

Settings

- η Stationary Poisson Hyperplane Process in \mathbb{R}^d
- Θ intensity measure of η
 - intensity γ
 - directional distribution φ
- X hyperplane mosaic associated to η
- Z_0 Zero cell of X
 $Z_0 \in \mathcal{P}_0$ with distribution \mathbb{Q}_0
- $Z_{0,n}$ Zero cell of X conditioned on the event $\{Z_0 \text{ has } n \text{ facets}\}$
 $Z_{0,n} \in \mathcal{P}_{0,n}$ with distribution $\mathbb{Q}_{0,n}$

Problem

We want to give asymptotic bounds when $n \rightarrow \infty$ for
 $q_{0,n} := \mathbb{P}(Z_0 \text{ has } n \text{ facets.})$

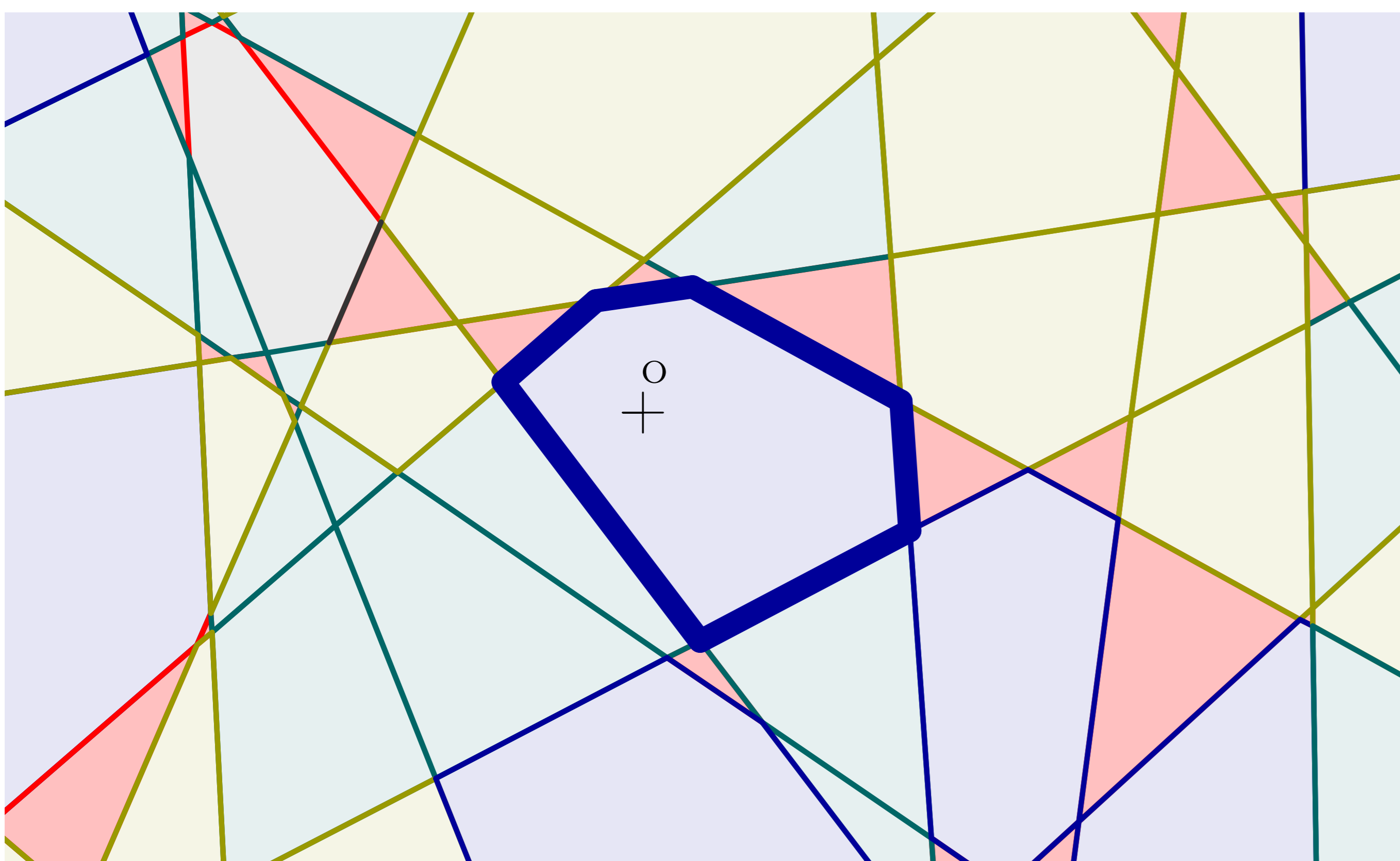


Figure: Zero cell of an isotropic mosaic

Φ -content and Shape

- ▶ Φ -content of a convex body K : $\Phi(K)$.
 $\Phi : \mathcal{C} \rightarrow (0, \infty)$
 $K \mapsto \Phi(K) := \int_{S^{d-1}} \text{width}(u, K) \varphi(du)$
Remarks: In the isotropic case Φ is the mean width. In general:
 $\mathbb{P}(K \text{ is inside a cell of } X) = e^{-\gamma \Phi(K)}$
- ▶ Shape of a convex body K : $s(K)$.
 $s : \mathcal{C} \rightarrow \mathcal{C}_1 := \{K \in \mathcal{C} \mid \Phi(K) = 1\}$
 $K \mapsto s(K) := \frac{1}{\Phi(K)} K$
- ▶ Φ and s (restricted to $\mathcal{P}_{0,n}$) induce the following natural homeomorphism
 $h : \mathcal{P}_{0,n} \rightarrow (0, \infty) \times \mathcal{P}_{0,n,1}$
 $P \mapsto h(P) := (\Phi(P), s(P))$

Θ_n : A measure on the space of n -tope \mathcal{P}_n

$$\Theta_n(\cdot) := \frac{1}{n!} \int \dots \int_{\mathcal{H}^n} \sum_{\epsilon \in \{\pm 1\}^n} \mathbb{1} \left(\bigcap_{i=1}^n H_i^{\epsilon_i} \in \cdot \right) \Theta(dH_1) \dots \Theta(dH_n)$$

We have that

$$q_{0,n} := \mathbb{P}(Z_0 \text{ has } n \text{ facets}) = \int_{\mathcal{P}_{0,n}} e^{-\gamma \Phi(P)} \Theta_n(dP)$$

and

$$\mathbb{Q}_{0,n}(\cdot) = \frac{1}{q_{0,n}} \int_{\mathcal{P}_{0,n}} \mathbb{1}(P \in \cdot) e^{-\gamma \Phi(P)} \Theta_n(dP)$$

Complementary theorem

In the literature we find that

- ▶ $\Phi(Z_{0,n})$ and $s(Z_{0,n})$ are independent
- ▶ $\Phi(Z_{0,n})$ is gamma distributed with parameters n and γ

We get additionally a description of the shape distribution:

- ▶ The probability distribution of $s(Z_{0,n}) \in \mathcal{P}_{0,n,1}$ is the normalisation of

$$\Theta_{n,1}(\cdot) := \Theta_n(h^{-1}((0, 1), \cdot))$$

Notations

- \mathcal{H} hyperplanes in \mathbb{R}^d
- H^- half space bounded by H containing the origin
- \mathcal{C} convex bodies with more than 2 points
- \mathcal{P}_0 polytopes containing 0
- $\mathcal{P}_{0,n} = \{P \in \mathcal{P}_0 \mid P \text{ has } n \text{ facets}\}$
- $\mathcal{C}_1 = \{K \in \mathcal{C} \mid \Phi(K) = 1\}$
- $\mathcal{P}_{0,1} = \{P \in \mathcal{P}_0 \mid \Phi(P) = 1\}$
- $\mathcal{P}_{0,n,1} = \{P \in \mathcal{P}_{0,n} \mid \Phi(P) = 1\}$

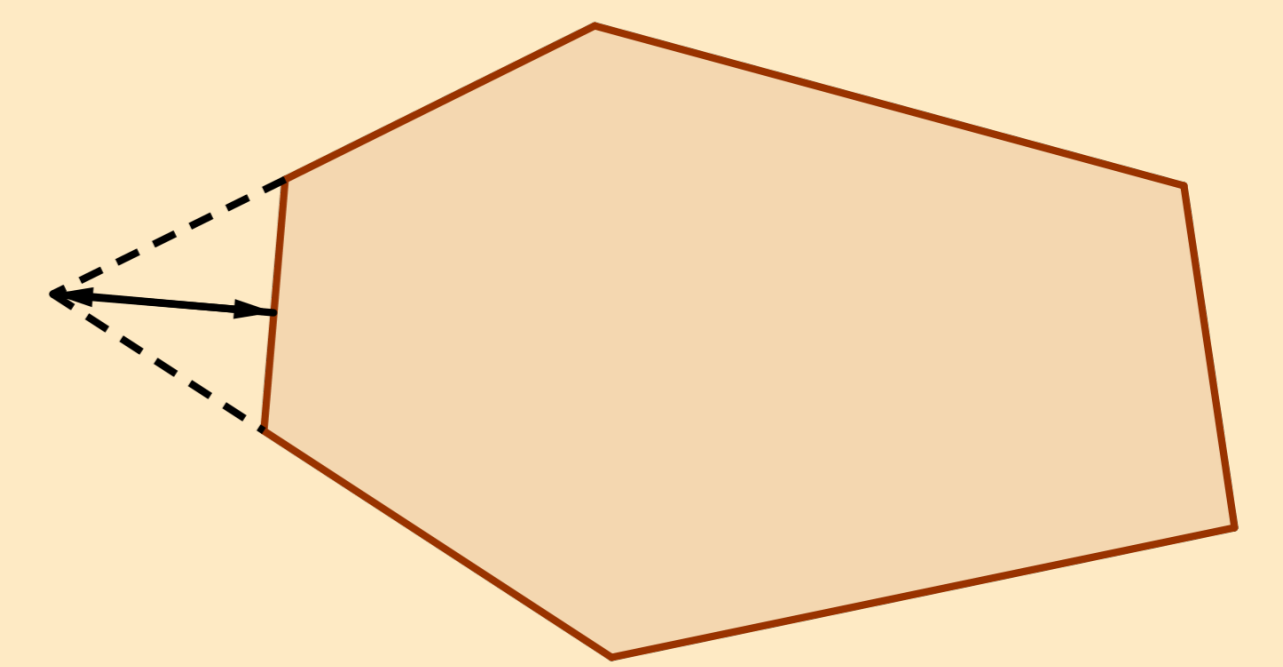
Approximation of a polytope by a polytope with less facets

There exist constants $c, n_0 > 0$ such that the following holds.
Let $P = \cap_{i=1}^n H_i^-$ be a simple polytope with n facets. If $n > n_0$, then there exists a subset $J \subset \{1, \dots, n\}$ of cardinality $n/4$ such that for any $j \in J$, the polytope $P_j = \cap_{i \neq j} H_i^-$ satisfies

$$d_H(P, P_j) < c n^{-2/(d-1)} \Phi(P),$$

and

$$\Phi(P_j) < \exp \{c n^{-1-2/(d-1)}\} \Phi(P).$$



Using the analytic expression of q_n described on the left column, the complementary theorem and the polytope approximation result above we get the following.

Proposition : Recurrence relation

There exists a constant c_0 such that for n big enough

$$q_{0,n+1} < c_0 n^{-2/(d-1)} q_{0,n}.$$

Iterating this relation implies the next upper bound.

Theorem : Upper bound

For any directional distribution φ , there exists a constant c_1 such that for n big enough

$$q_{0,n} < c_1^n n^{-2n/(d-1)}.$$

We obtain a lower bound of the q_n by construction.

Theorem : Lower bound

If φ is nice (e.g. isotropic), there exists a constant c_2 such that for n big enough

$$c_2^n n^{-2n/(d-1)} < q_{0,n}.$$

Extensions

Our work holds as well for the typical cell with the same arguments.
We would like to extend these results to

- ▶ the zero and the typical cell of other mosaics (e.g. Voronoi).
- ▶ the non stationary case.
- ▶ the lower dimensional faces of the mosaic.

A big open problem:

What is the asymptotic of the shape distribution of $Z_{0,n}$ when $n \rightarrow \infty$.

References / Further reading : a really short list

- Random mosaics:** P. Calka, chapter *Tesselations in New perspectives in stochastic geometry*, 2010
- Polytope approximation:** S. Reisner, C. Schütt, E. Werner *Dropping a vertex of a facet from a convex polytope*, 2001
- Complementary theorem:** R. Cowan, *A more comprehensive complementary theorem for analysis of Poisson point processes*, 2006
- Asymptotic of q_n in a special case:** H. J. Hilhorst, P. Calka, *Random line tessellations of the plane: statistical properties of the many-sided cells*, 2008