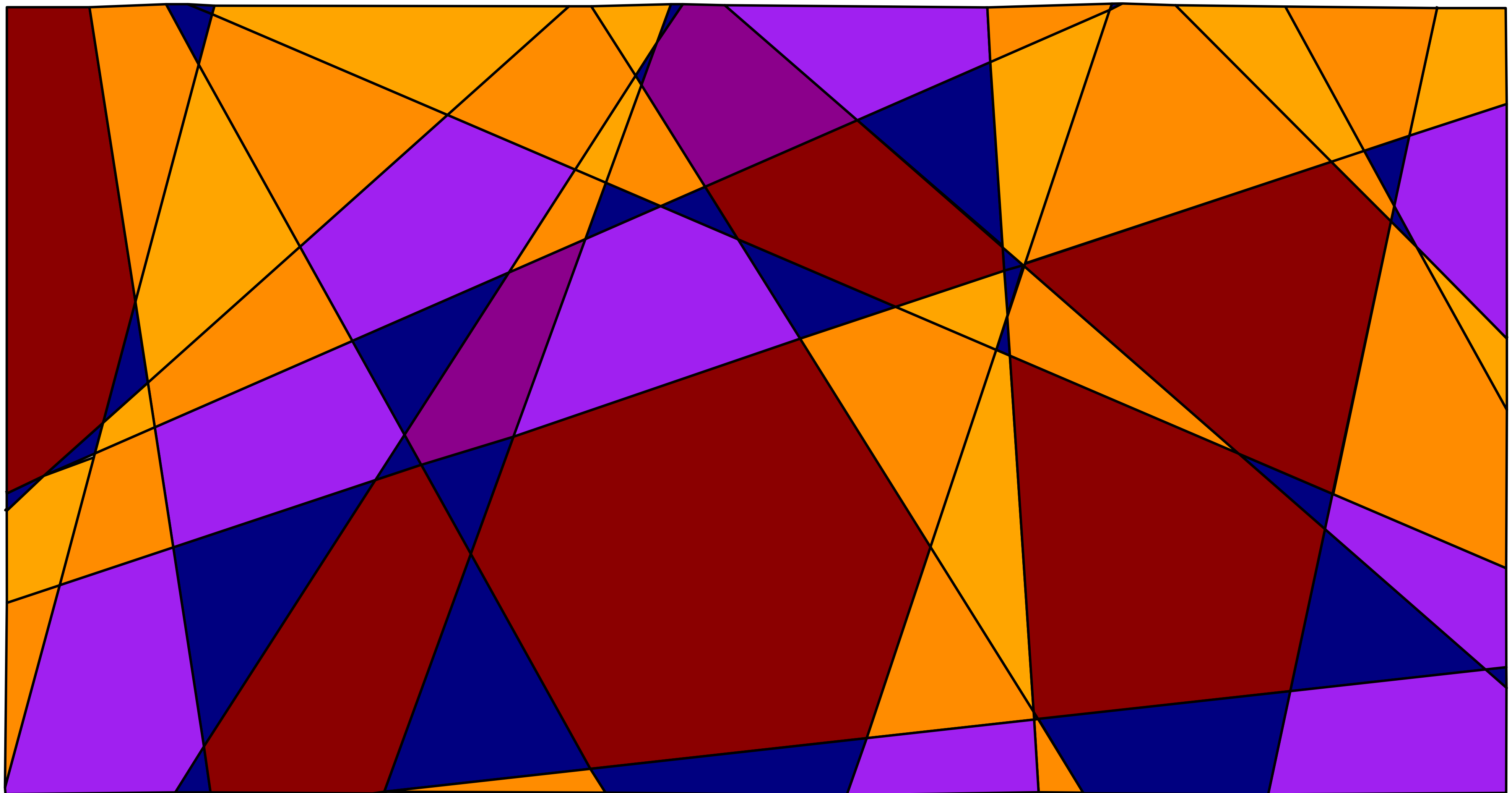


Stationary Poisson Hyperplane Mosaic in $\mathbb{R}^d$

η Poisson Hyperplane Process of **intensity measure** Θ

φ **directional distribution** (even measure on $\mathbb{S}^{d-1}$)

$$\Theta(\cdot) = \int_{\mathbb{S}^{d-1}} \int_0^\infty \mathbb{1}\{H(\boldsymbol{u}, t) \in \cdot\} \, \mathrm{d}t \, \varphi(\mathrm{d}\boldsymbol{u})$$



Goal

$$\mathbb{P}(Z \text{ has } n \text{ facets})?$$

when $n \to \infty$

typical cell

Goal

$$\mathbb{P}(Z \text{ has } n \text{ facets})?$$ when $n \to \infty$

typical cell

In a specific case it is already known:

Theorem [Calka and Hilhorst 2008]

In the 2-dimensional isotropic case we have that

$$\mathbb{P}(Z \text{ has } n \text{ facets}) \sim \alpha\beta^n n^{-2n}\sqrt{n} \text{ when } n \to \infty$$

with $\alpha = (6\pi^{5/2})^{-1}$ and $\beta = \pi^2 e^2$

Goal

$$\mathbb{P}(Z \text{ has } n \text{ facets})? \quad \text{when } n \to \infty$$

typical cell

In a specific case it is already known:

Theorem [Calka and Hilhorst 2008]

In the 2-dimensional isotropic case we have that

$$\mathbb{P}(Z \text{ has } n \text{ facets}) \sim \alpha \beta^n n^{-2n} \sqrt{n} \text{ when } n \to \infty$$

with $\alpha = (6\pi^{5/2})^{-1}$ and $\beta = \pi^2 e^2$

We generalize this to any dimension and nice directional distribution:

Main Theorem

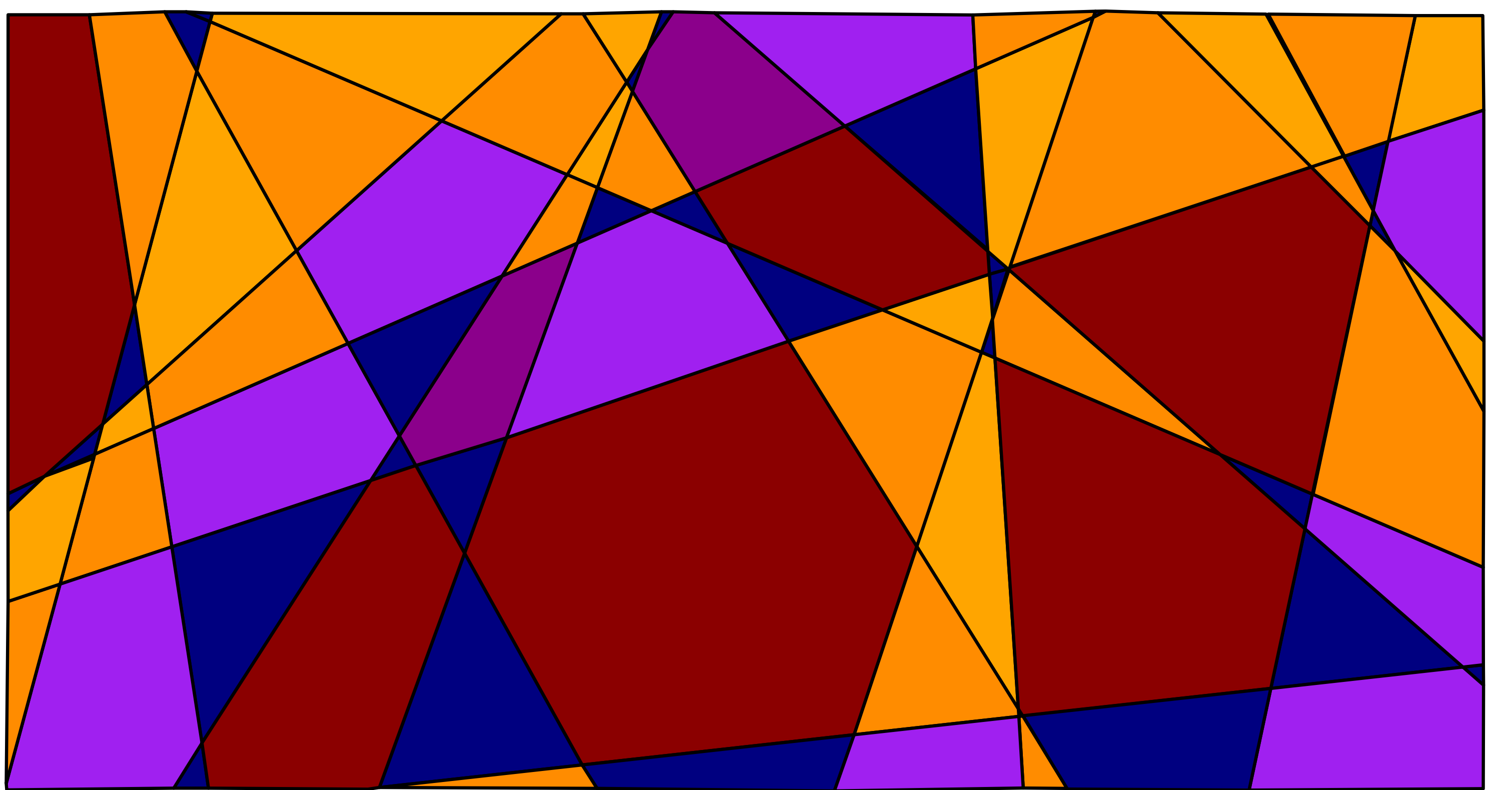
There exist constants c_1 and c_2 depending on d and φ such that for n big enough we have

$$c_1^n n^{-2n/(d-1)} < \mathbb{P}(Z \text{ has } n \text{ facets}) < c_2^n n^{-2n/(d-1)}$$

Typical Cell Z

$X = X_\eta \ \ldots$ **Mosaic**: Point Process in $\mathcal{P}$

Set of polytopes



Typical Cell Z

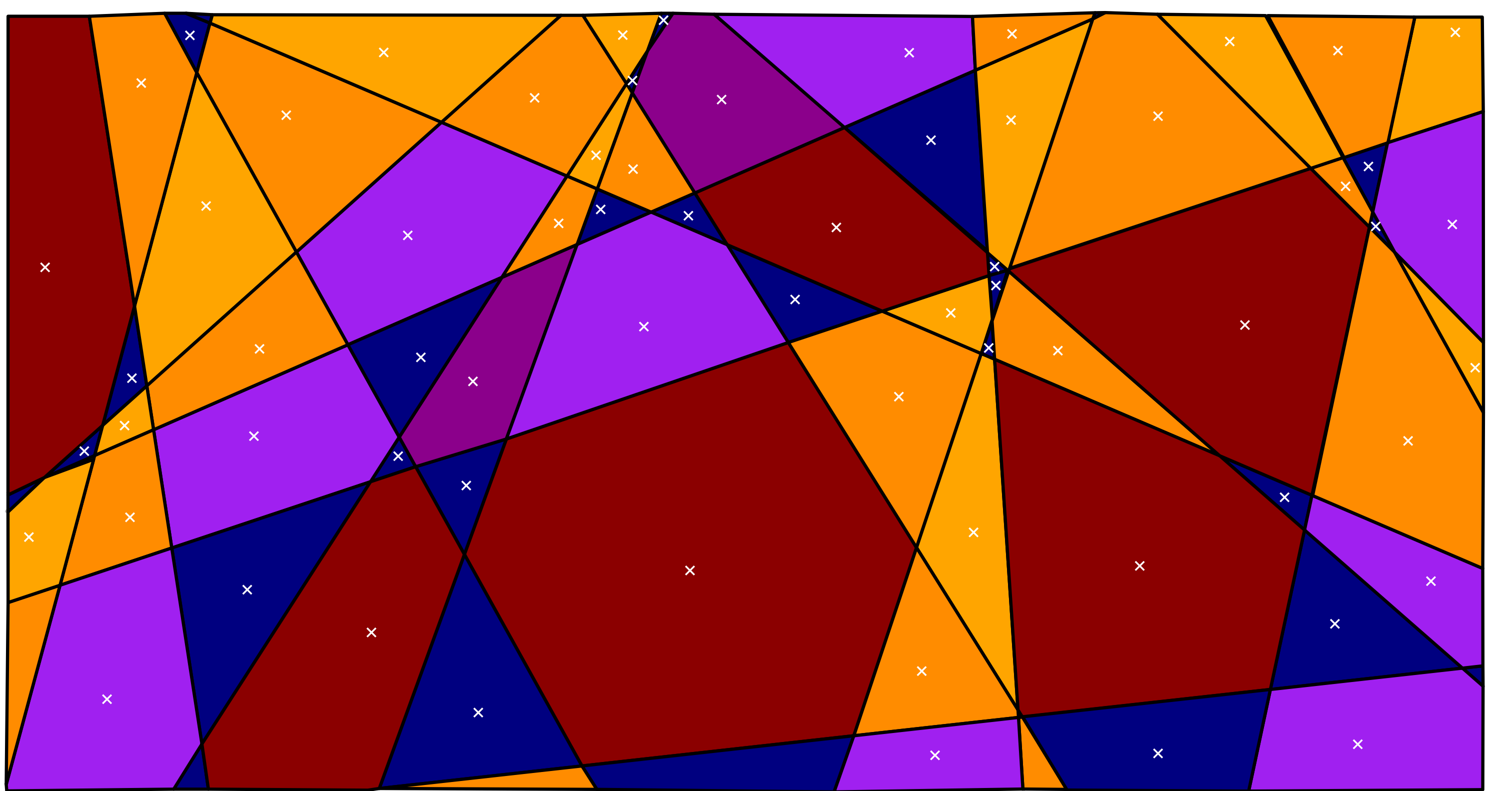
$X = X_\eta$ ... **Mosaic**: Point Process in $\mathcal{P}$

Set of polytopes

$\mathfrak{c} : \mathcal{P} \to \mathbb{R}^d$ a **center function**

e.g. center of mass, center of the circumball...

$$\mathfrak{c}(tP + x) = t\mathfrak{c}(P) + x$$



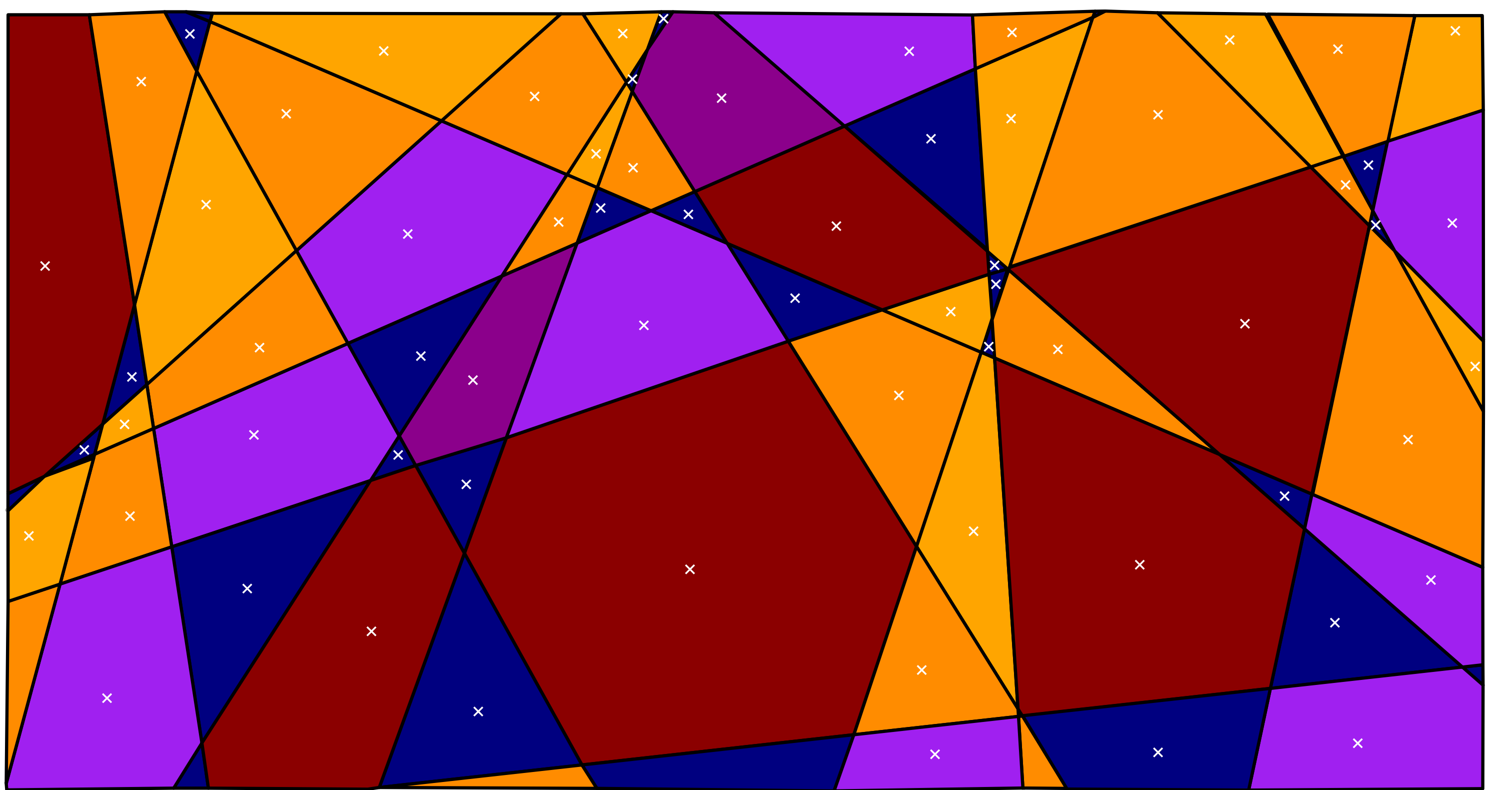
Typical Cell Z

Set of centred polytopes

$X = X_\eta \ldots$ **Mosaic**: Point Process in $\mathbb{R}^d \times \mathcal{P}_\mathfrak{c}$

$\mathfrak{c} : \mathcal{P} \to \mathbb{R}^d$ a **center function**

X is a germ-grain process with **germs** $\mathfrak{c}(P)$ and **grains** $P - \mathfrak{c}(P)$



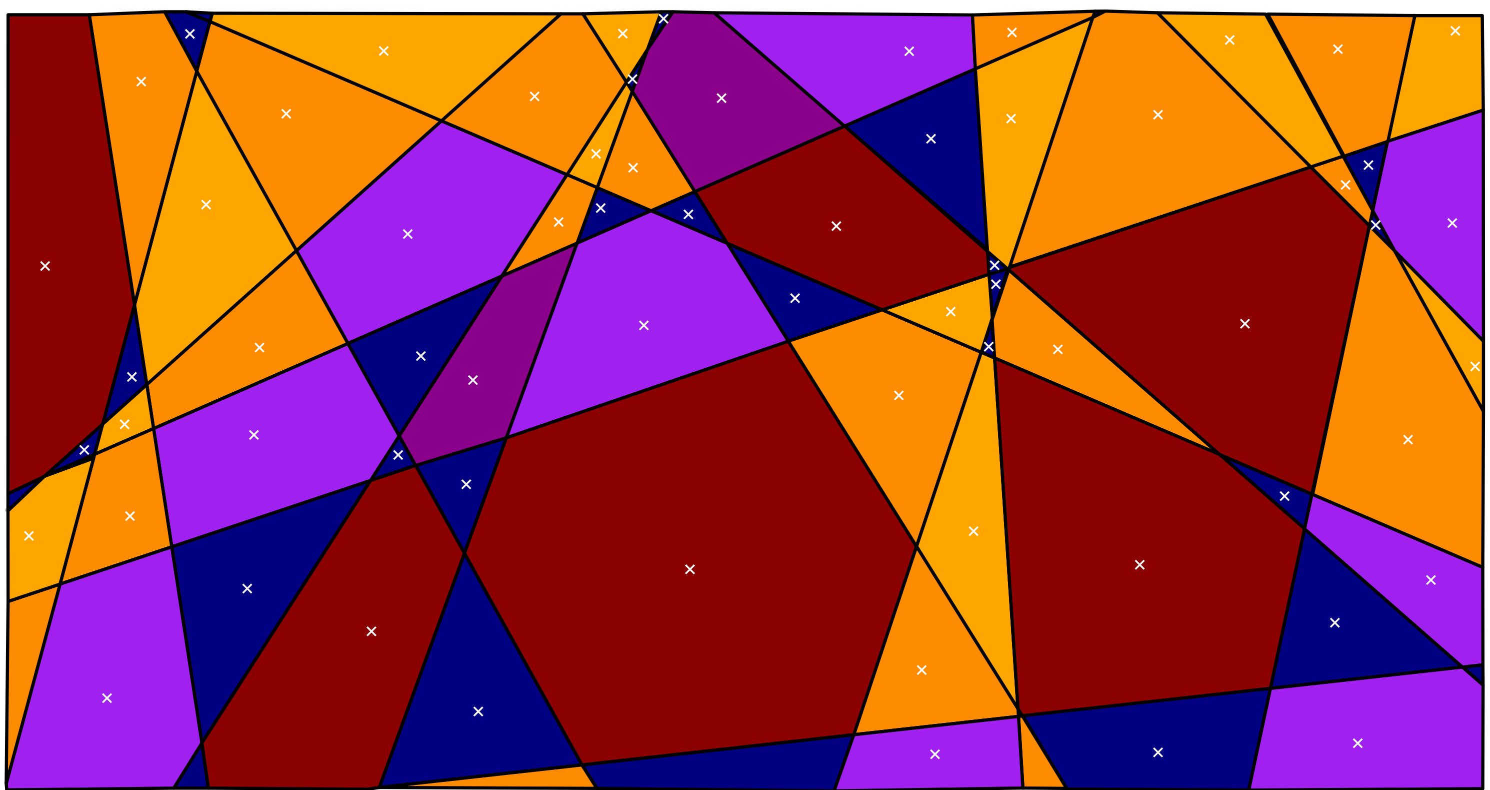
Typical Cell Z

$X = X_\eta \ldots$ **Mosaic**: Point Process in $\mathbb{R}^d \times \mathcal{P}_{\mathfrak{c}}$

$\mathfrak{c} : \mathcal{P} \to \mathbb{R}^d$ a **center function**

X is a germ-grain process with **germs** $\mathfrak{c}(P)$ and **grains** $P - \mathfrak{c}(P)$

Its intensity measure has the form $\gamma \lambda_d \otimes \mathbb{Q}$



Typical Cell Z

$X = X_\eta$ ... **Mosaic**: Point Process in $\mathbb{R}^d \times \mathcal{P}_\mathfrak{c}$ ← Set of centred polytopes

$\mathfrak{c} : \mathcal{P} \to \mathbb{R}^d$ a **center function**

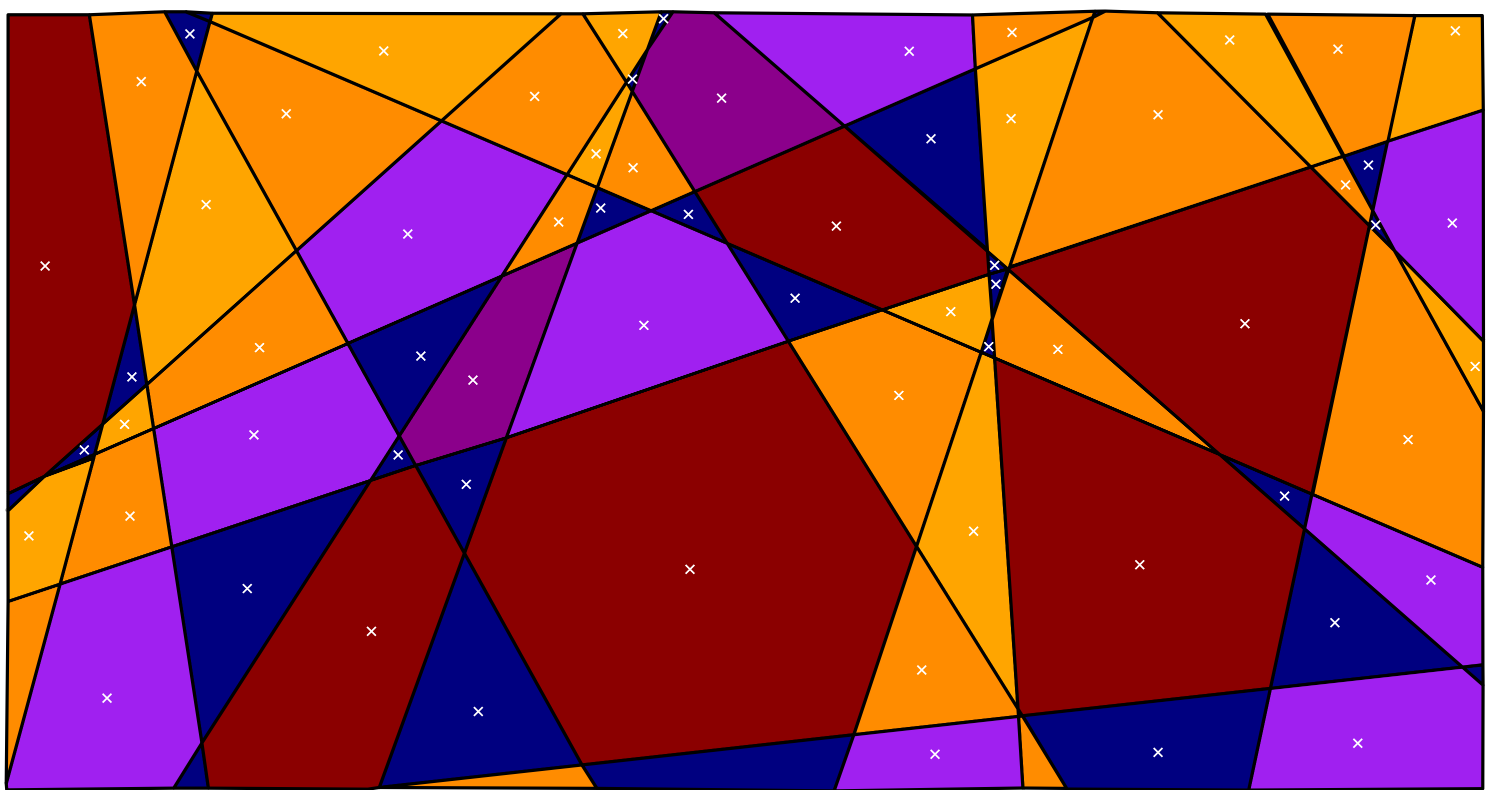
X is a germ-grain process with **germs** $\mathfrak{c}(P)$ and **grains** $P - \mathfrak{c}(P)$

Its intensity measure has the form $\gamma \lambda_d \otimes \mathbb{Q}$ ← grain distribution

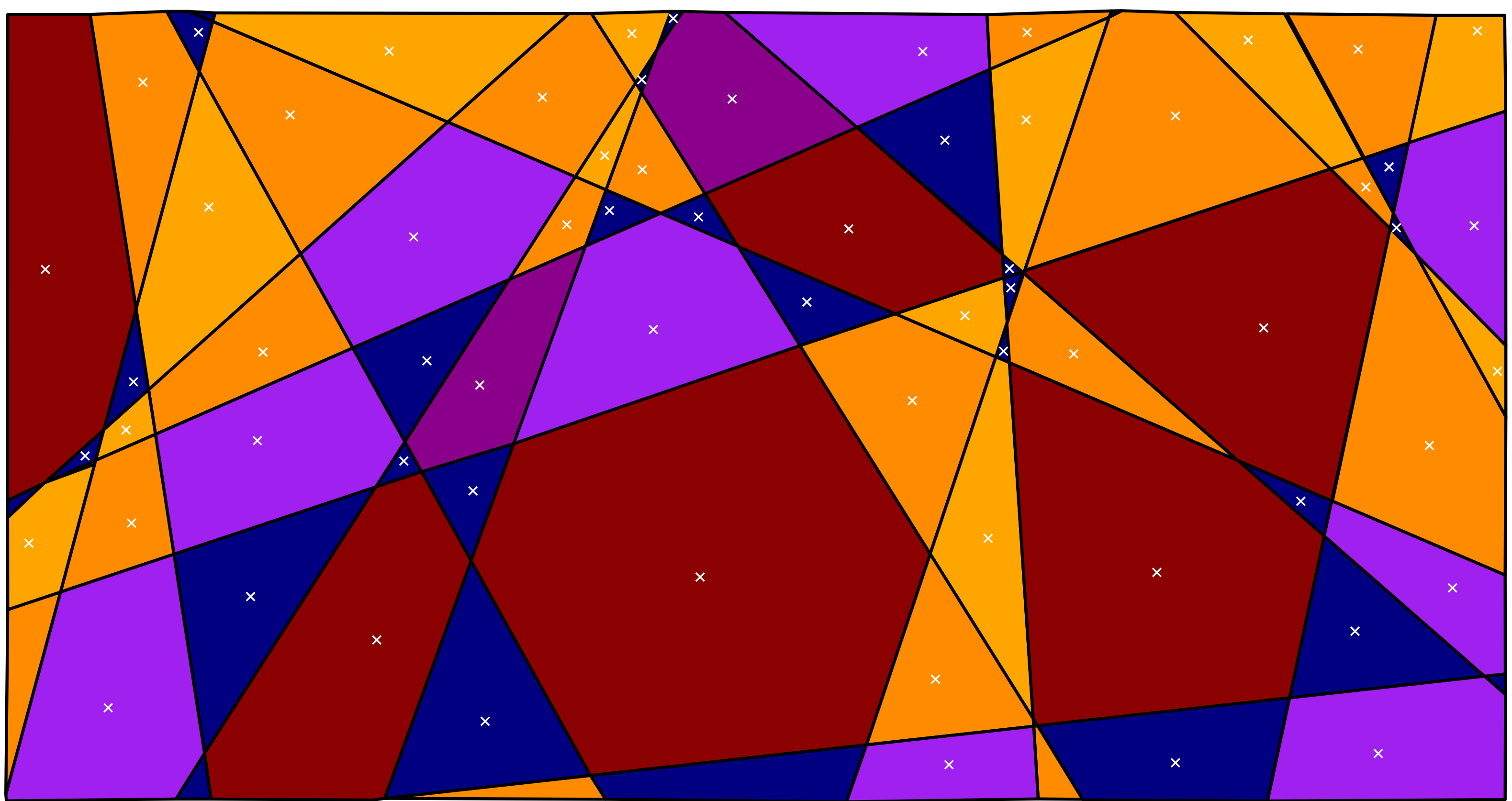
intensity

Lebesgue measure

Z ... **Typical cell** = random centred polytope with distribution $\mathbb{Q}$

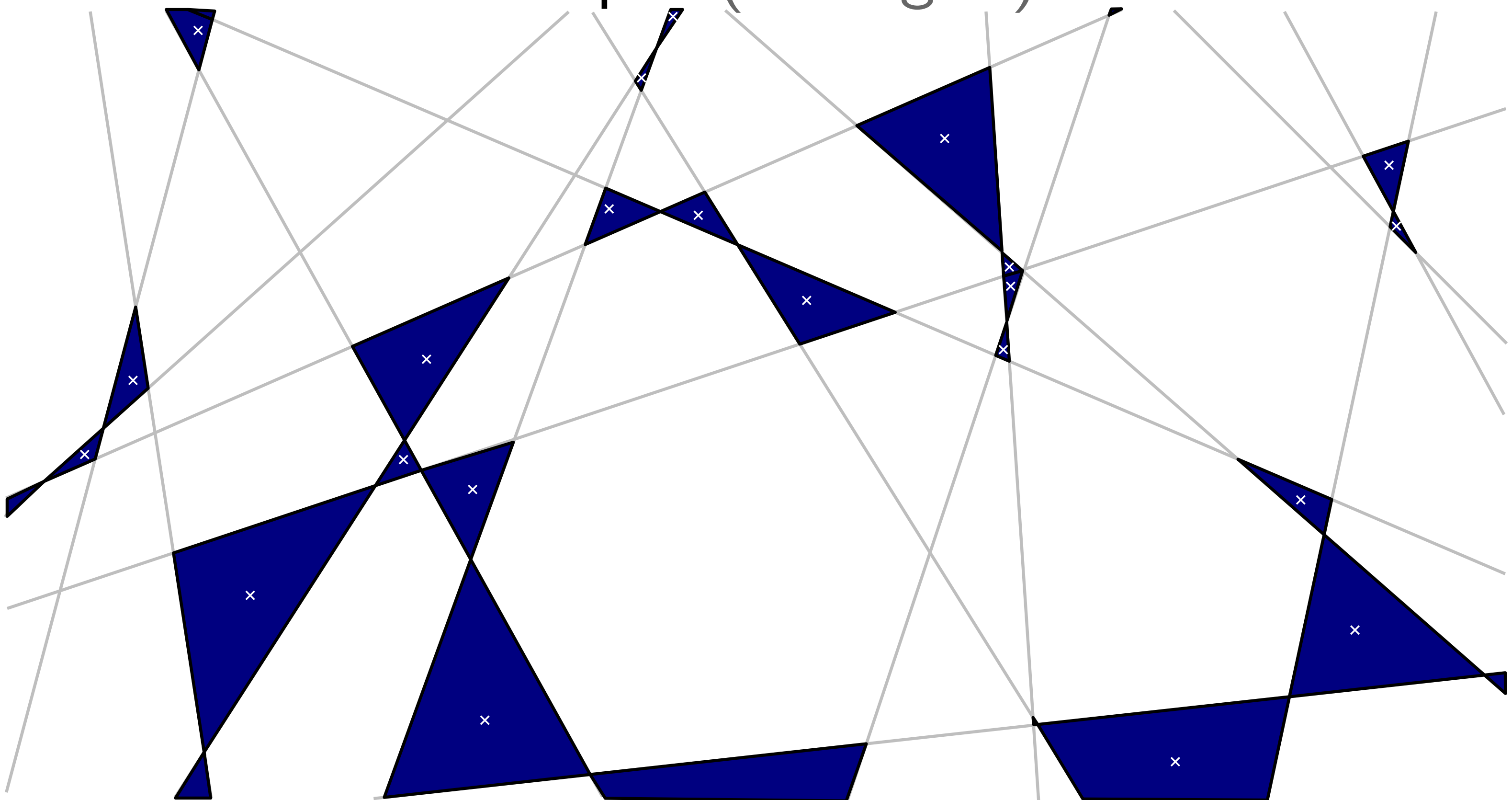


Cells With n Facets



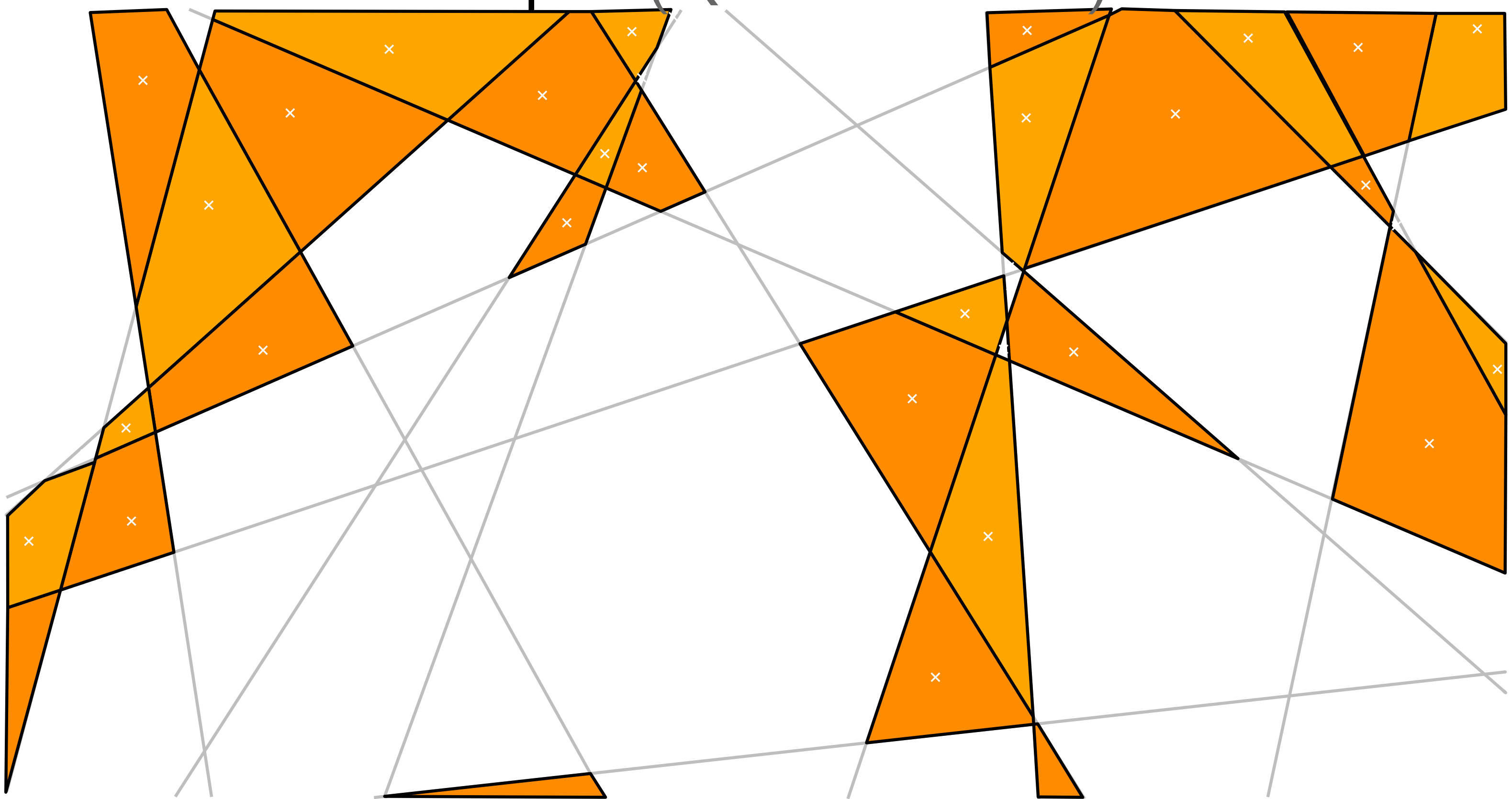
Cells With n Facets

3-topes (Triangles)



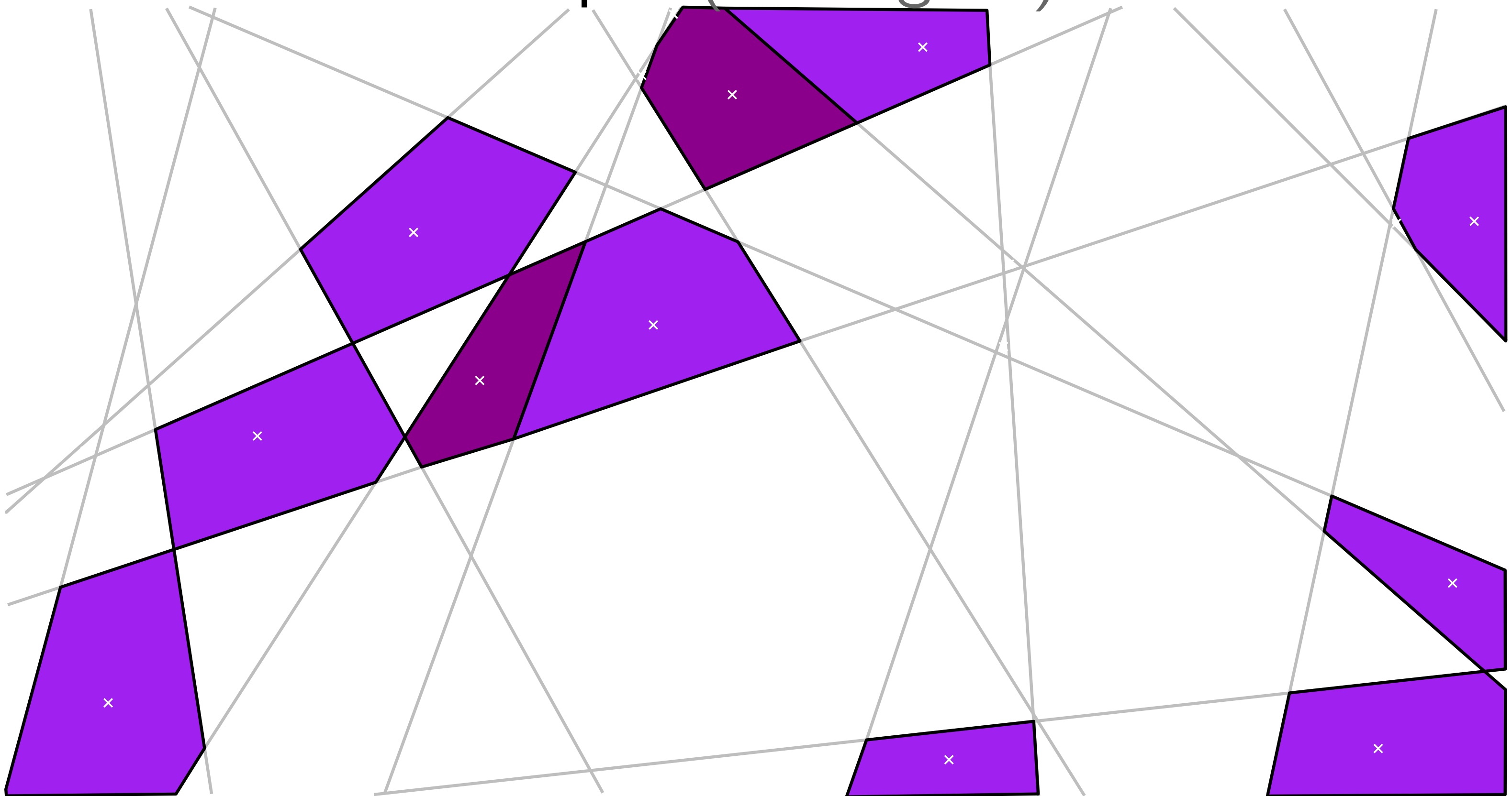
Cells With n Facets

4-topes (Quadrilaterals)



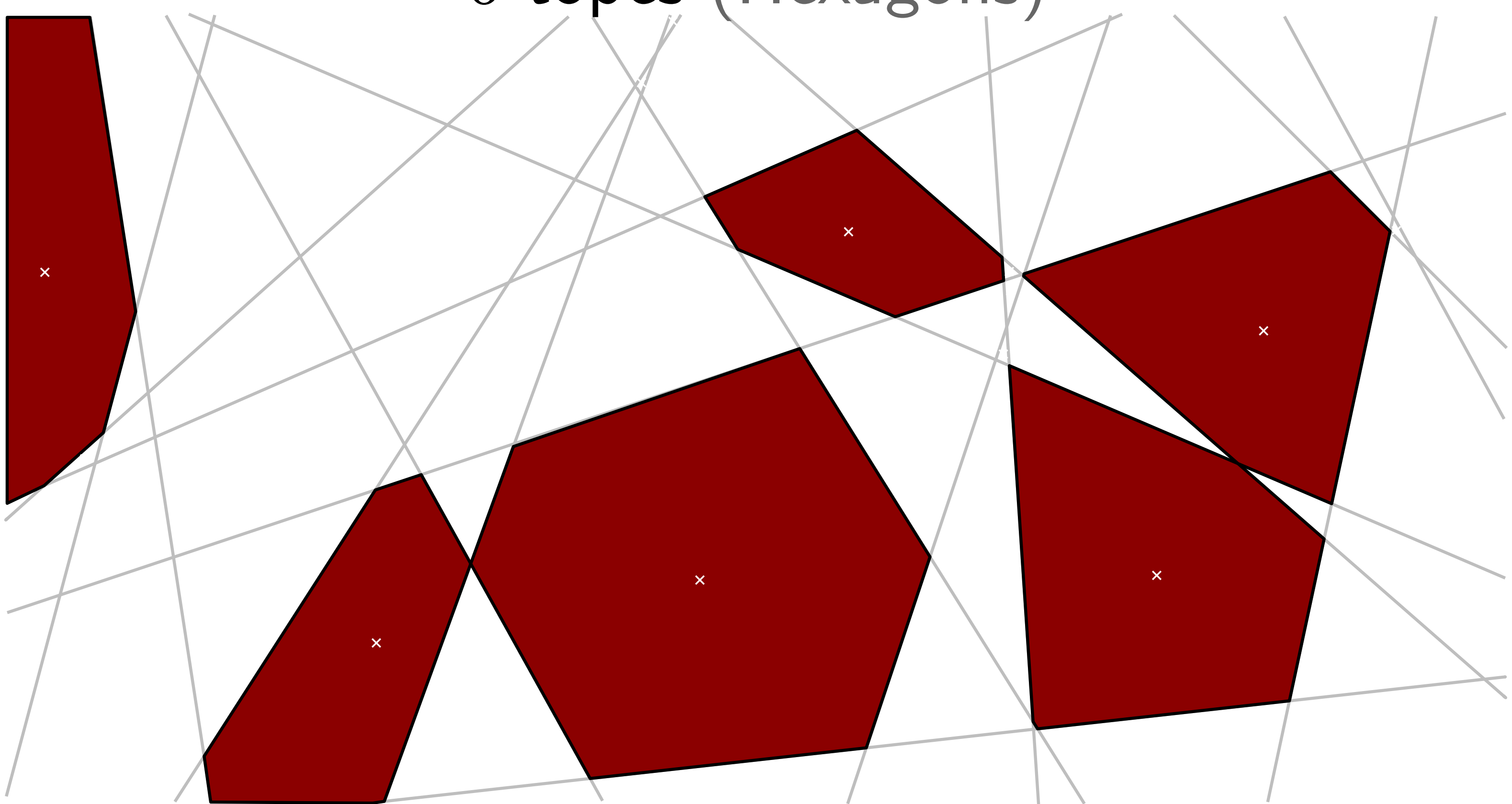
Cells With n Facets

5-topes (Pentagons)



Cells With n Facets

6-topes (Hexagons)

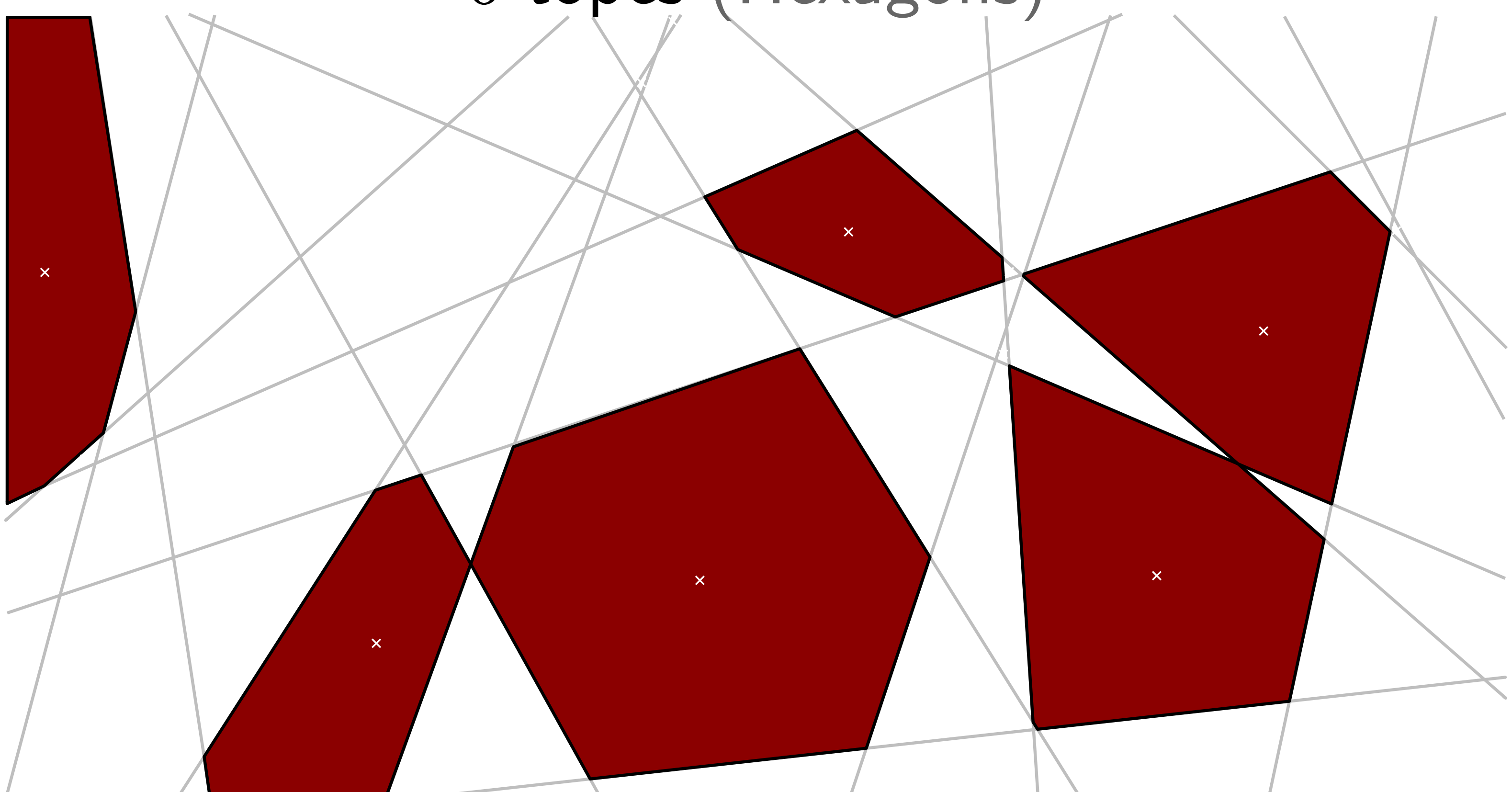


Cells With n Facets

Set of centred n-topes

$$\begin{aligned}\mathbb{P}(Z \text{ has } n \text{ facets}) &= \mathbb{Q}(\mathcal{P}_{n,\mathfrak{c}}) \\ &= \gamma^{-1}\gamma\lambda_d([0,1]^d)\mathbb{Q}(\mathcal{P}_{n,\mathfrak{c}}) \\ &= \gamma^{-1}\mathbb{E}\,\underbrace{X(\mathcal{P}_{n,[0,1]^d})}_{\text{number of } n\text{-topes of } X \text{ with center in } [0,1]^d}\end{aligned}$$

6-topes (Hexagons)

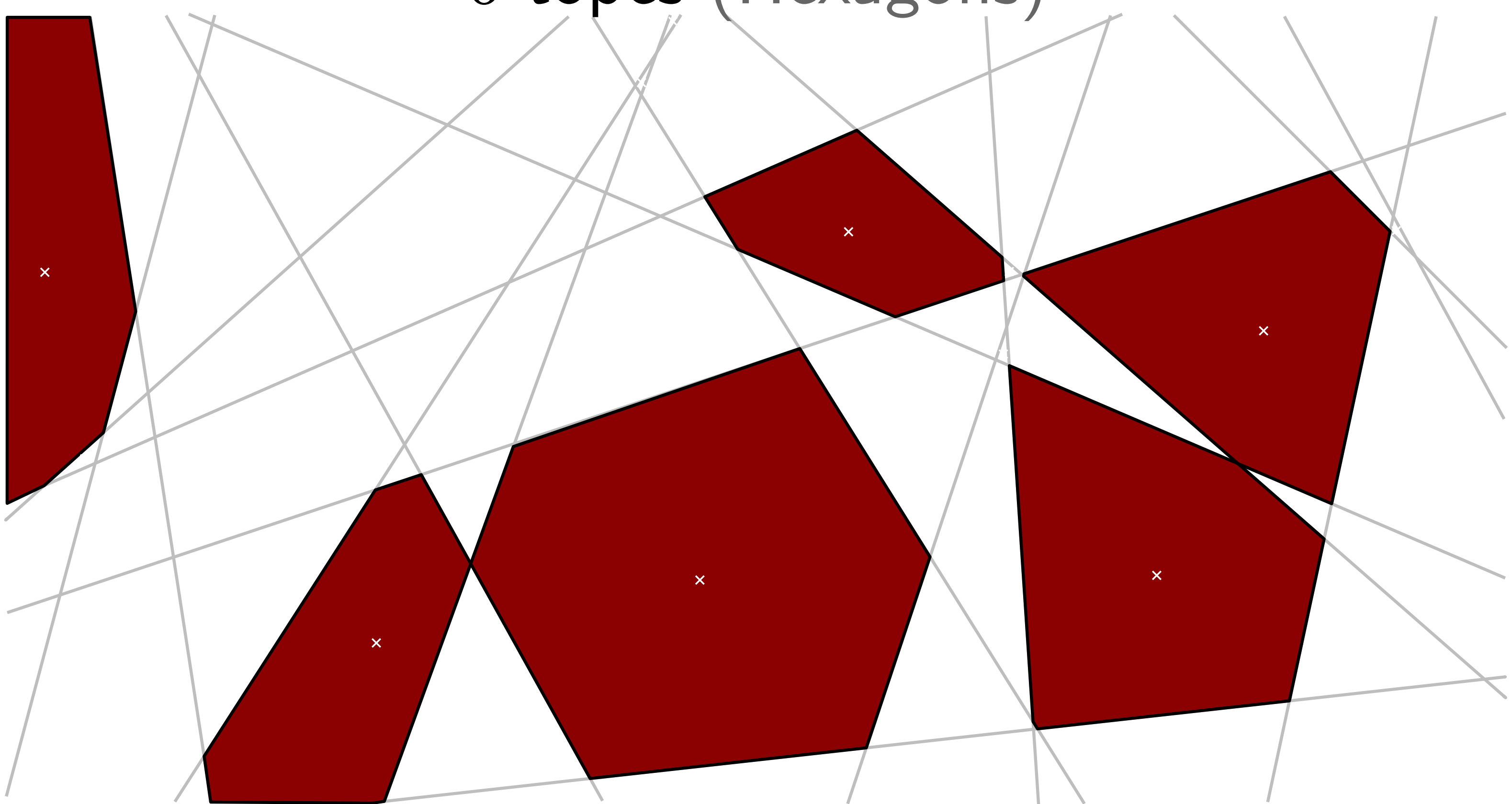


Cells With n Facets

Let $D \subset \mathcal{P}_n$. number of cells of X in D $\quad P = \bigcap_{i=1}^{n} H_i^{\epsilon_i}$

$$X(D) = \frac{1}{n!} \sum_{P \in \eta^n_{\neq} \times \{\pm 1\}^n} \mathbf{1}(P \in D)\,\mathbf{1}(\eta \cap P = \emptyset).$$

6-topes (Hexagons)

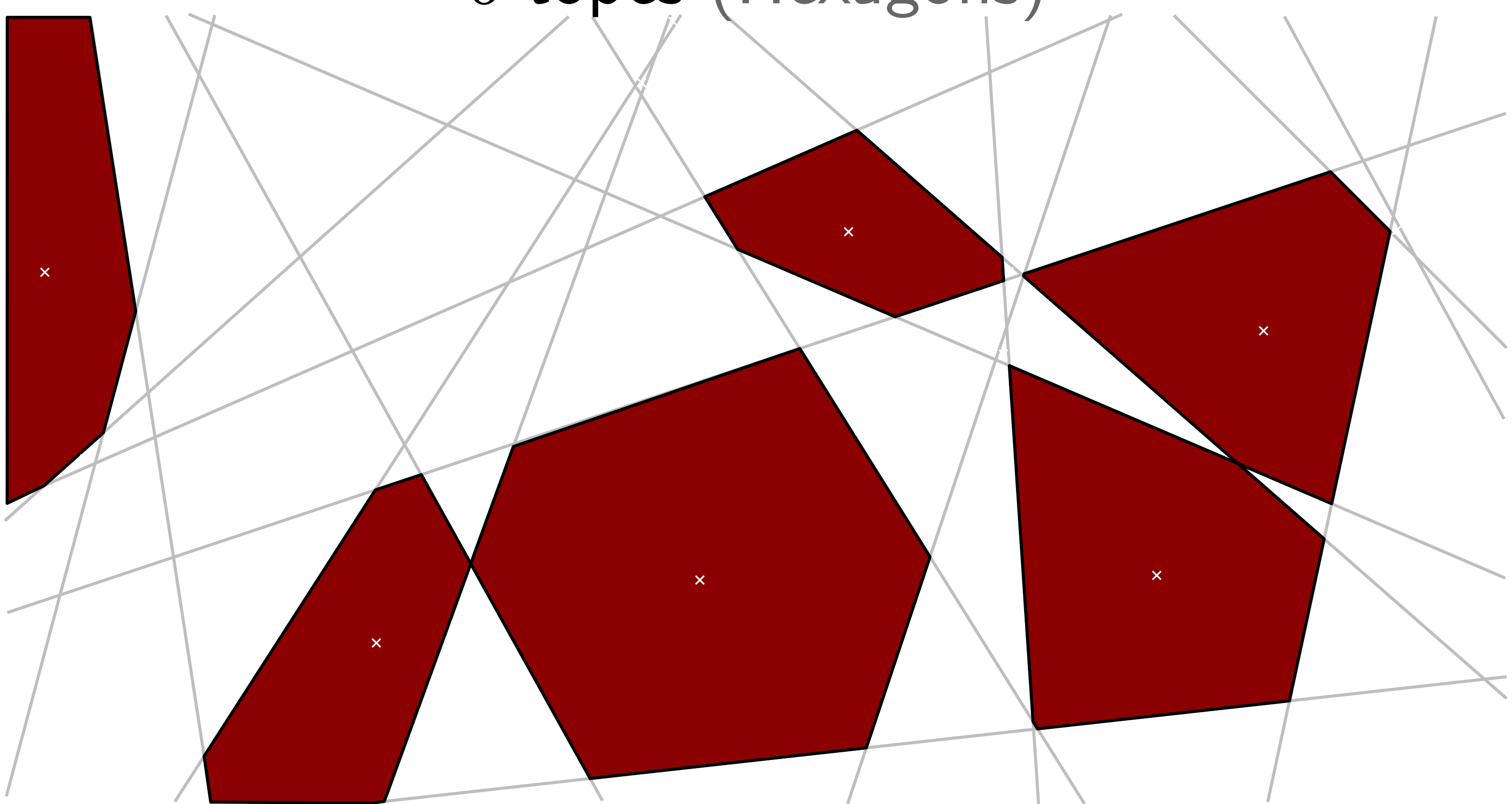


Cells With n Facets

Let $D \subset \mathcal{P}_n$. number of cells of X in D $\quad P = \bigcap_{i=1}^{n} H_i^{\epsilon_i}$

$$\mathbb{E}X(D) = \frac{1}{n!}\int \cdots \int_{\mathcal{H}^n} \sum_{\epsilon \in \{\pm 1\}^n} \mathbf{1}\left(P \in D\right) \mathbb{P}(\eta \cap P = \emptyset)\Theta(\mathrm{d}H_n)\cdots\Theta(\mathrm{d}H_1)$$

6-topes (Hexagons)



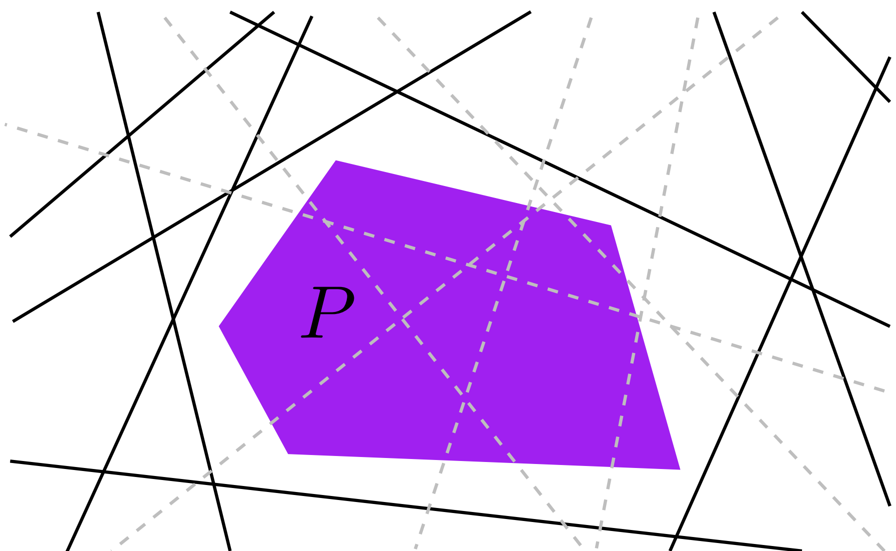
Cells With n Facets

Let $D \subset \mathcal{P}_n$. number of cells of X in D $P = \bigcap_{i=1}^{n} H_i^{\epsilon_i}$

$$\mathbb{E}X(D) = \frac{1}{n!}\int\cdots\int_{\mathcal{H}^n} \sum_{\boldsymbol{\epsilon}\in\{\pm 1\}^n} \mathbf{1}\,(P \in D)\,\mathbb{P}(\eta \cap P = \emptyset)\Theta(\mathrm{d}H_n)\cdots\Theta(\mathrm{d}H_1)$$

$$\mathbb{P}(\eta \cap P = \emptyset) = \mathbb{P}\left(\eta\left(\mathcal{H}_P\right) = 0\right)$$

$\mathcal{H}_P = \{H$ hyperplane $\mid H \cap P \neq \emptyset\}$



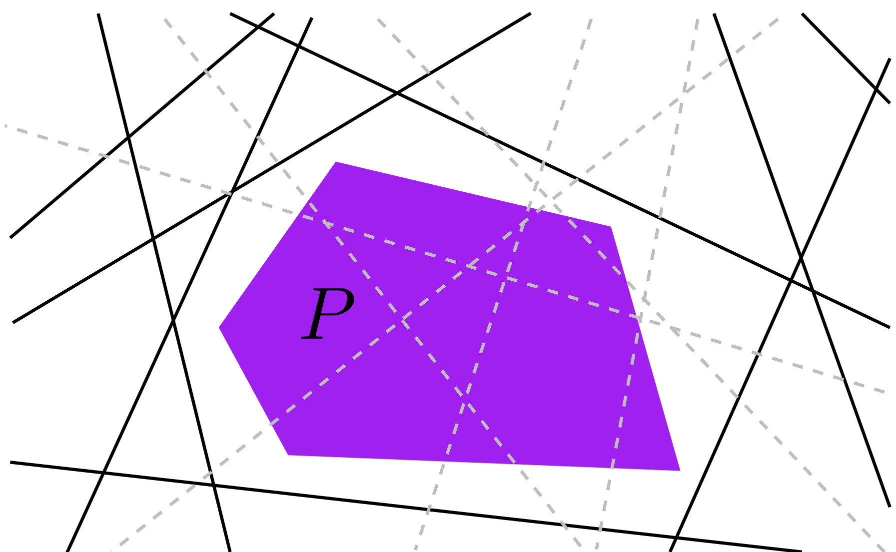
Cells With n Facets

Let $D \subset \mathcal{P}_n$. number of cells of X in D $\quad P = \bigcap_{i=1}^{n} H_i^{\epsilon_i}$

$$\mathbb{E}X(D) = \frac{1}{n!}\int \cdots \int_{\mathcal{H}^n} \sum_{\epsilon \in \{\pm 1\}^n} \mathbf{1}\,(P \in D)\,\mathbb{P}(\eta \cap P = \emptyset)\Theta(\mathrm{d}H_n)\cdots\Theta(\mathrm{d}H_1)$$

$$\mathbb{P}(\eta \cap P = \emptyset) = \mathbb{P}\left(\eta\left(\mathcal{H}_P\right) = 0\right) = \exp\left(-\Theta(\mathcal{H}_P)\right)$$

$\mathcal{H}_P = \{H \text{ hyperplane} \mid H \cap P \neq \emptyset\}$



Cells With n Facets

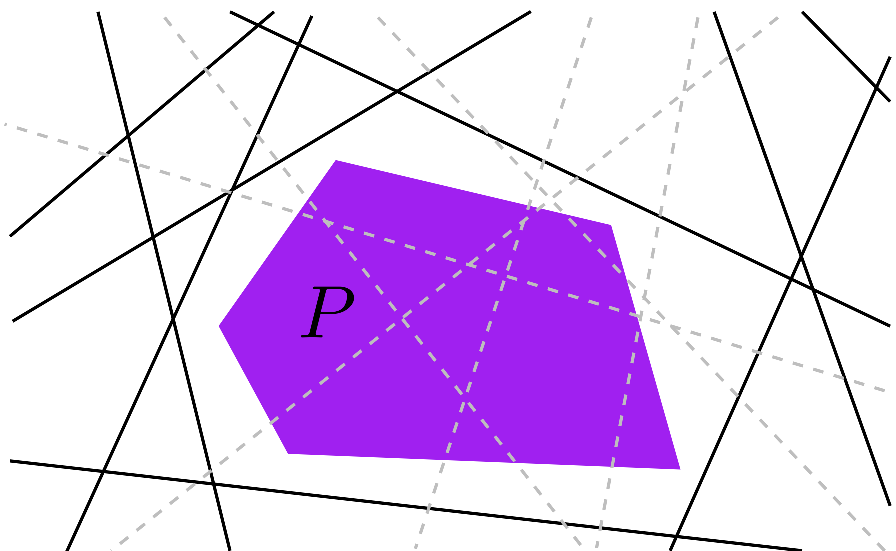
Let $D \subset \mathcal{P}_n$. number of cells of X in D $P = \bigcap_{i=1}^{n} H_i^{\epsilon_i}$

$$\mathbb{E}X(D) = \frac{1}{n!}\int \cdots \int_{\mathcal{H}^n} \sum_{\epsilon \in \{\pm 1\}^n} \mathbf{1}\,(P \in D)\, \mathbb{P}(\eta \cap P = \emptyset)\Theta(\mathrm{d}H_n)\cdots\Theta(\mathrm{d}H_1)$$

$$\mathbb{P}(\eta \cap P = \emptyset) = \mathbb{P}\left(\eta\left(\mathcal{H}_P\right) = 0\right) = \exp\left(-\Theta(\mathcal{H}_P)\right) = \exp\left(-\Phi(P)\right)$$

$$\Phi(P) := \Theta(\mathcal{H}_P) = \int_{\mathbb{S}^{d-1}} \int_0^\infty \mathbb{1}\{H(\boldsymbol{u}, t) \cap P \neq \emptyset\}\, \mathrm{d}t\, \varphi(\mathrm{d}\boldsymbol{u})$$

$\mathcal{H}_P = \{H$ hyperplane $\mid H \cap P \neq \emptyset\}$



Cells With n Facets

Let $D \subset \mathcal{P}_n$.

number of cells of X in D

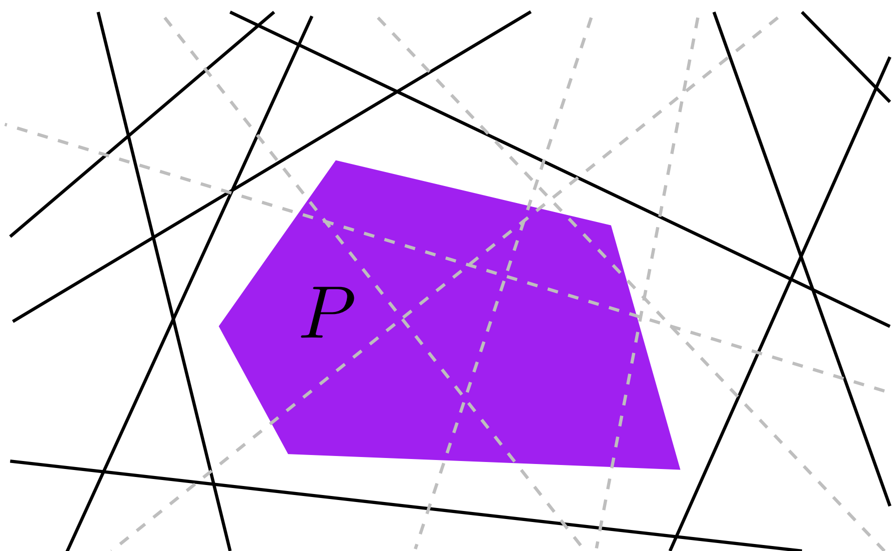
$P = \bigcap_{i=1}^{n} H_i^{\epsilon_i}$

$$\mathbb{E}X(D) = \frac{1}{n!}\int\cdots\int_{\mathcal{H}^n} \sum_{\epsilon\in\{\pm 1\}^n} \mathbf{1}\,(P \in D)\,\mathbb{P}(\eta \cap P = \emptyset)\Theta(\mathrm{d}H_n)\cdots\Theta(\mathrm{d}H_1)$$

$$\mathbb{P}(\eta \cap P = \emptyset) = \mathbb{P}\left(\eta\left(\mathcal{H}_P\right) = 0\right) = \exp\left(-\Theta(\mathcal{H}_P)\right) = \exp\left(-\Phi(P)\right)$$

$$\Phi(P) := \Theta(\mathcal{H}_P) = \int_{\mathbb{S}^{d-1}}\int_0^{\infty} \mathbb{1}\{H(\boldsymbol{u},t)\cap P \neq \emptyset\}\,\mathrm{d}t\,\varphi(\mathrm{d}\boldsymbol{u})$$

$\mathcal{H}_P = \{H$ hyperplane $\mid H \cap P \neq \emptyset\}$

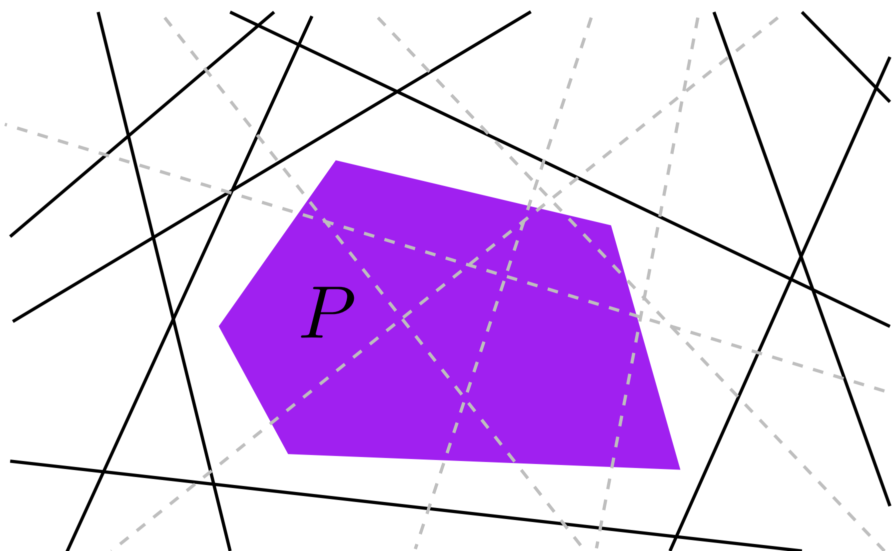


$$\Phi : \mathcal{P} \to (0,\infty)$$
$$\mathbb{P}(\eta \cap P = \emptyset) = \exp(-\Phi(P))$$
$$\Phi(tP + x) = t\Phi(P)$$

Cells With n Facets

Let $D \subset \mathcal{P}_n$. number of cells of X in D $\quad P = \bigcap_{i=1}^n H_i^{\epsilon_i}$

$$\mathbb{E}X(D) = \frac{1}{n!}\int \cdots \int_{\mathcal{H}^n} \sum_{\boldsymbol{\epsilon} \in \{\pm 1\}^n} \mathbf{1}\left(P \in D\right) \mathbb{P}(\eta \cap P = \emptyset) \Theta(\mathrm{d}H_n) \cdots \Theta(\mathrm{d}H_1)$$

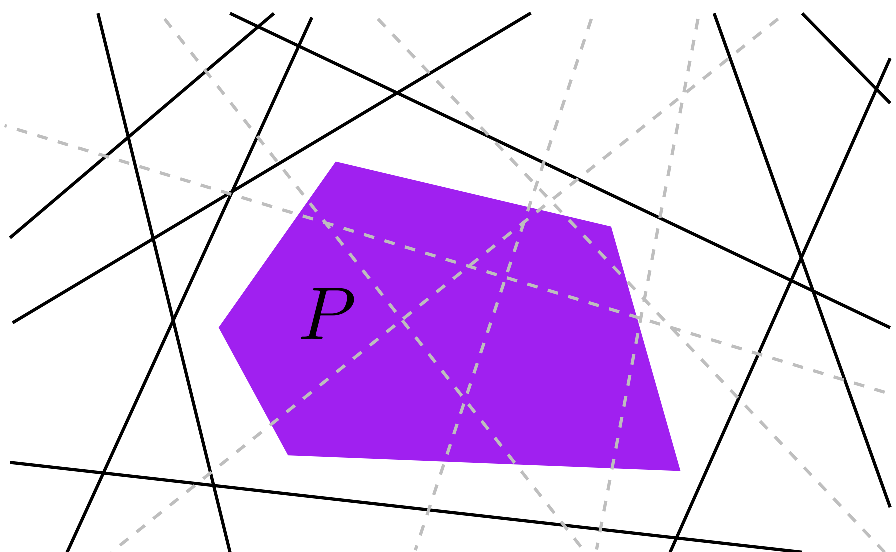


$$\Phi : \mathcal{P} \to (0, \infty)$$
$$\mathbb{P}(\eta \cap P = \emptyset) = \exp(-\Phi(P))$$
$$\Phi(tP + x) = t\Phi(P)$$

Cells With n Facets

Let $D \subset \mathcal{P}_n$. number of cells of X in D $P = \bigcap_{i=1}^n H_i^{\epsilon_i}$

$$\mathbb{E}X(D) = \frac{1}{n!}\int \cdots \int_{\mathcal{H}^n} \sum_{\boldsymbol{\epsilon} \in \{\pm 1\}^n} \mathbf{1}\left(P \in D\right) \exp\left(-\Phi\left(P\right)\right) \Theta(\mathrm{d}H_n) \cdots \Theta(\mathrm{d}H_1)$$



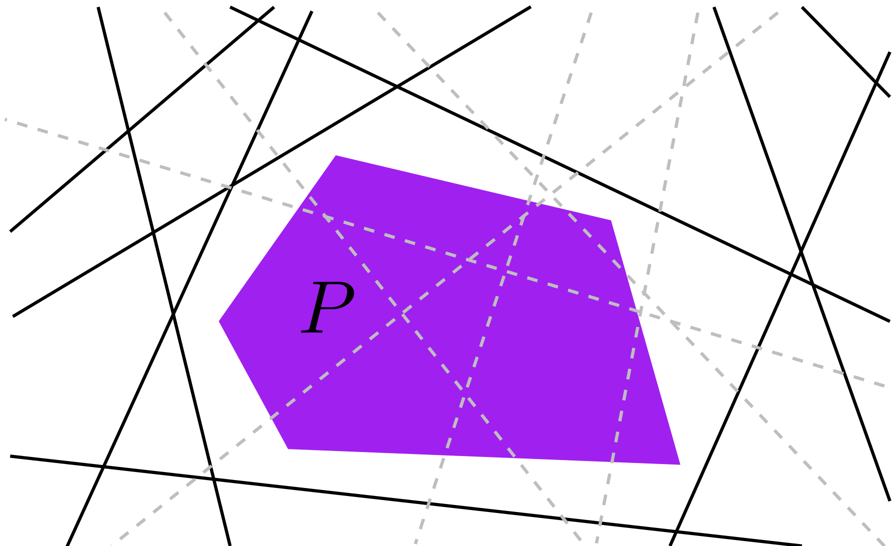
$$\Phi : \mathcal{P} \to (0, \infty)$$
$$\mathbb{P}(\eta \cap P = \emptyset) = \exp(-\Phi(P))$$
$$\Phi(tP + x) = t\Phi(P)$$

Cells With n Facets

Let $D \subset \mathcal{P}_n$. number of cells of X in D $P = \bigcap_{i=1}^n H_i^{\epsilon_i}$

$$\mathbb{E}X(D) = \int_{\mathcal{P}_n} \mathbf{1}\left(P \in D\right) \exp\left(-\Phi\left(P\right)\right) \Theta_n(\mathrm{d}P)$$

where $\Theta_n(\cdot) := \frac{1}{n!}\int \cdots \int_{\mathcal{H}^n} \sum_{\epsilon \in \{\pm 1\}^n} \mathbf{1}\left(P \in \cdot\right) \Theta(\mathrm{d}H_n) \cdots \Theta(\mathrm{d}H_1)$



Θ_n measure on $\mathcal{P}_n$
$\Theta_n(t \cdot + x) = t^n \Theta_n(\cdot)$

$\Phi : \mathcal{P} \to (0, \infty)$
$\mathbb{P}(\eta \cap P = \emptyset) = \exp(-\Phi(P))$
$\Phi(tP + x) = t\Phi(P)$

Decomposition of the Measure Θ_n

Θ_n measure on $\mathcal{P}_n$

$$\Theta_n(t\cdot+x) = t^n\Theta_n(\cdot)$$

$$\Phi : \mathcal{P} \to (0,\infty)$$

$$\mathbb{P}(\eta \cap P = \emptyset) = \exp(-\Phi(P))$$

$$\Phi(tP+x) = t\Phi(P)$$

Decomposition of the Measure Θ_n

Set of centred n-topes with Φ-content 1

$$\begin{aligned}
\mathfrak{h}_n : \mathcal{P}_n &\xrightarrow{\sim} \mathbb{R}^d \times (0,\infty) \times \mathcal{P}^1_{n,\mathfrak{c}} \\
P &\mapsto \left(\mathfrak{c}(P), \Phi(P), \underbrace{\Phi(P)^{-1}(P - \mathfrak{c}(P))}_{\mathfrak{s}(P)}\right),
\end{aligned}$$

Shape of P

Θ_n measure on $\mathcal{P}_n$

$\Theta_n(t \cdot + x) = t^n \Theta_n(\cdot)$

$\Phi : \mathcal{P} \to (0,\infty)$

$\mathbb{P}(\eta \cap P = \emptyset) = \exp(-\Phi(P))$

$\Phi(tP + x) = t\Phi(P)$

Decomposition of the Measure Θ_n

Set of centred n-topes with Φ-content 1

$$\begin{aligned} \mathfrak{h}_n : \mathcal{P}_n &\overset{\sim}{\to} \mathbb{R}^d \times (0,\infty) \times \mathcal{P}^1_{n,\mathfrak{c}} \\ P &\mapsto \Big(\mathfrak{c}(P), \Phi(P), \underbrace{\Phi(P)^{-1}(P - \mathfrak{c}(P))}_{\mathfrak{s}(P)}\Big), \end{aligned}$$

Shape of P

Θ_n measure on $\mathcal{P}_n$

$$\Theta_n(t \cdot + x) = t^n \Theta_n(\cdot)$$

$$\mathfrak{s}(tP + x) = \mathfrak{s}(P)$$

$$\Phi : \mathcal{P} \to (0,\infty)$$
$$\mathbb{P}(\eta \cap P = \emptyset) = \exp(-\Phi(P))$$
$$\Phi(tP + x) = t\Phi(P)$$

$$\mathfrak{c}(tP + x) = t\mathfrak{c}(P)$$

Decomposition of the Measure Θ_n

Set of centred n-topes with Φ-content 1

$$\begin{aligned}\mathfrak{h}_n : \mathcal{P}_n &\overset{\sim}{\to} \mathbb{R}^d \times (0,\infty) \times \mathcal{P}^1_{n,\mathfrak{c}} \\ P &\mapsto \Big(\mathfrak{c}(P), \Phi(P), \underbrace{\Phi(P)^{-1}(P - \mathfrak{c}(P))}_{\mathfrak{s}(P)}\Big),\end{aligned}$$

Shape of P

pushforward

Lebesgue measure

$$\boxed{\mathfrak{h}_n(\Theta_n) = \lambda_d \otimes \lambda_{1,n-d} \otimes \Theta^1_{n,\mathfrak{c}}}$$

$\lambda_{1,n-d}([0,a]) = a^{n-d}$

$\Theta^1_{n,\mathfrak{c}}(\cdot) = \Theta_n\big((0,1)\cdot + [0,1]^d\big)$

Θ_n measure on $\mathcal{P}_n$

$\Theta_n(t \cdot + x) = t^n \Theta_n(\cdot)$

$\Phi : \mathcal{P} \to (0,\infty)$

$\mathbb{P}(\eta \cap P = \emptyset) = \exp(-\Phi(P))$

$\Phi(tP + x) = t\Phi(P)$

$\mathfrak{s}(tP + x) = \mathfrak{s}(P)$

$\mathfrak{c}(tP + x) = t\mathfrak{c}(P)$

Decomposition of the Measure Θ_n

Set of centred n-topes with Φ-content 1

$$\begin{aligned} \mathfrak{h}_n : \mathcal{P}_n &\xrightarrow{\sim} \mathbb{R}^d \times (0,\infty) \times \mathcal{P}^1_{n,\mathfrak{c}} \\ P &\mapsto \Big(\mathfrak{c}(P), \Phi(P), \underbrace{\Phi(P)^{-1}(P - \mathfrak{c}(P))}_{\mathfrak{s}(P)}\Big), \end{aligned}$$

Shape of P

pushforward

Lebesgue measure

$$\boxed{\mathfrak{h}_n(\Theta_n) = \lambda_d \otimes \lambda_{1,n-d} \otimes \Theta^1_{n,\mathfrak{c}}}$$

$\lambda_{1,n-d}([0,a]) = a^{n-d}$

$\Theta^1_{n,\mathfrak{c}}(\cdot) = \Theta_n\big((0,1)\cdot + [0,1]^d\big)$

Decomposition of the Measure Θ_n

Set of centred n-topes with Φ-content 1

$$\begin{aligned}\mathfrak{h}_n : \mathcal{P}_n &\overset{\sim}{\to} \mathbb{R}^d \times (0,\infty) \times \mathcal{P}^1_{n,\mathfrak{c}} \\ P &\mapsto \Big(\mathfrak{c}(P), \Phi(P), \underbrace{\Phi(P)^{-1}(P - \mathfrak{c}(P))}_{\mathfrak{s}(P)}\Big),\end{aligned}$$

Shape of P

pushforward

Lebesgue measure

$$\boxed{\mathfrak{h}_n(\Theta_n) = \lambda_d \otimes \lambda_{1,n-d} \otimes \Theta^1_{n,\mathfrak{c}}}$$

$\lambda_{1,n-d}([0,a]) = a^{n-d}$

$\Theta^1_{n,\mathfrak{c}}(\cdot) = \Theta_n\big((0,1)\cdot + [0,1]^d\big)$

$$\mathbb{E}X(D) = \int_{\mathcal{P}_n} \mathbf{1}(P \in D) \exp(-\Phi(P))\, \Theta_n(\mathrm{d}P)$$

Decomposition of the Measure Θ_n

Set of centred n-topes with Φ-content 1

$$\begin{aligned} \mathfrak{h}_n : \mathcal{P}_n &\overset{\sim}{\to} \mathbb{R}^d \times (0,\infty) \times \mathcal{P}^1_{n,\mathfrak{c}} \\ P &\mapsto \Big(\mathfrak{c}(P), \Phi(P), \underbrace{\Phi(P)^{-1}(P - \mathfrak{c}(P))}_{\mathfrak{s}(P)}\Big), \end{aligned}$$

Shape of P

pushforward

Lebesgue measure

$$\boxed{\mathfrak{h}_n(\Theta_n) = \lambda_d \otimes \lambda_{1,n-d} \otimes \Theta^1_{n,\mathfrak{c}}}$$

$\lambda_{1,n-d}([0,a]) = a^{n-d}$

$\Theta^1_{n,\mathfrak{c}}(\cdot) = \Theta_n\big((0,1)\cdot + [0,1]^d\big)$

$$\mathbb{E}X(D) = \int_{\mathcal{P}^1_{n,\mathfrak{c}}} \int_{(0,\infty)} \int_{\mathbb{R}^d} \mathbf{1}\left(\mathfrak{h}_n^{-1}(c,t,s) \in D\right) e^{-t} t^{n-d-1} \mathrm{d}c \, \mathrm{d}t \, \Theta^1_{n,\mathfrak{c}}(\mathrm{d}P)$$

Decomposition of the Measure Θ_n

Set of centred n-topes with Φ-content 1

$$\mathfrak{h}_n : \mathcal{P}_n \overset{\sim}{\to} \mathbb{R}^d \times (0,\infty) \times \mathcal{P}^1_{n,\mathfrak{c}}$$

$$P \mapsto \left(\mathfrak{c}(P), \Phi(P), \underbrace{\Phi(P)^{-1}(P - \mathfrak{c}(P))}_{\mathfrak{s}(P)} \right),$$

Shape of P

pushforward

Lebesgue measure

$$\boxed{\mathfrak{h}_n(\Theta_n) = \lambda_d \otimes \lambda_{1,n-d} \otimes \Theta^1_{n,\mathfrak{c}}}$$

$\lambda_{1,n-d}([0,a]) = a^{n-d}$

$\Theta^1_{n,\mathfrak{c}}(\cdot) = \Theta_n\big((0,1)\cdot + [0,1]^d\big)$

$$\mathbb{E}X(D) = \int_{\mathcal{P}^1_{n,\mathfrak{c}}} \int_{(0,\infty)} \int_{\mathbb{R}^d} \mathbf{1}\left(\mathfrak{h}_n^{-1}(c,t,s) \in D\right) e^{-t} t^{n-d-1} \mathrm{d}c \, \mathrm{d}t \, \Theta^1_{n,\mathfrak{c}}(\mathrm{d}P)$$

Complementary Theorem (Miles 1971)

If we condition the typical cell Z to have n facets, then

- $\Phi(Z)$ and $\mathfrak{s}(Z)$ are independent
- $\Phi(Z)$ is Gamma distributed with parameter $n-d$

$\mathbb{P}(Z \text{ has } n \text{ facets})$

$$\gamma\mathbb{P}(Z \text{ has } n \text{ facets}) = \mathbb{E}X(\mathcal{P}_{n,[0,1]^d})$$

$$= \int_{\mathcal{P}^1_{n,\mathfrak{c}}} \int_{(0,\infty)} \int_{[0,1]^d} e^{-t} t^{n-d-1} \mathrm{d}c \, \mathrm{d}t \, \Theta^1_{n,\mathfrak{c}}(\mathrm{d}P)$$

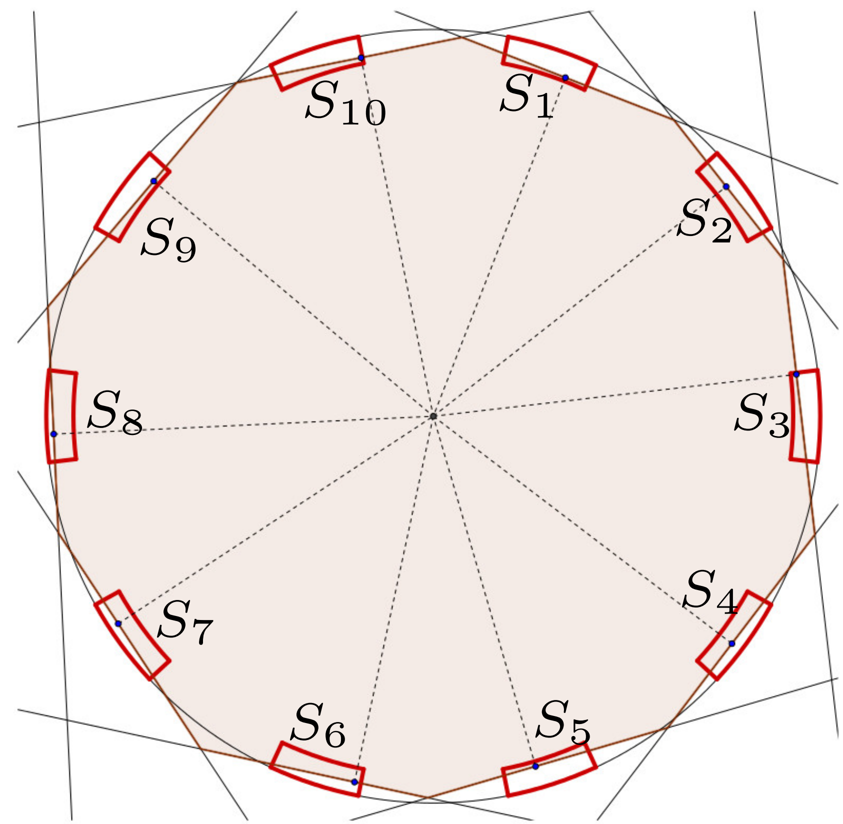
$$= (n-d-1)! \Theta^1_{n,\mathfrak{c}}(\mathcal{P}^1_{n,\mathfrak{c}})$$

$$= \frac{(n-d-1)!}{n!} \int \cdots \int_{\mathcal{H}^n} \sum_{\epsilon \in \{\pm 1\}^n} \mathbb{1}\left(\mathfrak{c}(P) \in [0,1]^d\right) \mathbb{1}\left(\Phi(P) < 1\right) \mathbb{1}\left(P \in \mathcal{P}_n\right) \Theta(\mathrm{d}H_n) \cdots \Theta(\mathrm{d}H_1)$$

Lower Bound

$$\gamma \frac{n!}{(n-d-1)!}\mathbb{P}(Z \text{ has } n \text{ facets})$$
$$= \int\cdots\int_{\mathcal{H}^n} \sum_{\epsilon\in\{\pm1\}^n} \mathbb{1}\left(\mathfrak{c}(P)\in[0,1]^d\right)\mathbb{1}\left(\Phi(P)<1\right)\mathbb{1}\left(P\in\mathcal{P}_n\right)\,\Theta(\mathrm{d}H_n)\cdots\Theta(\mathrm{d}H_1)$$
$$> n!\int\cdots\int_{\mathcal{H}^n}\mathbb{1}(H_1\in S_1)\cdots\mathbb{1}(H_n\in S_n)\,\Theta(\mathrm{d}H_n)\cdots\Theta(\mathrm{d}H_1)$$
$$= n!\Theta(S_1)^n$$
$$> n!\left(cn^{-(d-1)/(d+1)}\right)^n$$
$$\sim c^n n^{-2n/(d-1)}$$

Stirling

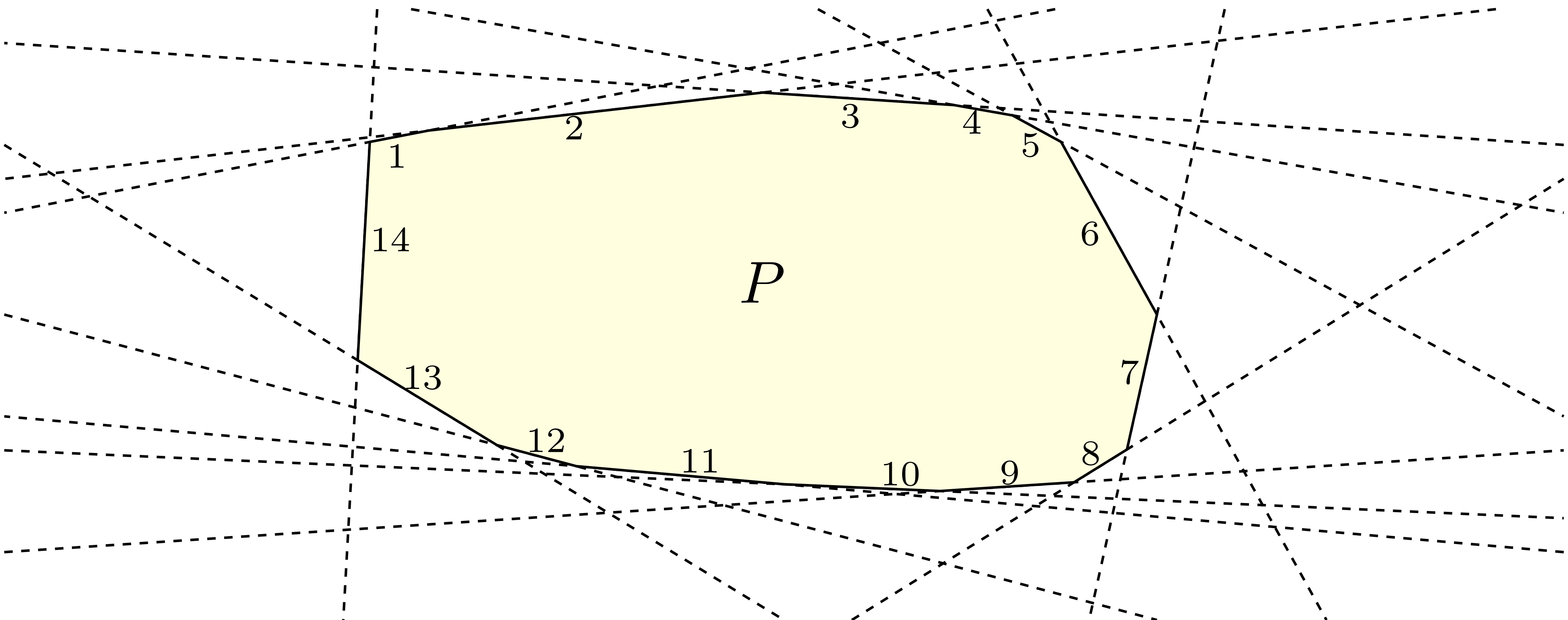


Theorem: Lower bound

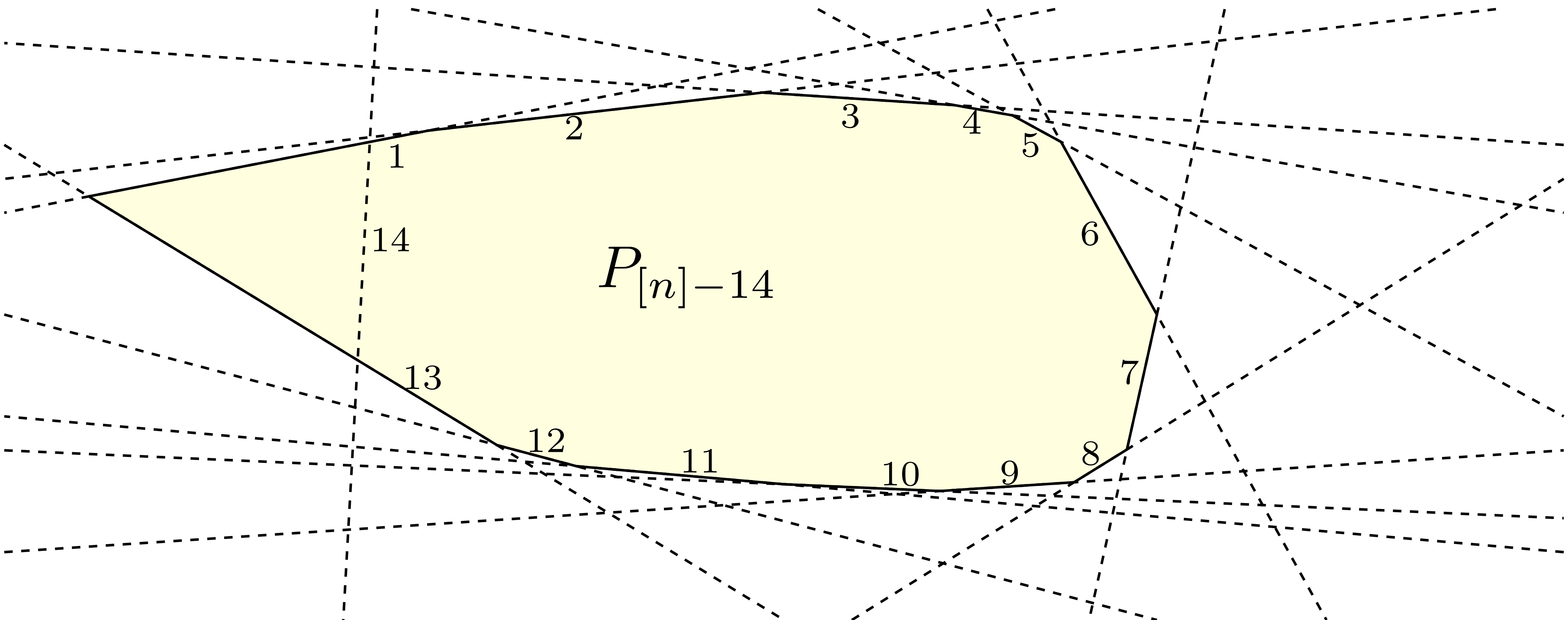
There exists a constant c_1 depending on d and φ such that for n big enough we have

$$\mathbb{P}(Z \text{ has } n \text{ facets}) > c_1^n n^{-2n/(d-1)}$$

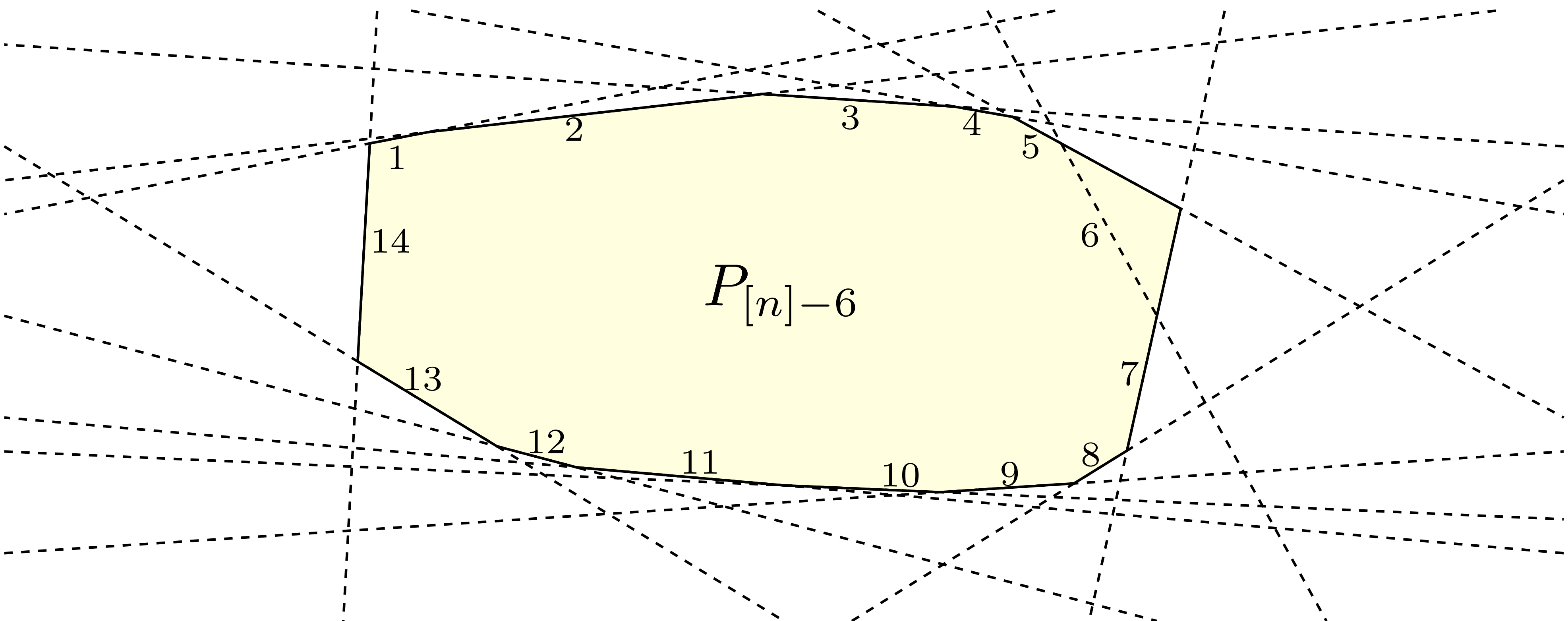
Approximation by Deleting One Facet



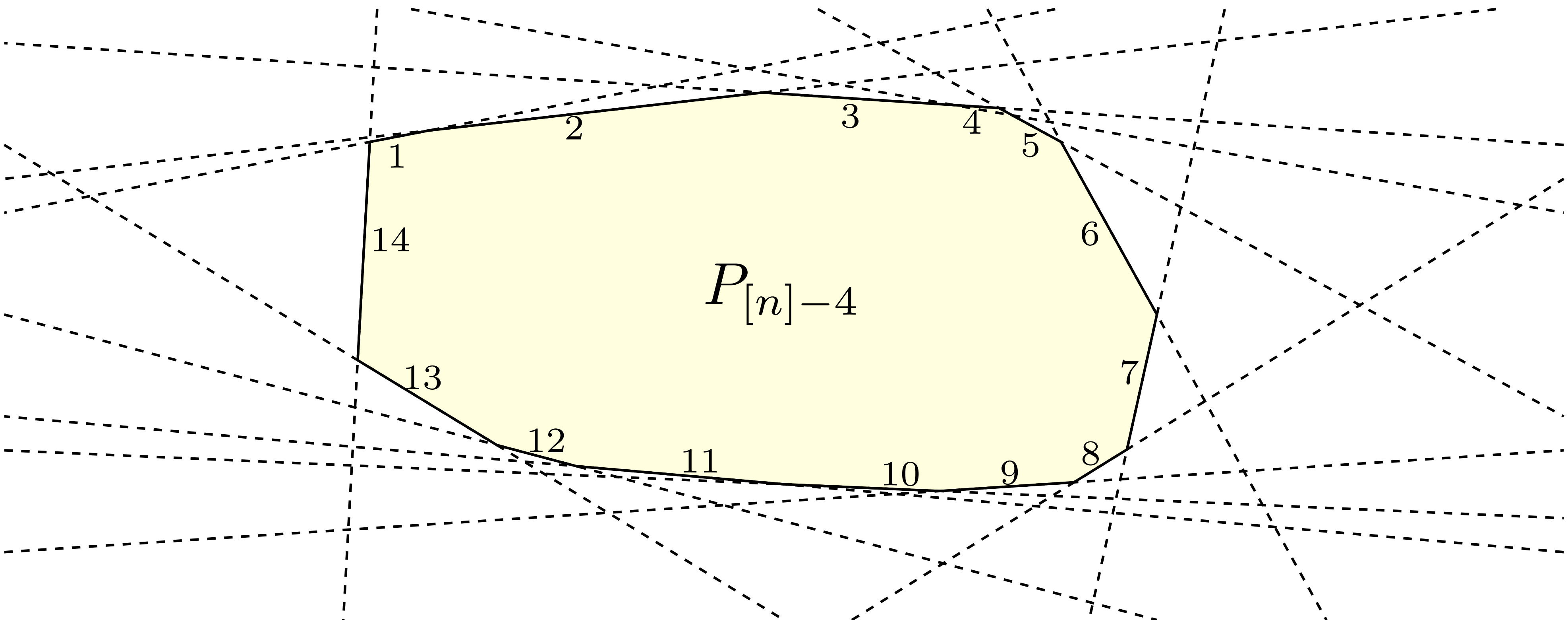
Approximation by Deleting One Facet



Approximation by Deleting One Facet



Approximation by Deleting One Facet



Approximation by Deleting One Facet

There exists a constant c_0 such that:

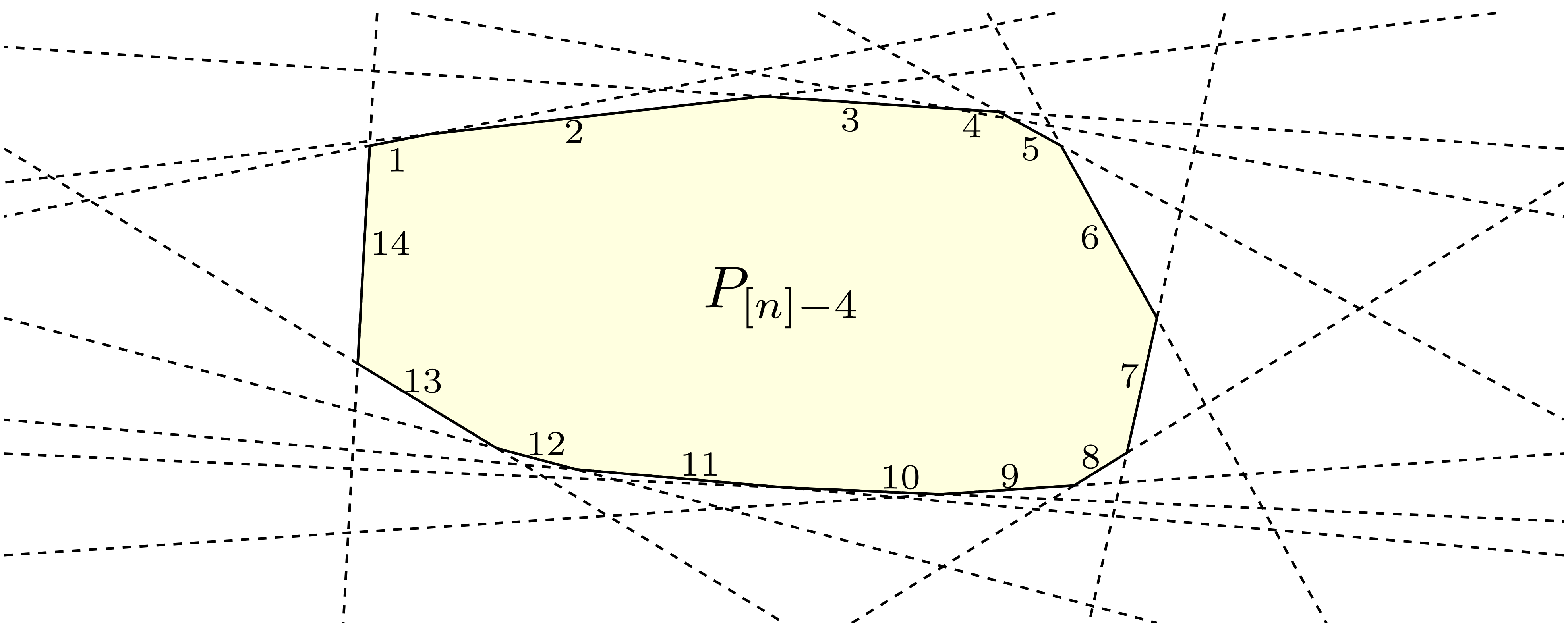
Theorem

Let $P = \cap_{i=1}^{n} H_i^-$ be a simple polytope with n big enough.
There exists a subset $J \subset [n]$ of cardinality at least $n/4$ such that for any $j \in J$ we have

$d_H \left(P, P_{[n]-j}\right) < c_0 n^{-2/(d-1)} \Phi\left(P\right)$

and

$\Phi\left(P_{[n]-j}\right) < \exp\left(c_0 n^{-1-2/(d-1)}\right) \Phi(P).$



Upper Bound

$$\gamma \frac{n!}{(n-d-1)!} \mathbb{P}(Z \text{ has } n \text{ facets})$$
$$= \int \cdots \int_{\mathcal{H}^n} \sum_{\boldsymbol{\epsilon} \in \{\pm 1\}^n} \mathbb{1}\left(\mathfrak{c}(P) \in [0,1]^d\right) \mathbb{1}\left(\Phi(P) < 1\right) \mathbb{1}\left(P \in \mathcal{P}_n\right) \Theta(\mathrm{d}H_n) \cdots \Theta(\mathrm{d}H_1)$$

Upper Bound

$$\frac{\gamma}{4}\frac{n!}{(n-d-1)!}\mathbb{P}(Z \text{ has } n \text{ facets})$$

$$< \int \cdots \int_{\mathcal{H}^n} \sum_{\epsilon \in \{\pm 1\}^n} \mathbb{1}\left(\mathfrak{c}(P) \in [0,1]^d\right) \mathbb{1}\left(\Phi(P) < 1\right) \mathbb{1}\left(P \in \mathcal{P}_n\right)$$

$$\mathbb{1}\left(d_H\left(P, P_{[n-1]}\right) < c_0 n^{-2/(d-1)}\right)$$

$$\mathbb{1}\left(\Phi\left(P_{[n-1]}\right) < \exp\left(c_0 n^{-1-2/(d-1)}\right)\right) \; \Theta(\mathrm{d}H_n) \cdots \Theta(\mathrm{d}H_1)$$

Upper Bound

$$\frac{\gamma}{4}\frac{n!}{(n-d-1)!}\mathbb{P}(Z \text{ has } n \text{ facets})$$

$$< \int \cdots \int_{\mathcal{H}^n} \sum_{\epsilon\in\{\pm1\}^n} \mathbb{1}\left(\mathfrak{c}(P) \in [0,1]^d\right) \mathbb{1}\left(\Phi(P) < 1\right) \mathbb{1}\left(P \in \mathcal{P}_n\right)$$

$$\mathbb{1}\left(d_H\left(P, P_{[n-1]}\right) < c_0 n^{-2/(d-1)}\right)$$

$$\mathbb{1}\left(\Phi\left(P_{[n-1]}\right) < \exp\left(c_0 n^{-1-2/(d-1)}\right)\right) \;\Theta(\mathrm{d}H_n)\cdots\Theta(\mathrm{d}H_1)$$

$$< \int \cdots \int_{\mathcal{H}^{n-1}} \sum_{\epsilon\in\{\pm1\}^{n-1}} \mathbb{1}\left(\mathfrak{c}(P_{[n-1]}) \in [0,2]^d\right) \mathbb{1}\left(\Phi\left(P_{[n-1]}\right) < \exp\left(c_0 n^{-1-2/(d-1)}\right)\right)$$

$$\mathbb{1}\left(P_{[n-1]} \in \mathcal{P}_{n-1}\right)$$

$$\left(\int_{\mathcal{H}} \sum_{\epsilon_n=\pm1} \mathbb{1}\left(d_H\left(P, P_{[n-1]}\right) < c_0 n^{-2/(d-1)}\right) \Theta(\mathrm{d}H_n)\right) \Theta(\mathrm{d}H_1)\cdots\Theta(\mathrm{d}H_n)$$

Upper Bound

$$\gamma \frac{n!}{(n-d-1)!} \mathbb{P}(Z \text{ has } n \text{ facets})$$
$$< \int \cdots \int_{\mathcal{H}^{n-1}} \sum_{\boldsymbol{\epsilon} \in \{\pm 1\}^{n-1}} \mathbb{1}\left(\mathfrak{c}(P_{[n-1]}) \in [0,2]^d\right) \mathbb{1}\left(\Phi\left(P_{[n-1]}\right) < \exp\left(c_0 n^{-1-2/(d-1)}\right)\right)$$
$$\mathbb{1}\left(P_{[n-1]} \in \mathcal{P}_{n-1}\right)$$
$$\left(\int_{\mathcal{H}} \sum_{\epsilon_n = \pm 1} \mathbb{1}\left(d_H\left(P, P_{[n-1]}\right) < c_0 n^{-2/(d-1)}\right) \Theta(\mathrm{d}H_n)\right) \Theta(\mathrm{d}H_{n-1}) \cdots \Theta(\mathrm{d}H_1)$$

Upper Bound

$$\gamma \frac{n!}{(n-d-1)!} \mathbb{P}(Z \text{ has } n \text{ facets})$$

$$< \int \cdots \int_{\mathcal{H}^{n-1}} \sum_{\boldsymbol{\epsilon} \in \{\pm 1\}^{n-1}} \mathbb{1}\left(\mathfrak{c}(P_{[n-1]}) \in [0,2]^d\right) \mathbb{1}\left(\Phi\left(P_{[n-1]}\right) < \exp\left(c_0 n^{-1-2/(d-1)}\right)\right)$$

$$\mathbb{1}\left(P_{[n-1]} \in \mathcal{P}_{n-1}\right)$$

$$\left(\int_{\mathcal{H}} \sum_{\epsilon_n = \pm 1} \mathbb{1}\left(d_H\left(P, P_{[n-1]}\right) < c_0 n^{-2/(d-1)}\right) \Theta(\mathrm{d}H_n)\right) \Theta(\mathrm{d}H_{n-1}) \cdots \Theta(\mathrm{d}H_1)$$

$$< 2^d \exp\left(c_0 n^{-1-2/(d-1)}\right)^{n-d-1} c_0 n^{-2/(d-1)}$$

$$\int \cdots \int_{\mathcal{H}^{n-1}} \sum_{\boldsymbol{\epsilon} \in \{\pm 1\}^{n-1}} \mathbb{1}\left(\mathfrak{c}(P_{[n-1]}) \in [0,1]^d\right) \mathbb{1}\left(\Phi\left(P_{[n-1]}\right) < 1\right) \mathbb{1}\left(P_{[n-1]} \in \mathcal{P}_{n-1}\right)$$

$$\Theta(\mathrm{d}H_{n-1}) \cdots \Theta(\mathrm{d}H_1)$$

Upper Bound

$$\gamma \frac{n!}{(n-d-1)!}\mathbb{P}(Z \text{ has } n \text{ facets})$$

$$< \int \cdots \int_{\mathcal{H}^{n-1}} \sum_{\boldsymbol{\epsilon} \in \{\pm 1\}^{n-1}} \mathbb{1}\left(\mathfrak{c}(P_{[n-1]}) \in [0,2]^d\right) \mathbb{1}\left(\Phi\left(P_{[n-1]}\right) < \exp\left(c_0 n^{-1-2/(d-1)}\right)\right)$$

$$\mathbb{1}\left(P_{[n-1]} \in \mathcal{P}_{n-1}\right)$$

$$\left(\int_{\mathcal{H}} \sum_{\epsilon_n = \pm 1} \mathbb{1}\left(d_H\left(P, P_{[n-1]}\right) < c_0 n^{-2/(d-1)}\right) \Theta(\mathrm{d}H_n)\right) \Theta(\mathrm{d}H_{n-1}) \cdots \Theta(\mathrm{d}H_1)$$

$$< 2^d \exp\left(c_0 n^{-1-2/(d-1)}\right)^{n-d-1} c_0 n^{-2/(d-1)}$$

$$\int \cdots \int_{\mathcal{H}^{n-1}} \sum_{\boldsymbol{\epsilon} \in \{\pm 1\}^{n-1}} \mathbb{1}\left(\mathfrak{c}(P_{[n-1]}) \in [0,1]^d\right) \mathbb{1}\left(\Phi\left(P_{[n-1]}\right) < 1\right) \mathbb{1}\left(P_{[n-1]} \in \mathcal{P}_{n-1}\right)$$

$$\Theta(\mathrm{d}H_{n-1}) \cdots \Theta(\mathrm{d}H_1)$$

$$< 2^d \exp\left(c_0 n^{-1-2/(d-1)}\right)^{n-d-1} c_0 n^{-2/(d-1)} \gamma \frac{(n-1)!}{(n-d-2)!}\mathbb{P}(Z \text{ has } n-1 \text{ facets})$$

Upper Bound

$$\gamma \frac{n!}{(n-d-1)!}\mathbb{P}(Z \text{ has } n \text{ facets})$$

$$< \int \cdots \int_{\mathcal{H}^{n-1}} \sum_{\epsilon \in \{\pm 1\}^{n-1}} \mathbb{1}\left(\mathfrak{c}(P_{[n-1]}) \in [0,2]^d\right) \mathbb{1}\left(\Phi\left(P_{[n-1]}\right) < \exp\left(c_0 n^{-1-2/(d-1)}\right)\right)$$

$$\mathbb{1}\left(P_{[n-1]} \in \mathcal{P}_{n-1}\right)$$

$$\left(\int_{\mathcal{H}} \sum_{\epsilon_n = \pm 1} \mathbb{1}\left(d_H\left(P, P_{[n-1]}\right) < c_0 n^{-2/(d-1)}\right) \Theta(\mathrm{d}H_n)\right) \Theta(\mathrm{d}H_{n-1}) \cdots \Theta(\mathrm{d}H_1)$$

$$< 2^d \exp\left(c_0 n^{-1-2/(d-1)}\right)^{n-d-1} c_0 n^{-2/(d-1)}$$

$$\int \cdots \int_{\mathcal{H}^{n-1}} \sum_{\epsilon \in \{\pm 1\}^{n-1}} \mathbb{1}\left(\mathfrak{c}(P_{[n-1]}) \in [0,1]^d\right) \mathbb{1}\left(\Phi\left(P_{[n-1]}\right) < 1\right) \mathbb{1}\left(P_{[n-1]} \in \mathcal{P}_{n-1}\right)$$

$$\Theta(\mathrm{d}H_{n-1}) \cdots \Theta(\mathrm{d}H_1)$$

$$< 2^d \exp\left(c_0 n^{-1-2/(d-1)}\right)^{n-d-1} c_0 n^{-2/(d-1)} \gamma \frac{(n-1)!}{(n-d-2)!} \mathbb{P}(Z \text{ has } n-1 \text{ facets})$$

Theorem: Upper bound

There exists a constant c_2 depending on d and φ such that for n big enough we have

$$\mathbb{P}(Z \text{ has } n \text{ facets}) < c_2^n n^{-2n/(d-1)}$$

Shape of a Cell With Many Facets

Hug, Reitzner and Schneider [2004,2007] studied big cells. For example in the *isotropic case* they show that for any $j = 2, \cdots, d$ and any $\varepsilon > 0$ we have

j-th intrinsic volume

$$\boxed{\mathbb{P}\left(d_H\left(\mathfrak{s}(Z), \mathbb{B}^d\right) > \varepsilon \mid V_j(Z) > a\right) \to 0 \text{ when } a \to \infty}$$

'*Big*' cells are almost spherical.

Shape of a Cell With Many Facets

Hug, Reitzner and Schneider [2004,2007] studied big cells.
For example in the *isotropic case* they show that for any $j = 2, \cdots, d$ and any $\varepsilon > 0$ we have

j-th intrinsic volume

$$\boxed{\mathbb{P}\left(d_H\left(\mathfrak{s}(Z), \mathbb{B}^d\right) > \varepsilon \mid V_j(Z) > a\right) \to 0 \text{ when } a \to \infty}$$

'*Big*' cells are almost spherical.

Conjecture

The result above remains true if you change $V_j(Z)$ by $V_1(Z)$ or $\mathrm{NumberOfFaces(Z)}$.

Shape of a Cell With Many Facets

Hug, Reitzner and Schneider [2004,2007] studied big cells. For example in the *isotropic case* they show that for any $j = 2, \cdots, d$ and any $\varepsilon > 0$ we have

j-th intrinsic volume

$$\boxed{\mathbb{P}\left(d_H\left(\mathfrak{s}(Z), \mathbb{B}^d\right) > \varepsilon \mid V_j(Z) > a\right) \to 0 \text{ when } a \to \infty}$$

'*Big*' cells are almost spherical.

Conjecture

The result above remains true if you change $V_j(Z)$ by $V_1(Z)$ or $\mathrm{NumberOfFaces(Z)}$.

We did a small step in the direction of this conjecture:

Theorem

There exists $\epsilon > 0$ such that for any $j \leq \lceil (d-1)/2 \rceil$ we have

$$\mathbb{P}\left(\frac{V_j(Z)}{V_1(Z)^j} < \epsilon \mid Z \text{ has } n \text{ facets}\right) \to 0 \text{ when } n \to \infty.$$

Cell with many facets are not '*too flat*'.

Take Home Message and Perspectives

Main Theorem

There exist constants c_1 and c_2 depending on d and φ such that for n big enough we have

$$c_1^n n^{-2n/(d-1)} < \mathbb{P}(Z \text{ has } n \text{ facets}) < c_2^n n^{-2n/(d-1)}$$

$\Rightarrow$ Explicit the dependence to d in the constants c_1 and c_2.

$\Rightarrow$ Generalization : Other kind of mosaics, the zero cell...

Theorem

There exists $\epsilon > 0$ such that for any $j \leq \lceil (d-1)/2 \rceil$ we have

$$\mathbb{P}\left(\frac{V_j(Z)}{V_1(Z)^j} < \epsilon \mid Z \text{ has } n \text{ facets} \right) \to 0 \text{ when } n \to \infty.$$

$\Rightarrow$ The conjecture is still open

Take Home Message and Perspectives

Main Theorem

There exist constants c_1 and c_2 depending on d and φ such that for n big enough we have

$$c_1^n n^{-2n/(d-1)} < \mathbb{P}(Z \text{ has } n \text{ facets}) < c_2^n n^{-2n/(d-1)}$$

$\Rightarrow$ Explicit the dependence to d in the constants c_1 and c_2.

$\Rightarrow$ Generalization : Other kind of mosaics, the zero cell...

Theorem

There exists $\epsilon > 0$ such that for any $j \leq \lceil (d-1)/2 \rceil$ we have

$$\mathbb{P}\left(\frac{V_j(Z)}{V_1(Z)^j} < \epsilon \mid Z \text{ has } n \text{ facets} \right) \to 0 \text{ when } n \to \infty.$$

$\Rightarrow$ The conjecture is still open

THANK YOU!