

Cells With Many Facets in a Hyperplane Mosaic

Gilles Bonnet,
joint work with Matthias Reitzner and Pierre Calka

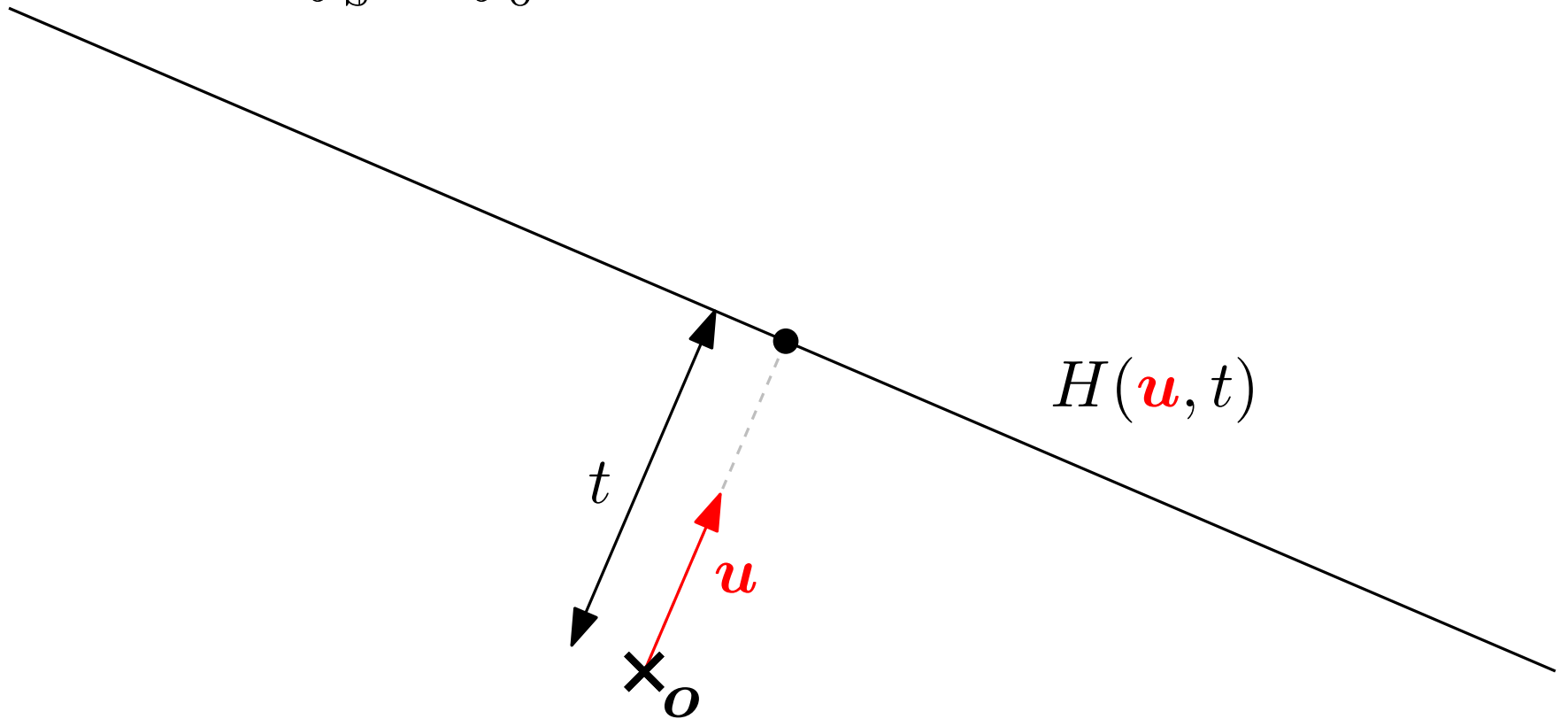
Université Littoral Côtes d'Opal
Thursday 19th November 2015



Stationary Poisson Hyperplane Mosaic in \mathbb{R}^d

η Poisson Hyperplane Process of **intensity measure** Θ
 φ **directional distribution** (even measure on \mathbb{S}^{d-1})

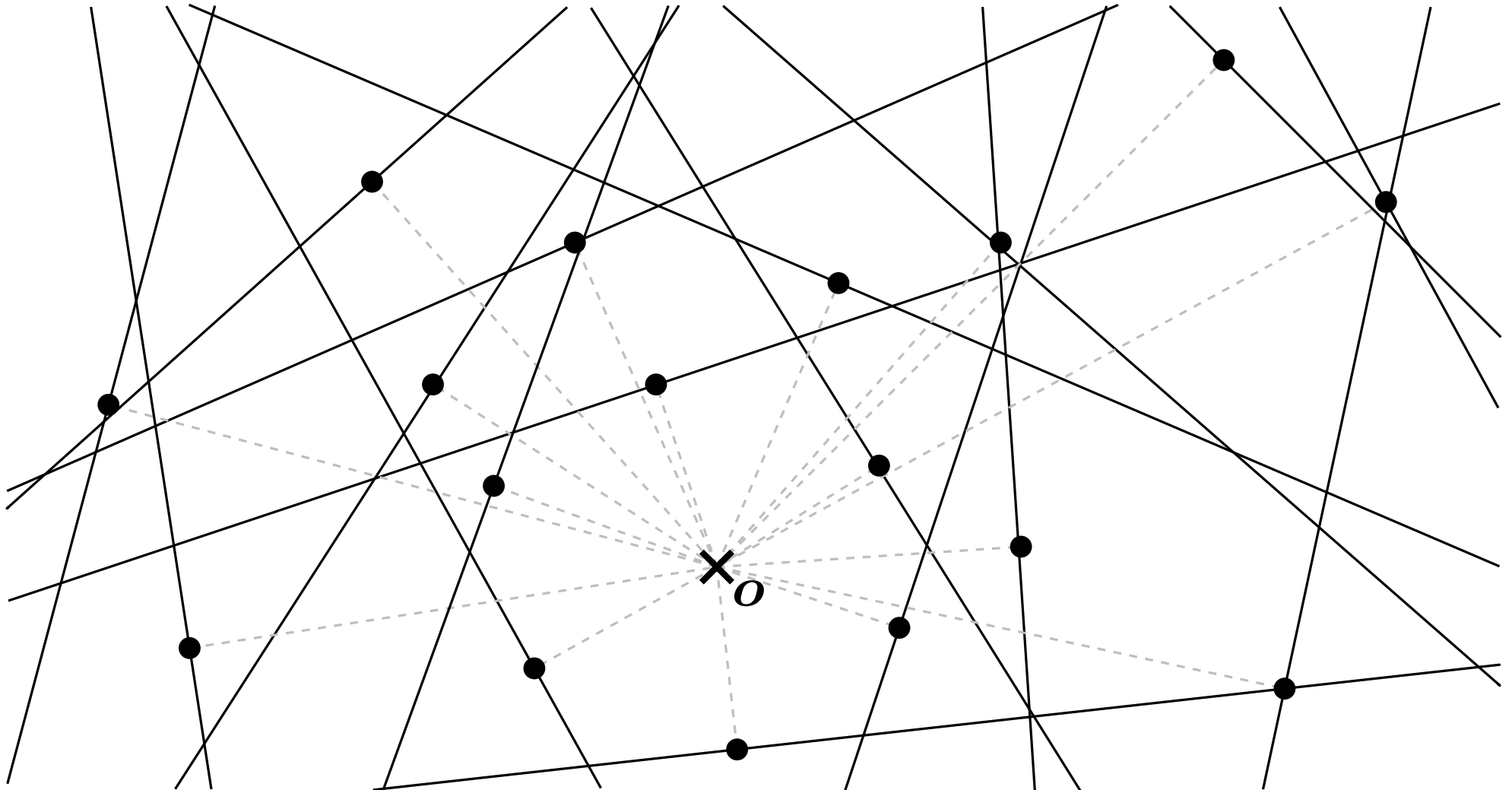
$$\Theta(\cdot) = \int_{\mathbb{S}^{d-1}} \int_0^\infty \mathbb{1}\{H(\mathbf{u}, t) \in \cdot\} dt \varphi(d\mathbf{u})$$



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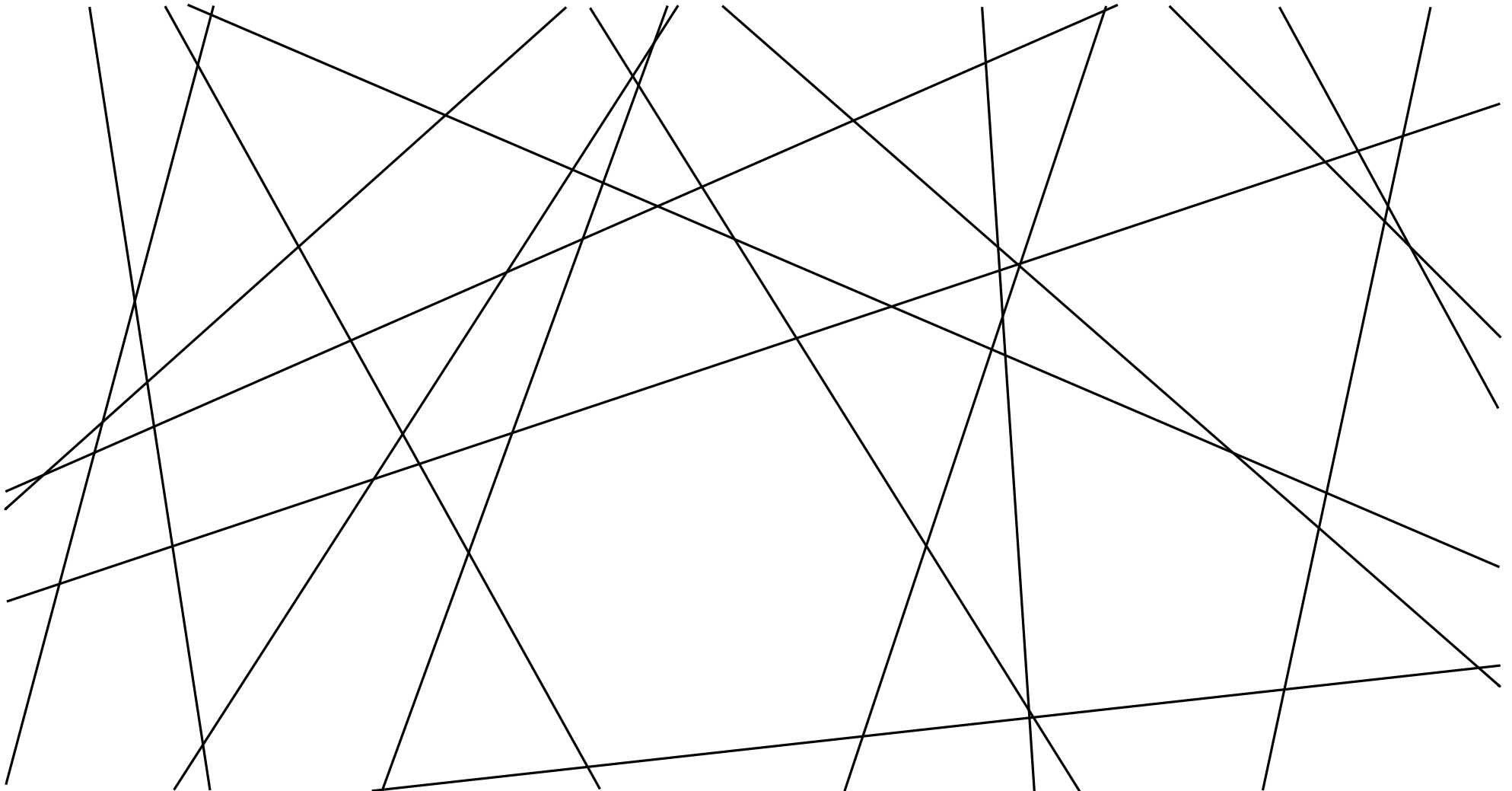
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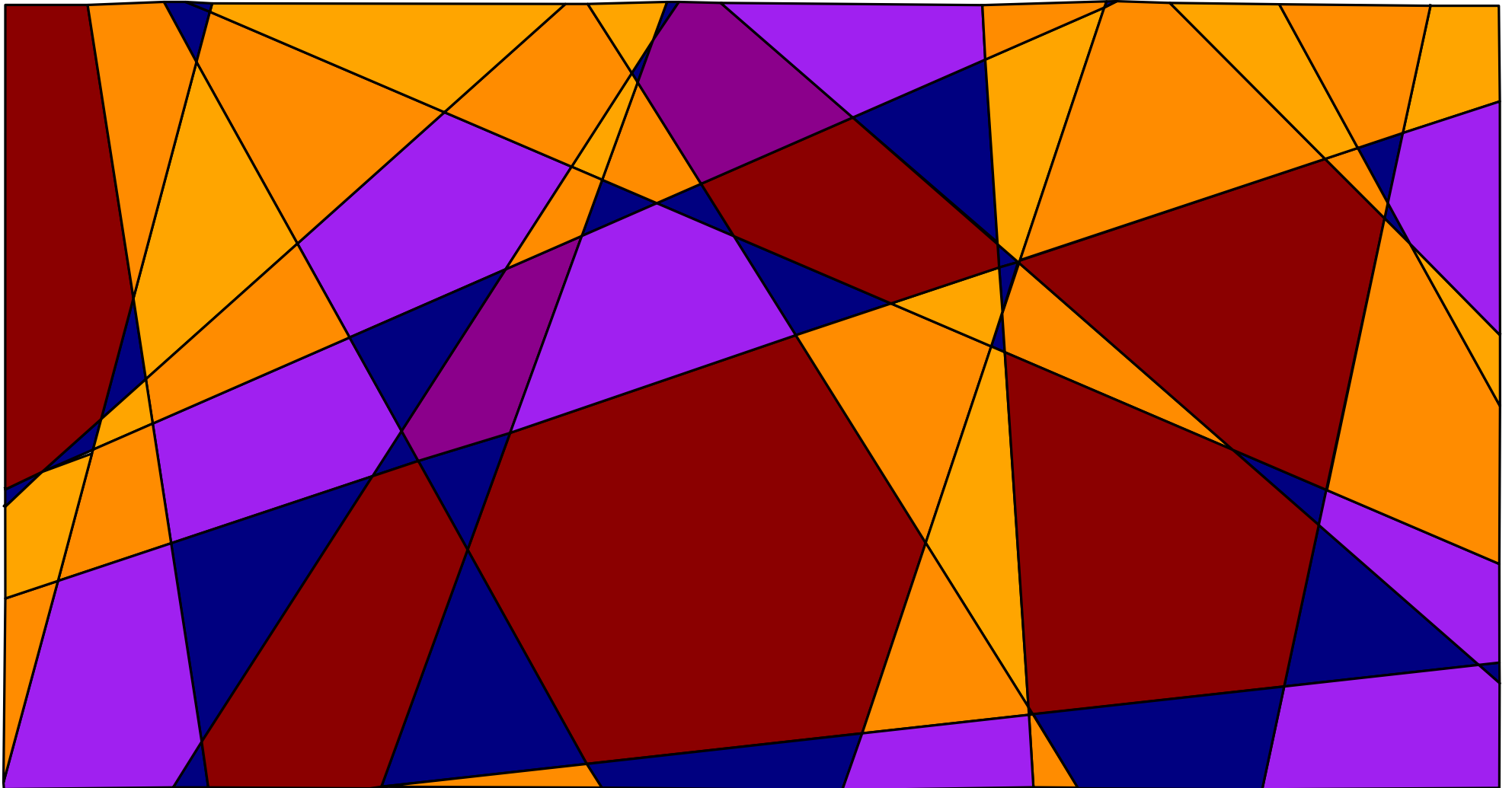
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Goal

$\mathbb{P}(Z \text{ has } n \text{ facets})?$ when $n \rightarrow \infty$

typical cell

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In a specific case it is already known:

Theorem [Calka and Hilhorst 2008]

In the 2-dimensional isotropic case we have that

$$\mathbb{P}(Z \text{ has } n \text{ facets}) \sim \alpha \beta^n n^{-2n} \sqrt{n} \text{ when } n \rightarrow \infty$$

with $\alpha = (6\pi^{5/2})^{-1}$ and $\beta = \pi^2 e^2$

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We generalize this to any dimension and nice directional distribution:

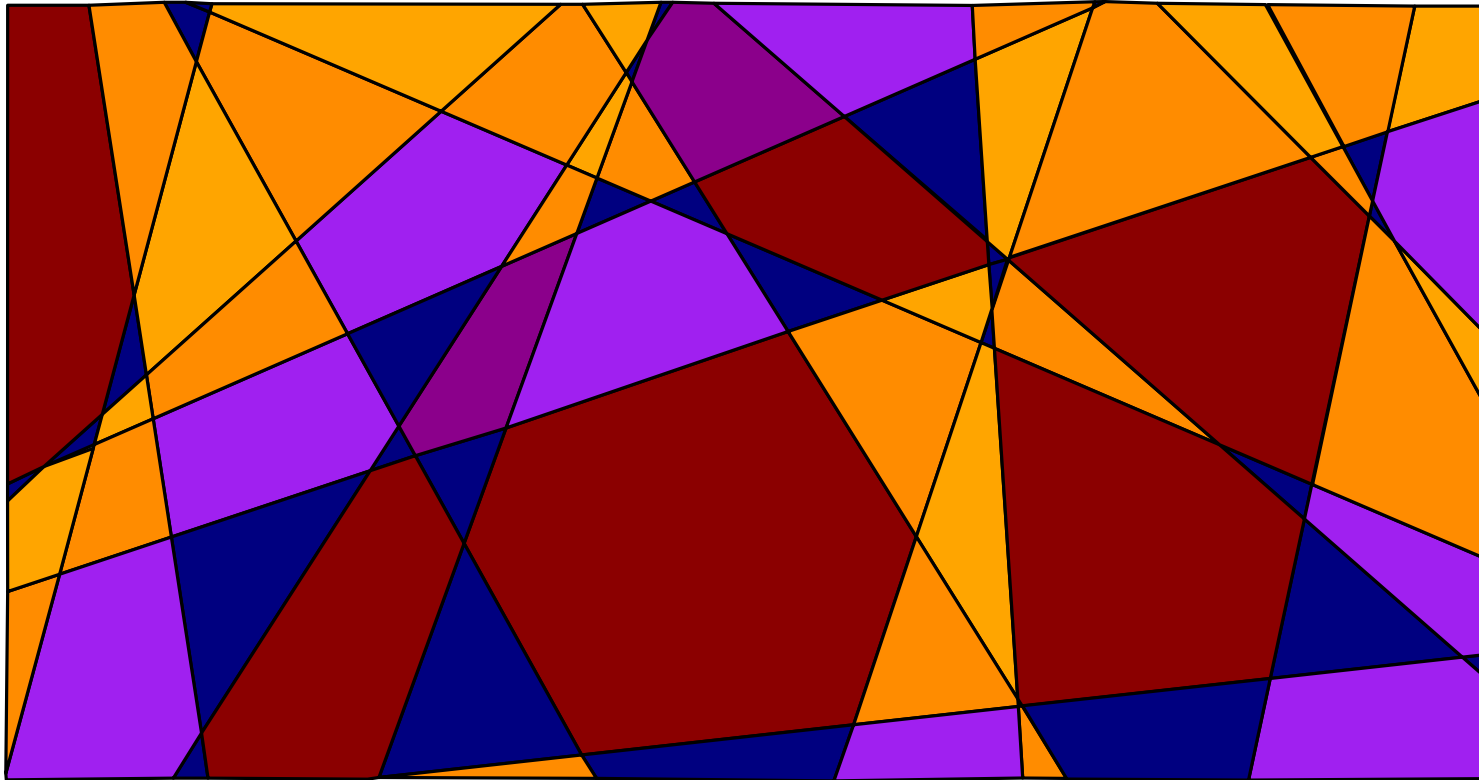
Main Theorem

There exist constants c_1 and c_2 depending on d and φ such that for n big enough we have

$$c_1^n n^{-2n/(d-1)} < \mathbb{P}(Z \text{ has } n \text{ facets}) < c_2^n n^{-2n/(d-1)}$$

Typical Cell Z

$X = X_\eta \dots$ **Mosaic:** Point Process in \mathcal{P}  Set of polytopes



Typical Cell Z

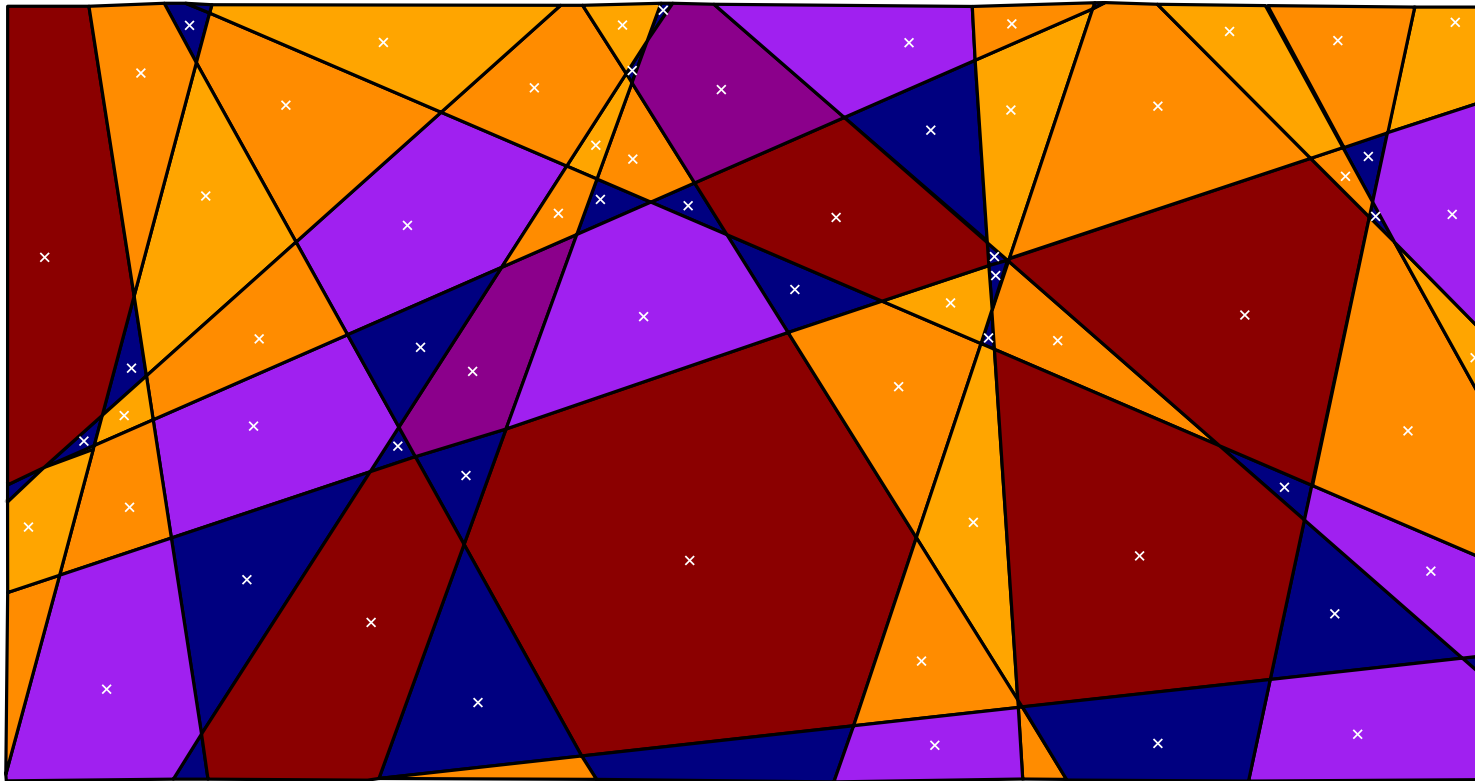
$X = X_\eta \dots$ **Mosaic:** Point Process in \mathcal{P}

$\mathbf{c} : \mathcal{P} \rightarrow \mathbb{R}^d$ a **center function**

e.g. center of mass, center of the circumball...

Set of polytopes

$$\mathbf{c}(tP + x) = t\mathbf{c}(P) + x$$



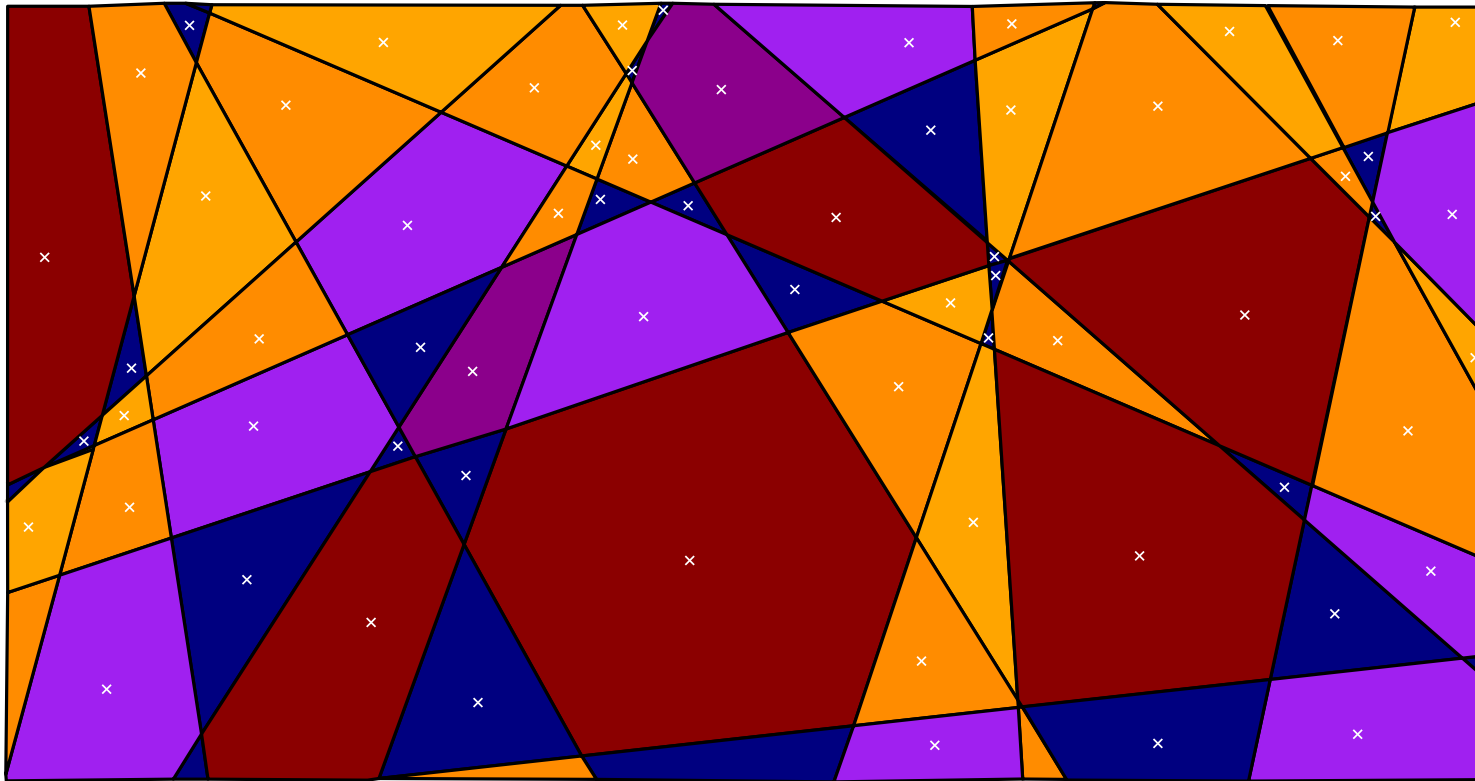
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Set of centred polytopes

$X = X_\eta \dots$ **Mosaic:** Point Process in $\mathbb{R}^d \times \mathcal{P}_c$

$\mathfrak{c} : \mathcal{P} \rightarrow \mathbb{R}^d$ a **center function**

X is a **germ-grain** process with **germs** $\mathfrak{c}(P)$ and **grains** $P - \mathfrak{c}(P)$



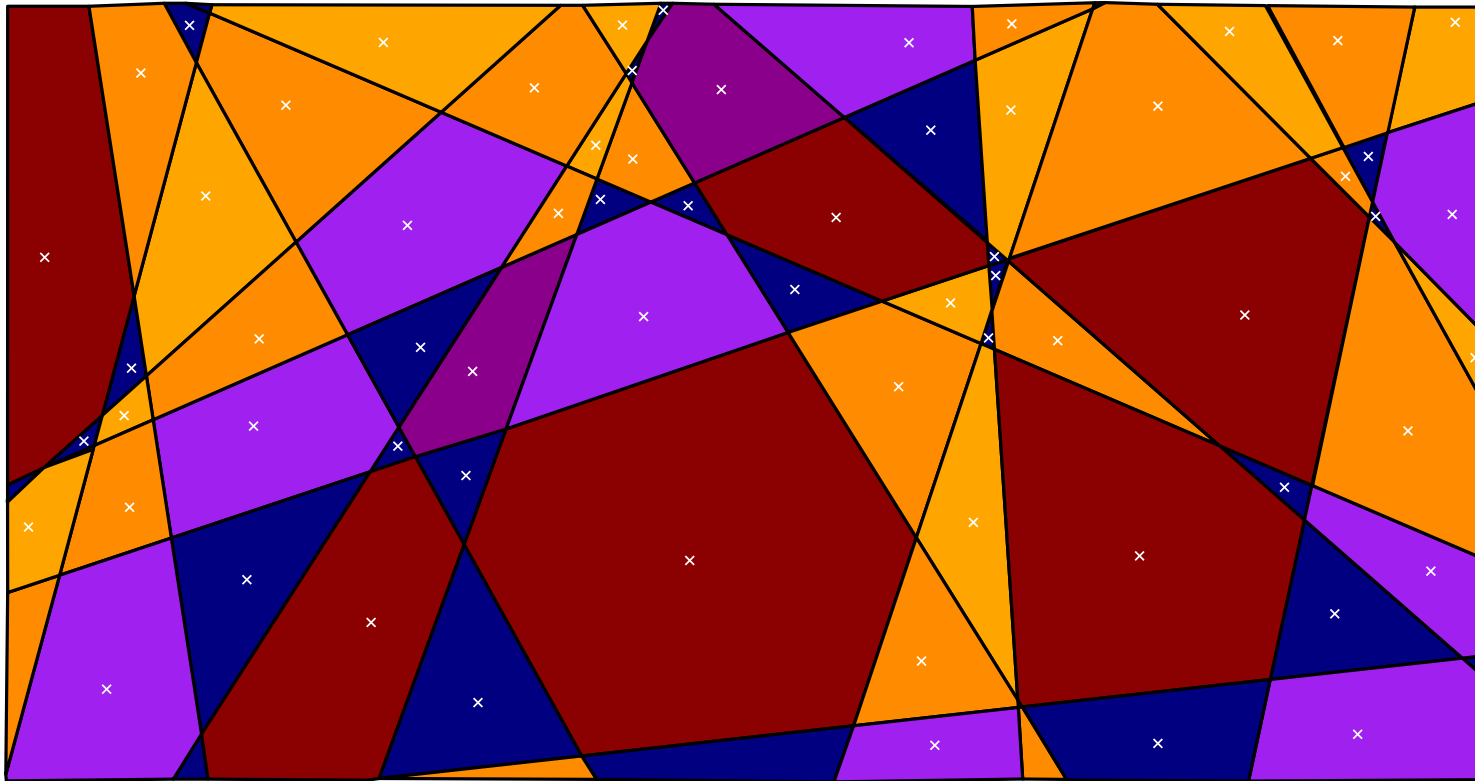
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X is a **germ-grain** process with **germs** $c(P)$ and **grains** $P - c(P)$

Its intensity measure has the form $\gamma \lambda_d \otimes \mathbb{Q}$ ← grain distribution
intensity ← Lebesgue measure



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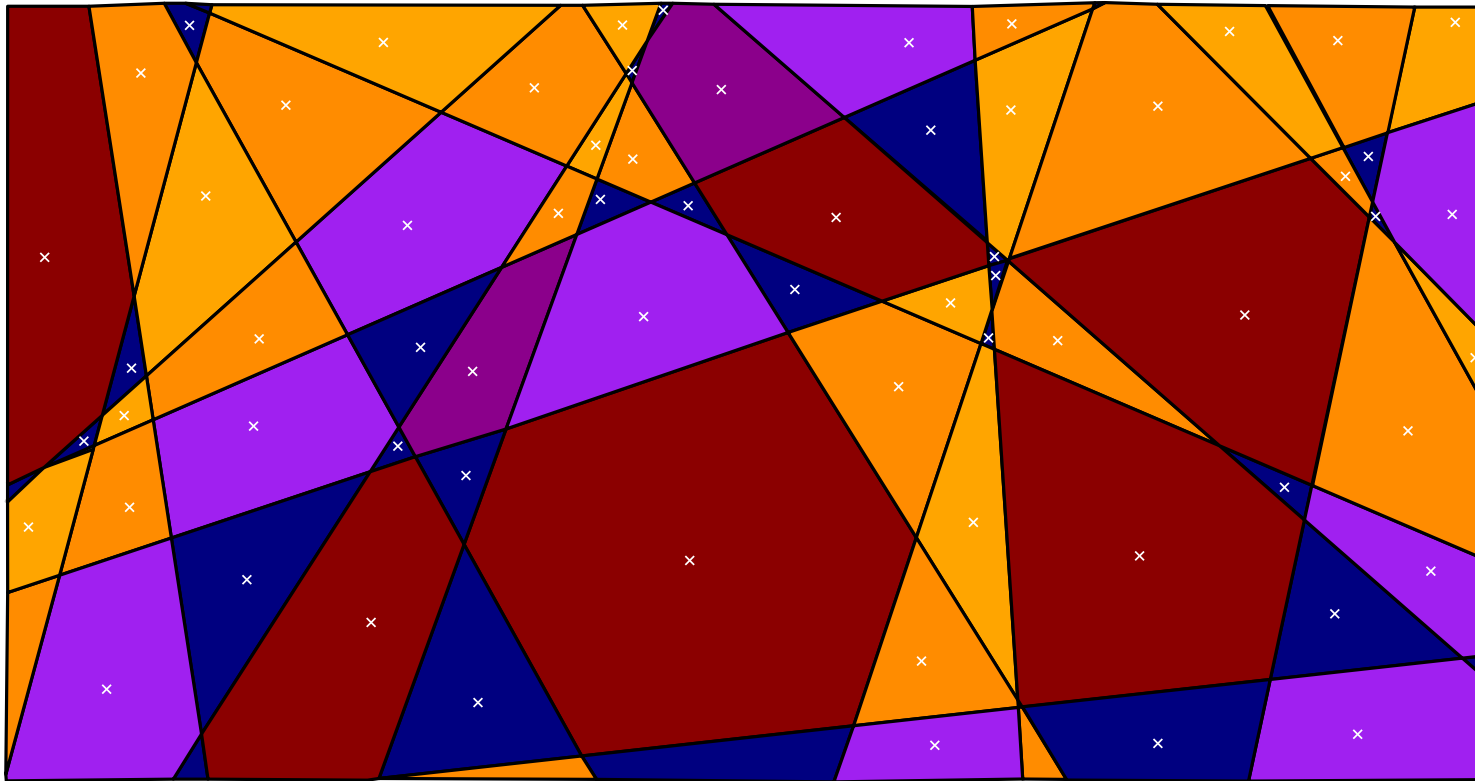
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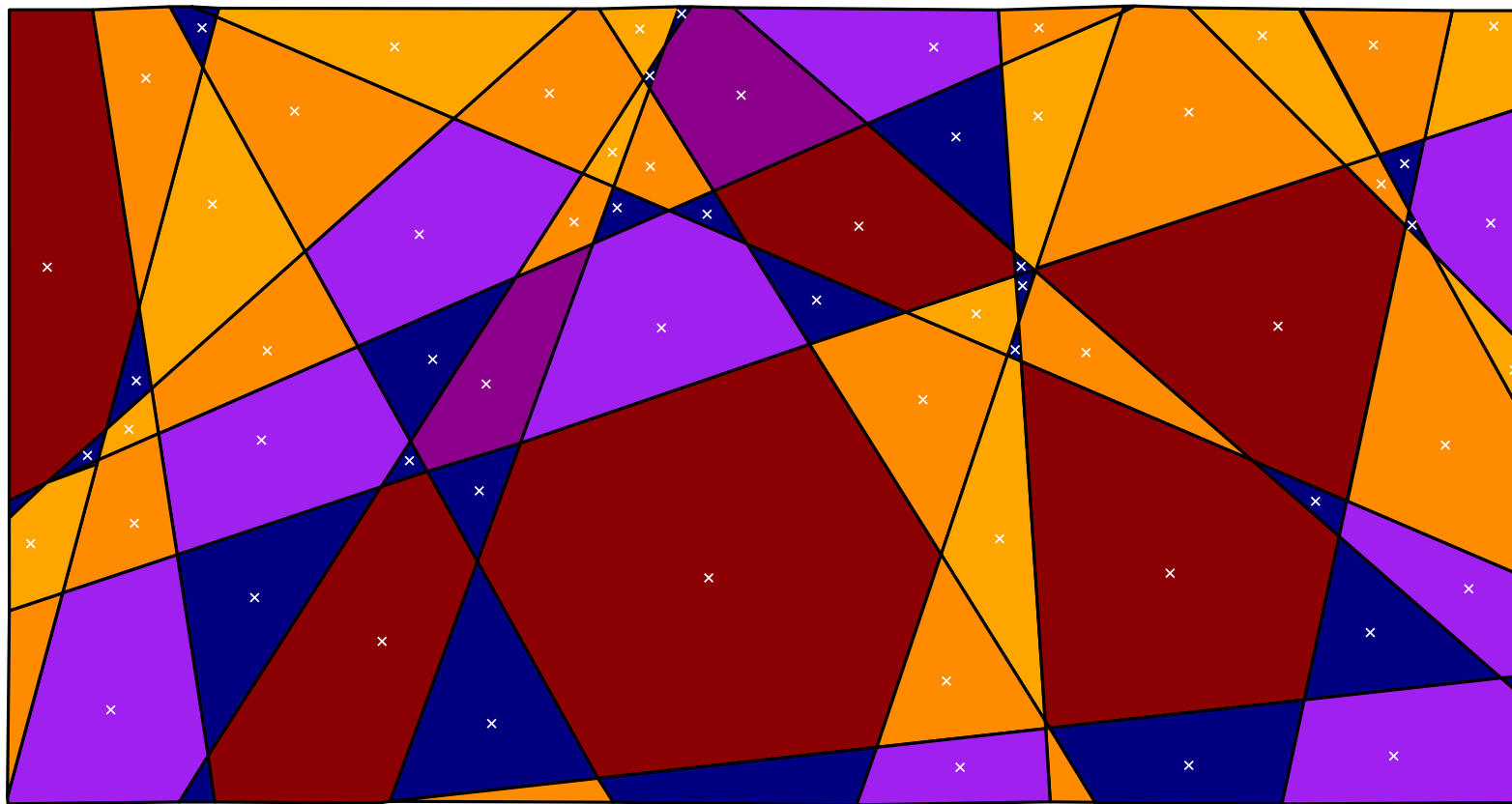
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$Z \dots$ **Typical cell** = random centred polytope with distribution \mathbb{Q}

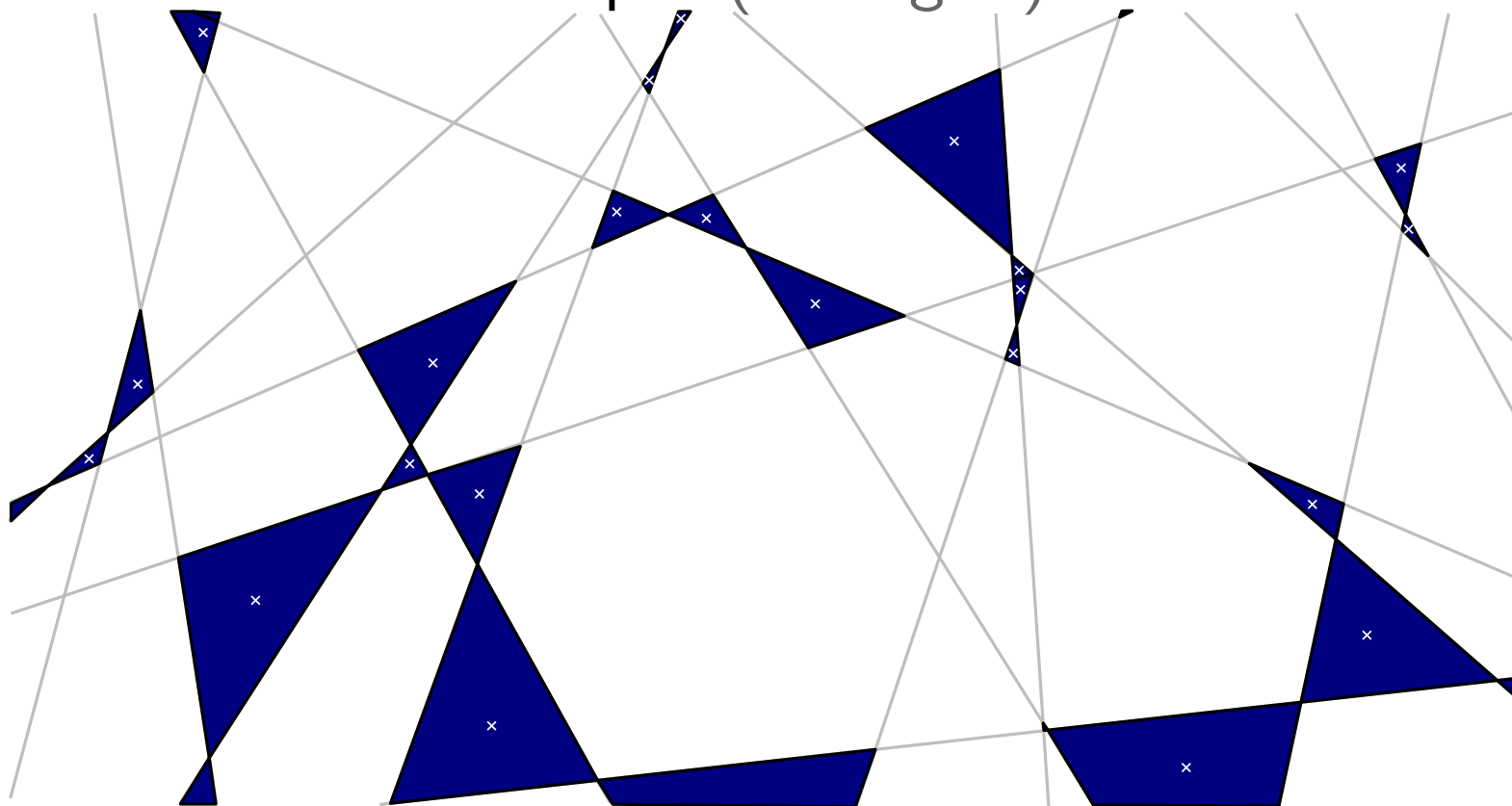


Cells With n Facets



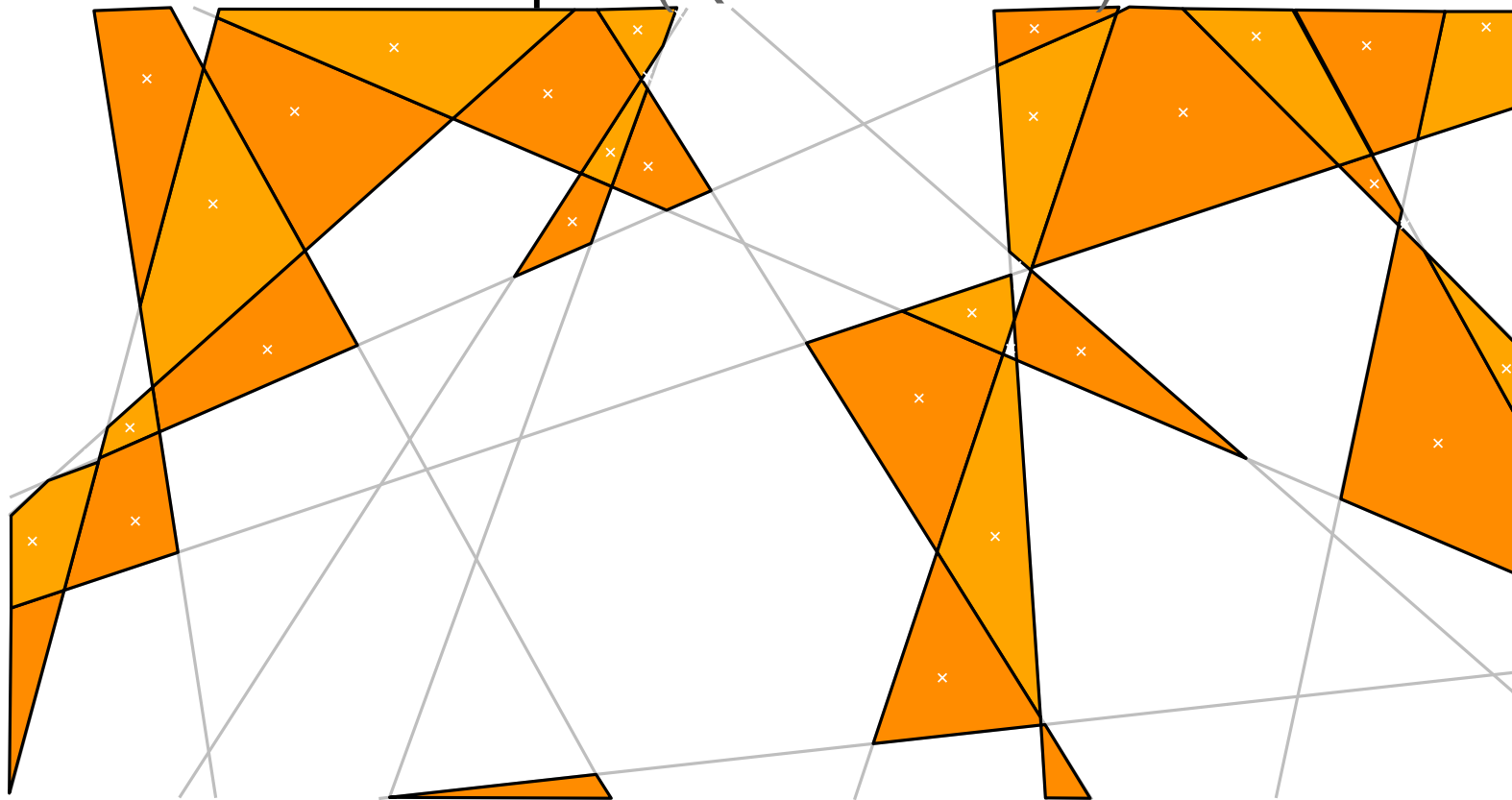
Cells With n Facets

3-topes (Triangles)



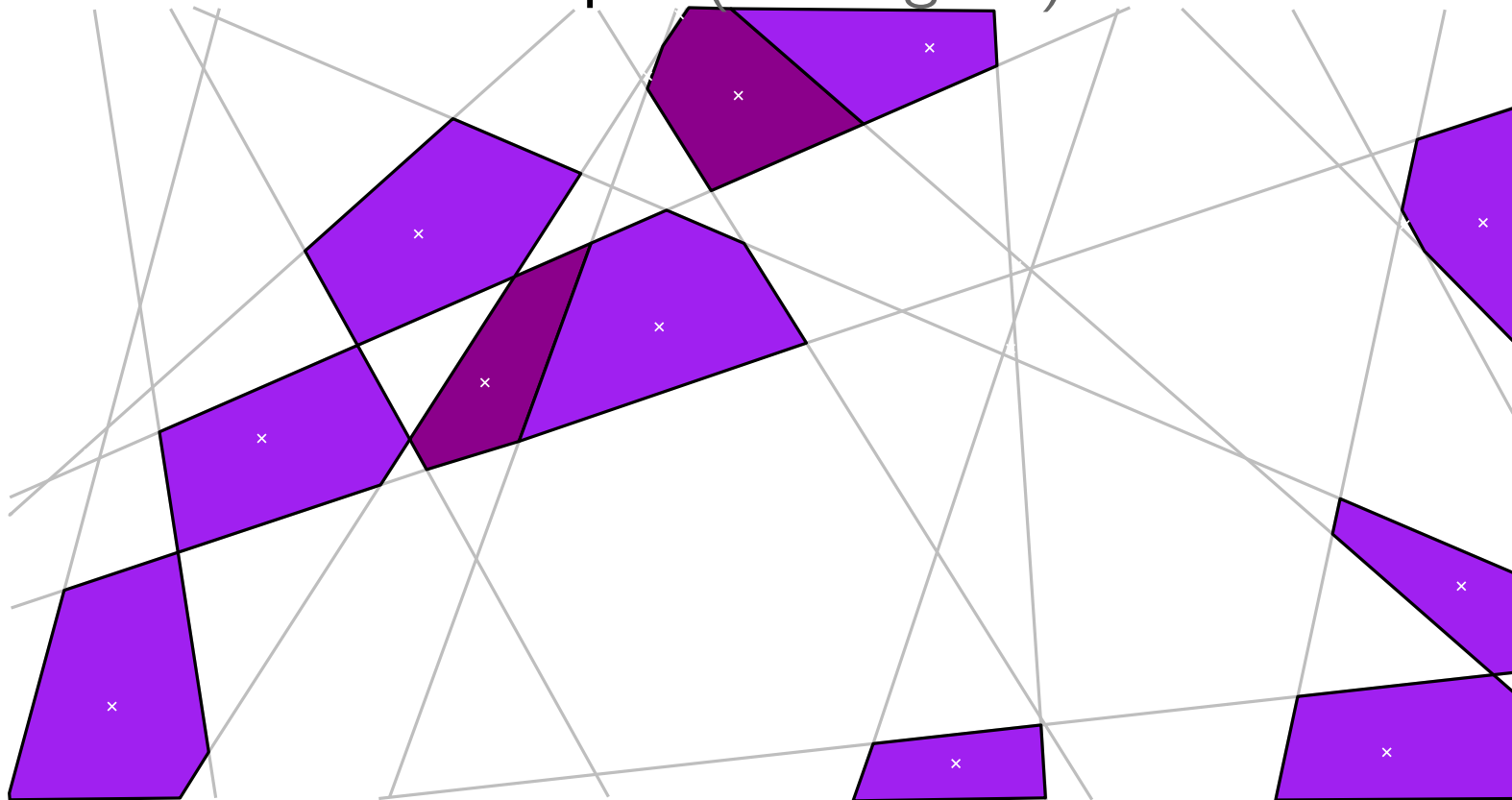
Cells With n Facets

4-topes (Quadrilaterals)



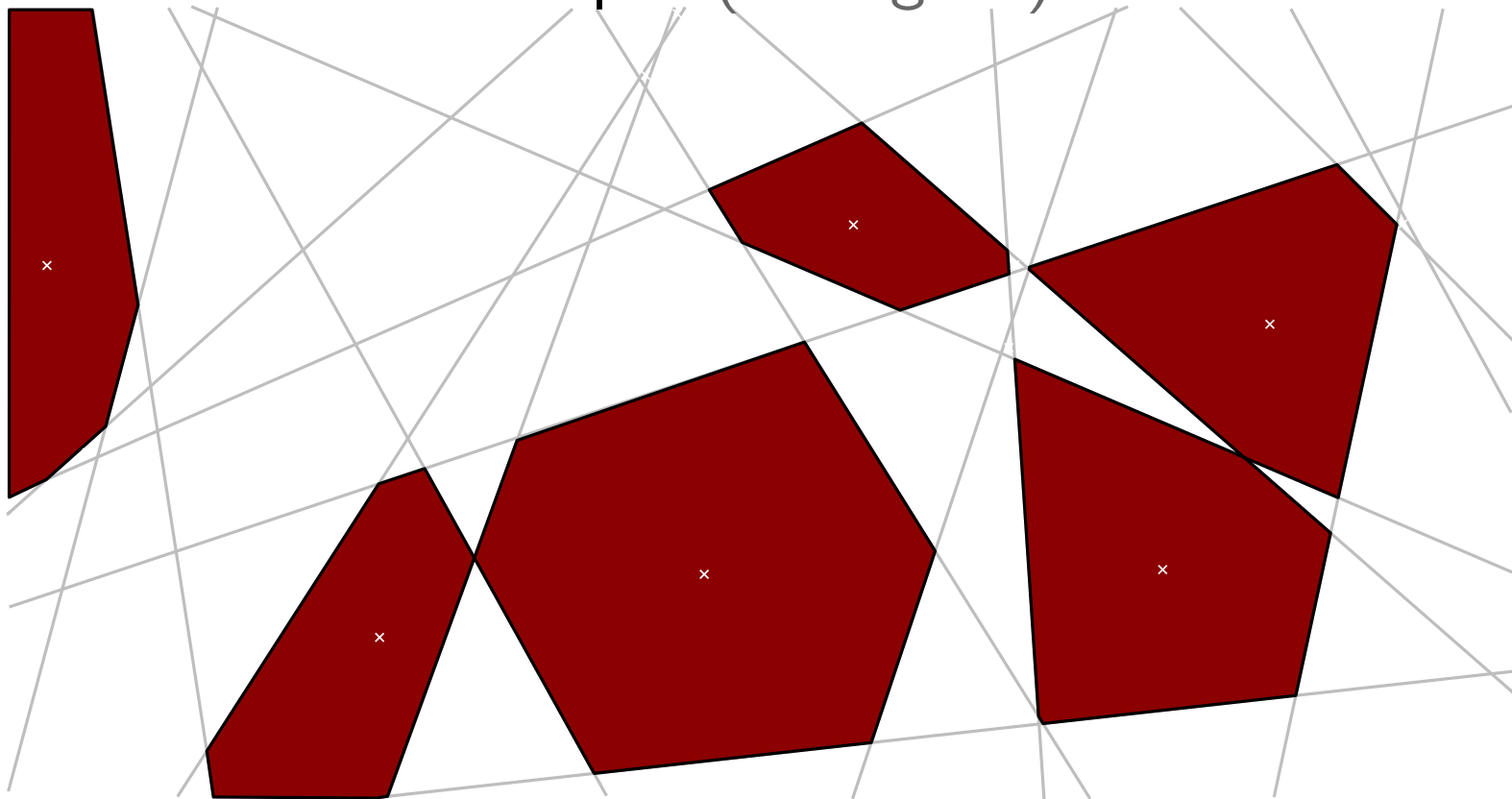
Cells With n Facets

5-topes (Pentagons)



Cells With n Facets

6-topes (Hexagons)



Cells With n Facets

$$\mathbb{P}(Z \text{ has } n \text{ facets}) = \mathbb{Q}(\mathcal{P}_{n,c})$$

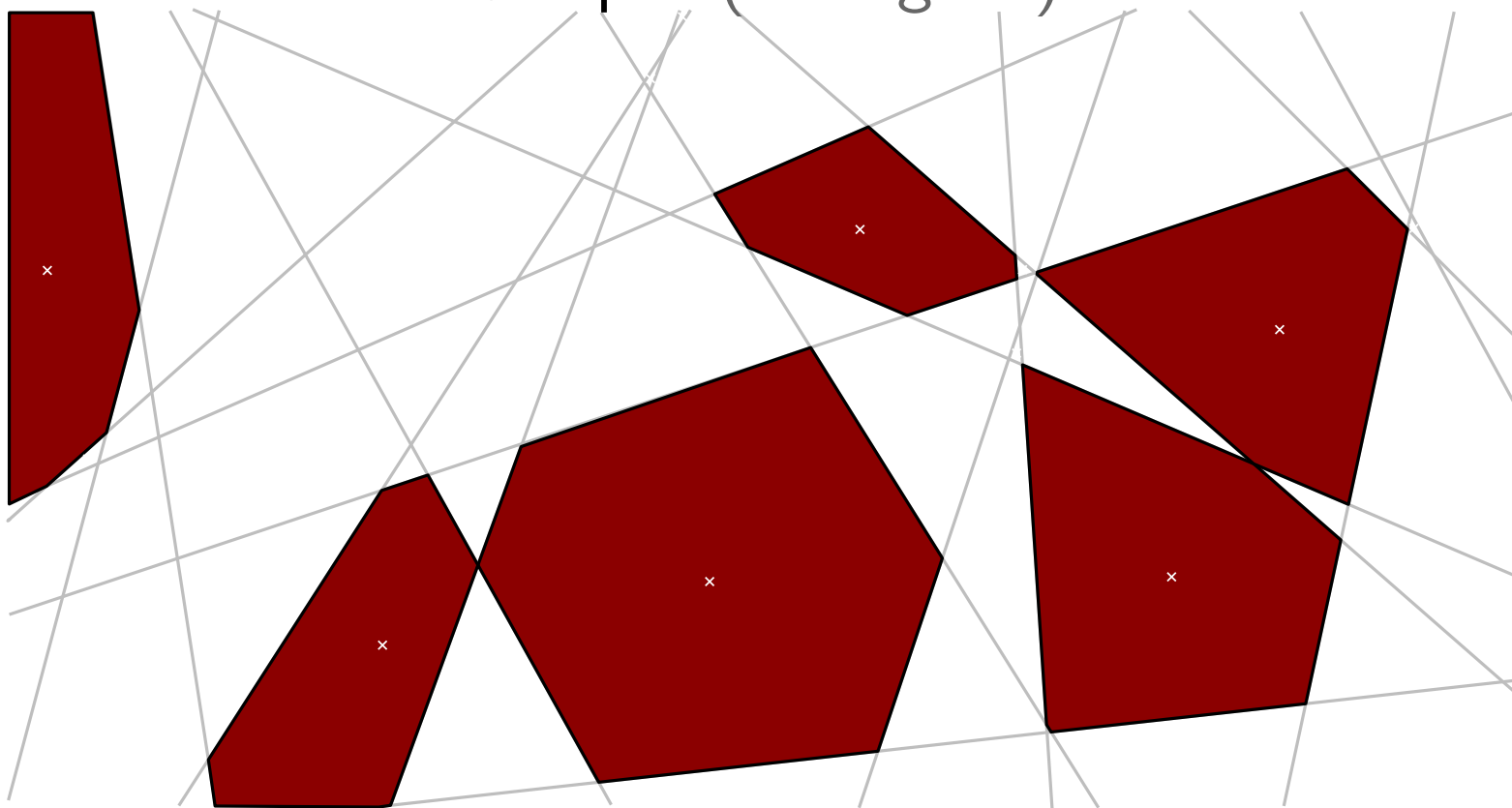
Set of centred n -topes

$$= \gamma^{-1} \gamma \lambda_d([0, 1]^d) \mathbb{Q}(\mathcal{P}_{n,c})$$

$$= \gamma^{-1} \mathbb{E} X(\mathcal{P}_{n,[0,1]^d})$$

number of n -topes of X with center in $[0, 1]^d$

6-topes (Hexagons)



Cells With n Facets

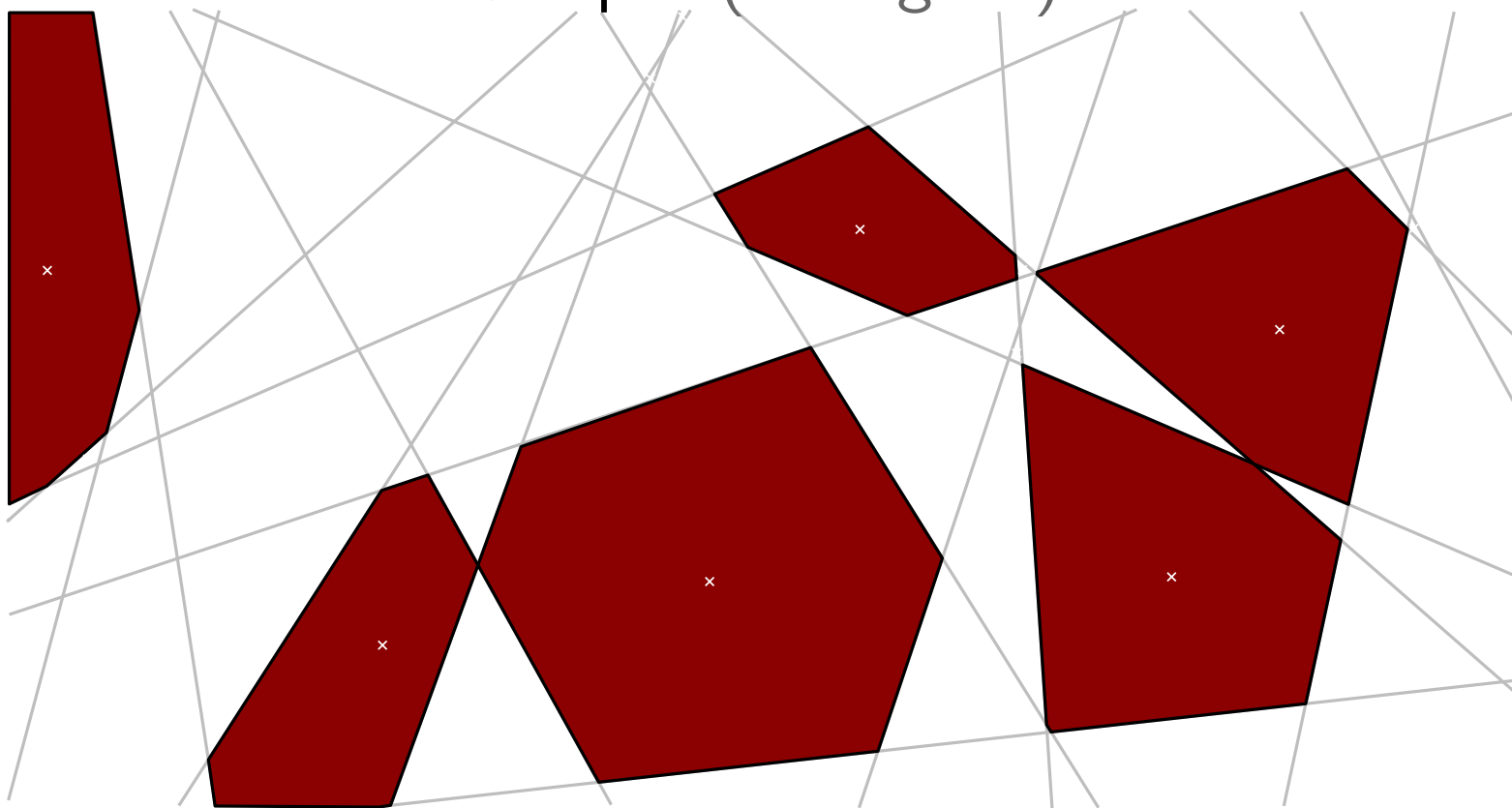
Let $D \subset \mathcal{P}_n$.

number of cells of X in D

$$P = \bigcap_{i=1}^n H_i^{\epsilon_i}$$

$$X(D) = \frac{1}{n!} \sum_{P \in \eta_{\neq}^n \times \{\pm 1\}^n} \mathbf{1}(P \in D) \mathbf{1}(\eta \cap P = \emptyset).$$

6-topes (Hexagons)



Cells With n Facets

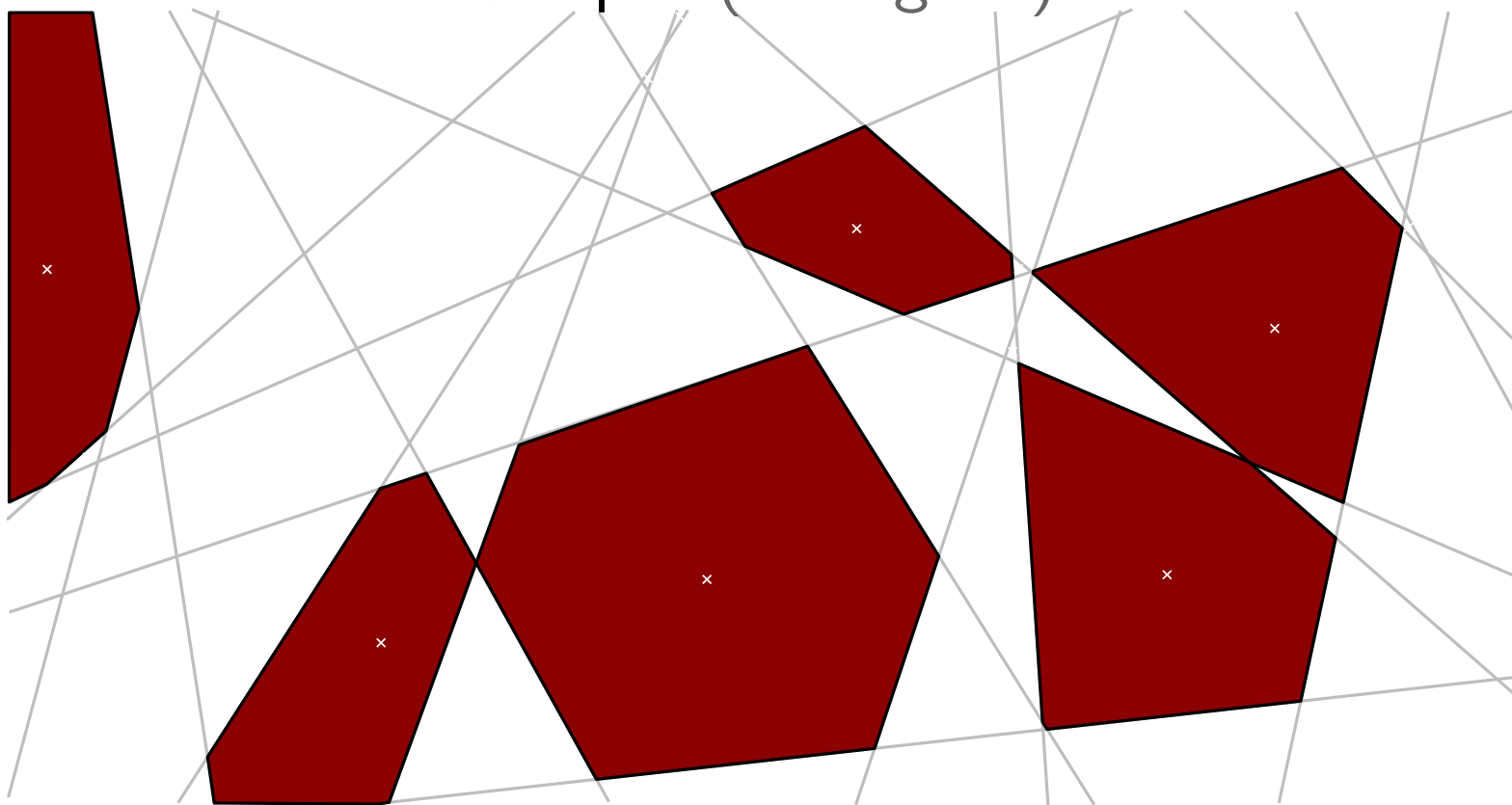
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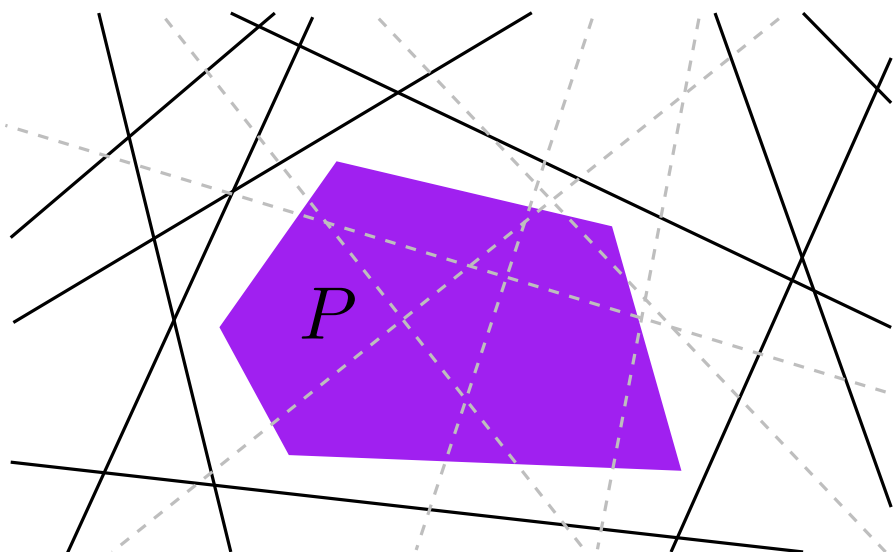
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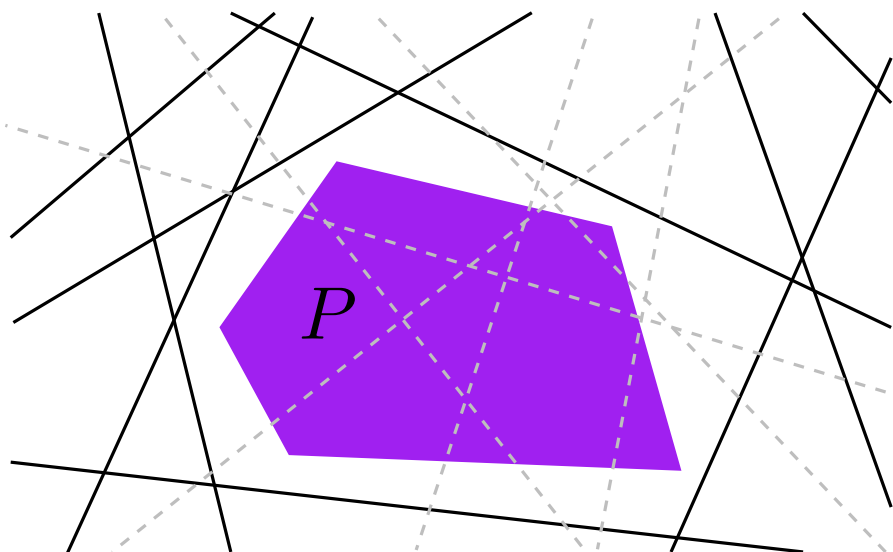
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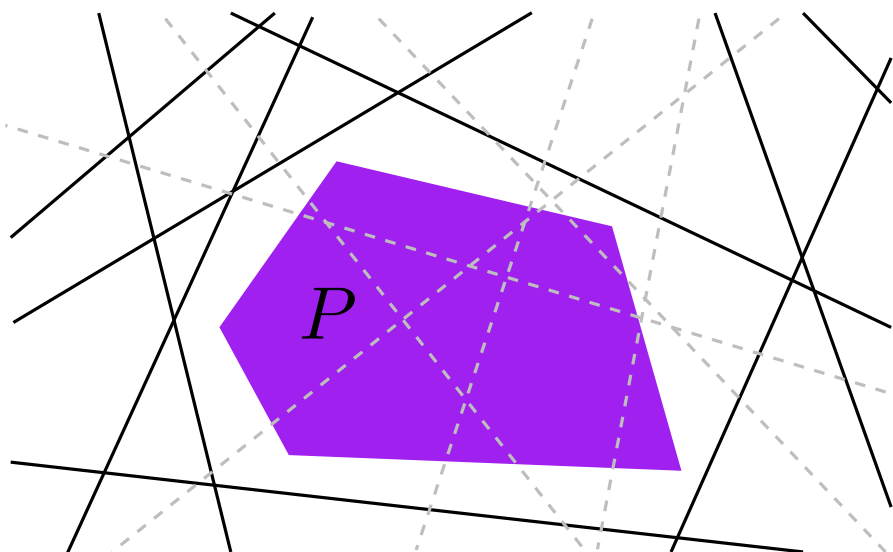
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$$\Phi(P) := \Theta(\mathcal{H}_P) = \int_{\mathbb{S}^{d-1}} \int_0^\infty \mathbb{1}\{H(\mathbf{u}, t) \cap P \neq \emptyset\} dt \varphi(d\mathbf{u})$$

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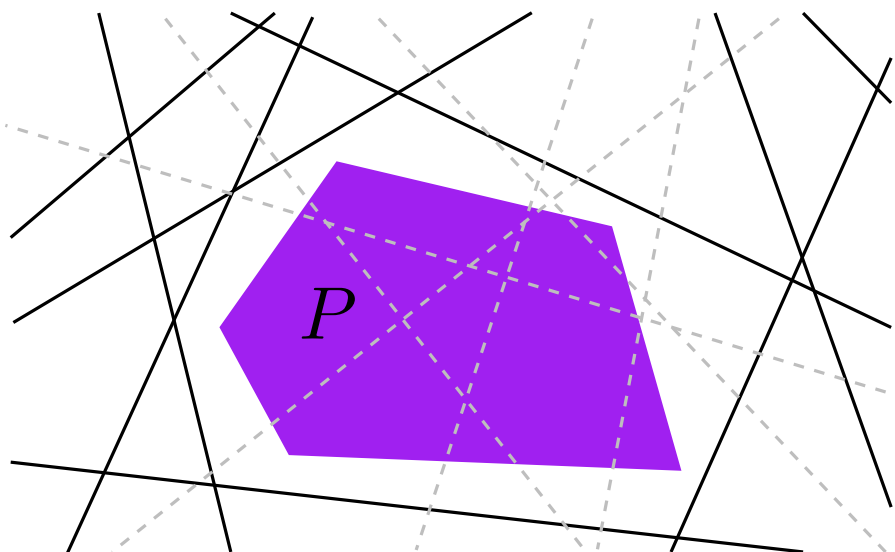
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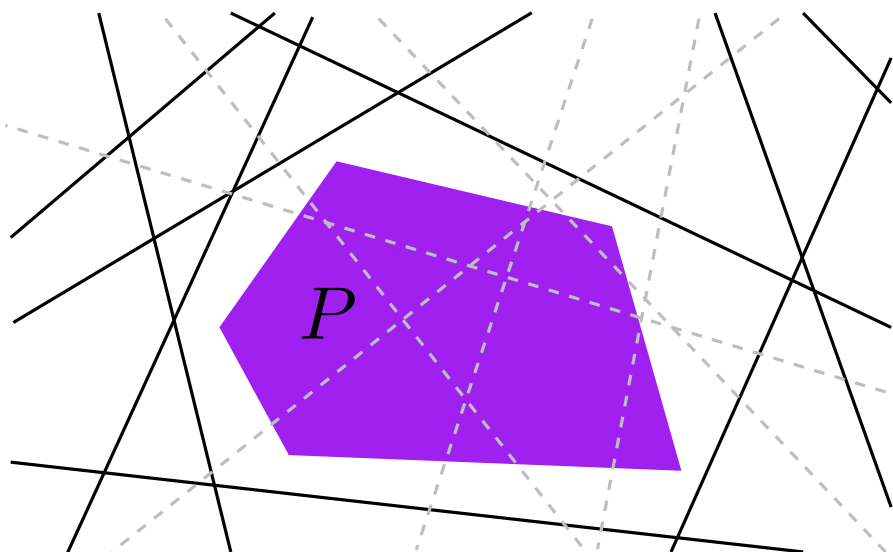
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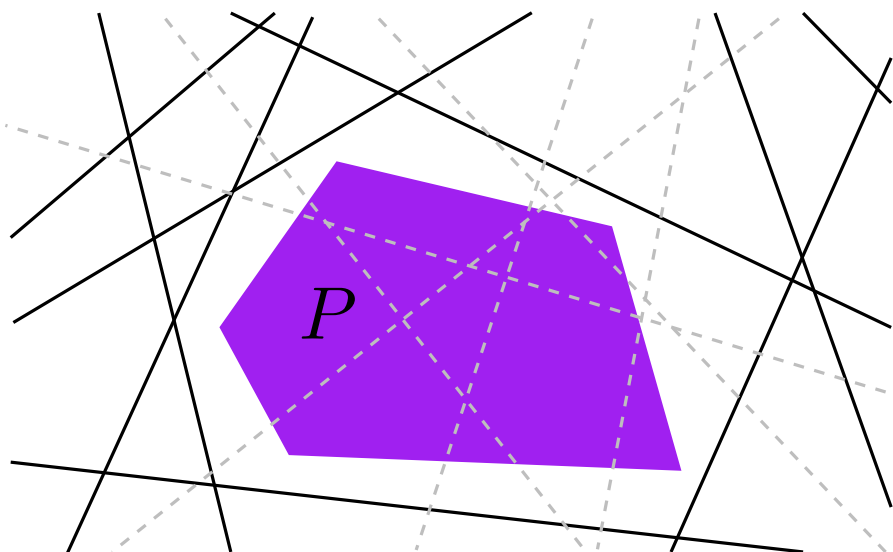
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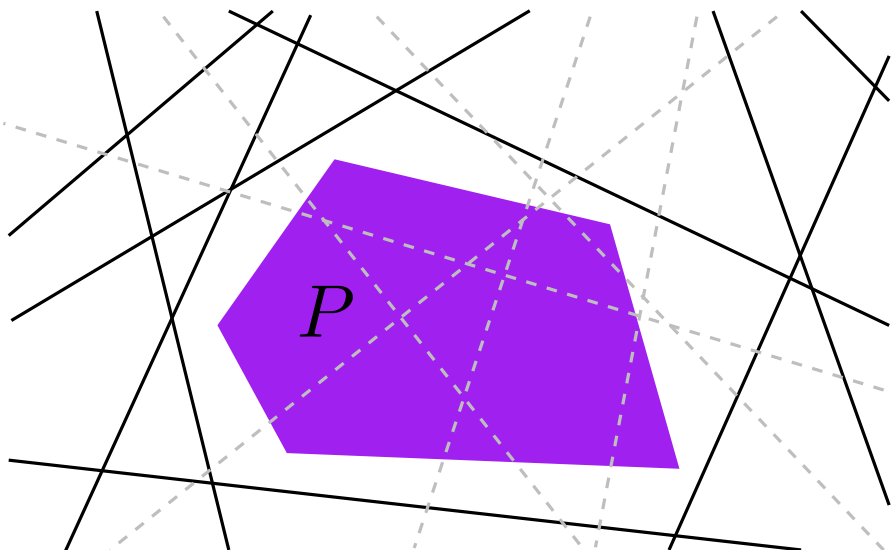
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$$\begin{aligned} &\Theta_n \text{ measure on } \mathcal{P}_n \\ &\Theta_n(t \cdot + x) = t^n \Theta_n(\cdot) \end{aligned}$$

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Decomposition of the Measure Θ_n

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Decomposition of the Measure Θ_n

$$\mathfrak{h}_n : \mathcal{P}_n \xrightarrow{\sim} \mathbb{R}^d \times (0, \infty) \times \mathcal{P}_{n, \mathfrak{c}}^1$$

Set of centred n -topes with Φ -content 1

$$P \mapsto \left(\mathfrak{c}(P), \Phi(P), \underbrace{\Phi(P)^{-1}(P - \mathfrak{c}(P))}_{\mathfrak{s}(P)} \right),$$

$\mathfrak{s}(P)$ ← Shape of P

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pushforward

Lebesgue measure

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$$\mathfrak{h}_n(\Theta_n) = \lambda_d \otimes \lambda_{1,n-d} \otimes \Theta_{n,c}^1$$

$$\lambda_{1,n-d}([0,a]) = a^{n-d}$$

$$\Theta_{n,c}^1(\cdot) = \Theta_n((0,1) \cdot + [0,1]^d)$$

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Complementary Theorem (Miles 1971)

If we condition the typical cell Z to have n facets, then

- $\Phi(Z)$ and $\mathfrak{s}(Z)$ are independent
- $\Phi(Z)$ is Gamma distributed with parameter $n - d$

$\mathbb{P}(Z \text{ has } n \text{ facets})$

$$\begin{aligned}
 \gamma \mathbb{P}(Z \text{ has } n \text{ facets}) &= \mathbb{E}X(\mathcal{P}_{n,[0,1]^d}) \\
 &= \int_{\mathcal{P}_{n,c}^1} \int_{(0,\infty)} \int_{[0,1]^d} e^{-t} t^{n-d-1} dc dt \Theta_{n,c}^1(dP) \\
 &= (n-d-1)! \Theta_{n,c}^1(\mathcal{P}_{n,c}^1) \\
 &= \frac{(n-d-1)!}{n!} \int \cdots \int_{\mathcal{H}^n} \sum_{\epsilon \in \{\pm 1\}^n} \mathbf{1}(\mathbf{c}(P) \in [0,1]^d) \mathbf{1}(\Phi(P) < 1) \mathbf{1}(P \in \mathcal{P}_n) \Theta(dH_n) \cdots \Theta(dH_1)
 \end{aligned}$$

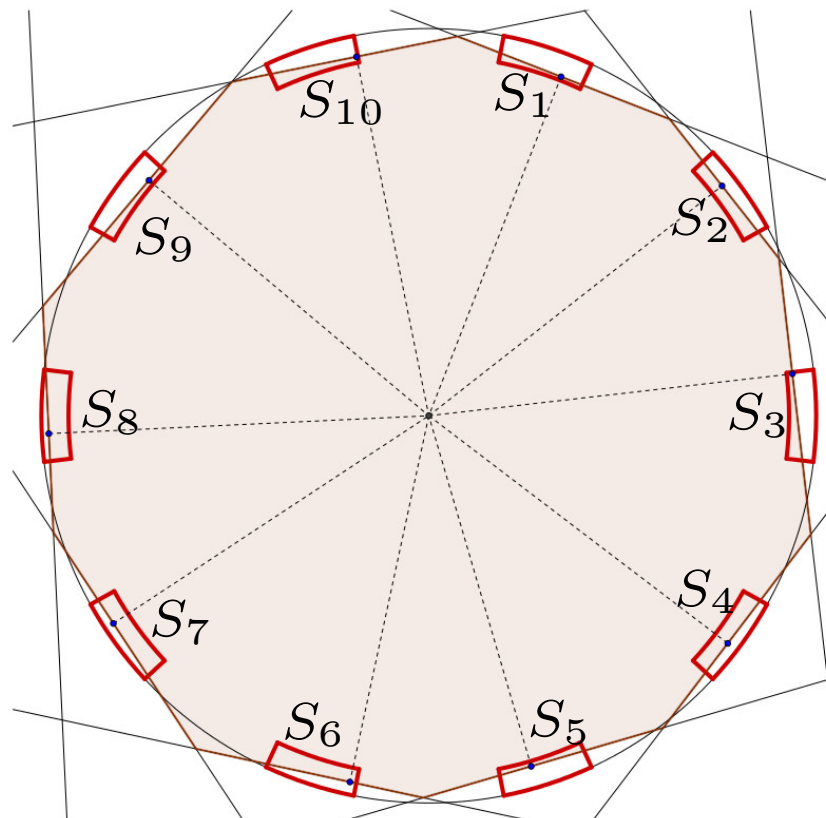
Lower Bound

$$\begin{aligned}
 & \gamma \frac{n!}{(n-d-1)!} \mathbb{P}(Z \text{ has } n \text{ facets}) \\
 &= \int \cdots \int_{\mathcal{H}^n} \sum_{\epsilon \in \{\pm 1\}^n} \mathbb{1}(\mathbf{c}(P) \in [0, 1]^d) \mathbb{1}(\Phi(P) < 1) \mathbb{1}(P \in \mathcal{P}_n) \Theta(dH_n) \cdots \Theta(dH_1) \\
 &> n! \int \cdots \int_{\mathcal{H}^n} \mathbb{1}(H_1 \in S_1) \cdots \mathbb{1}(H_n \in S_n) \Theta(dH_n) \cdots \Theta(dH_1) \\
 &= n! \Theta(S_1)^n \\
 &> n! \left(c n^{-(d-1)/(d+1)} \right)^n \\
 &\sim c^n n^{-2n/(d-1)} \\
 &\quad \swarrow \text{Stirling}
 \end{aligned}$$

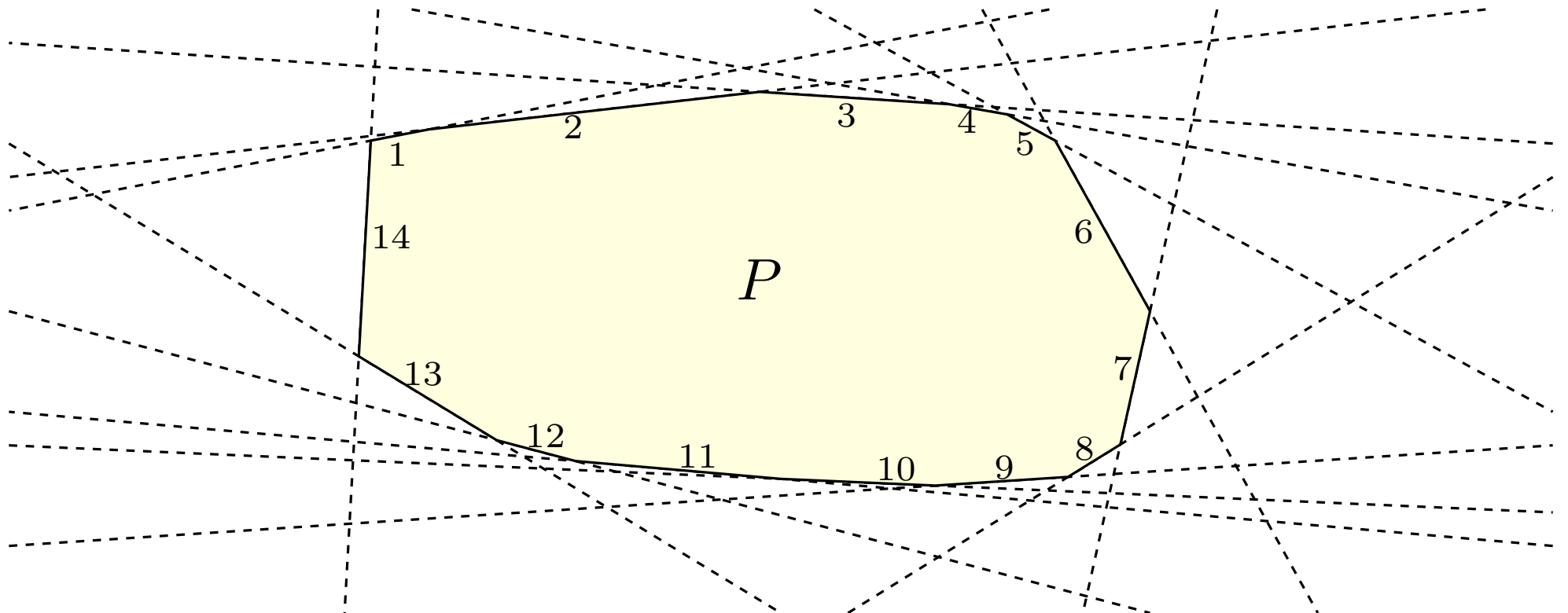
Theorem: Lower bound

There exists a constant c_1 depending on d and φ such that for n big enough we have

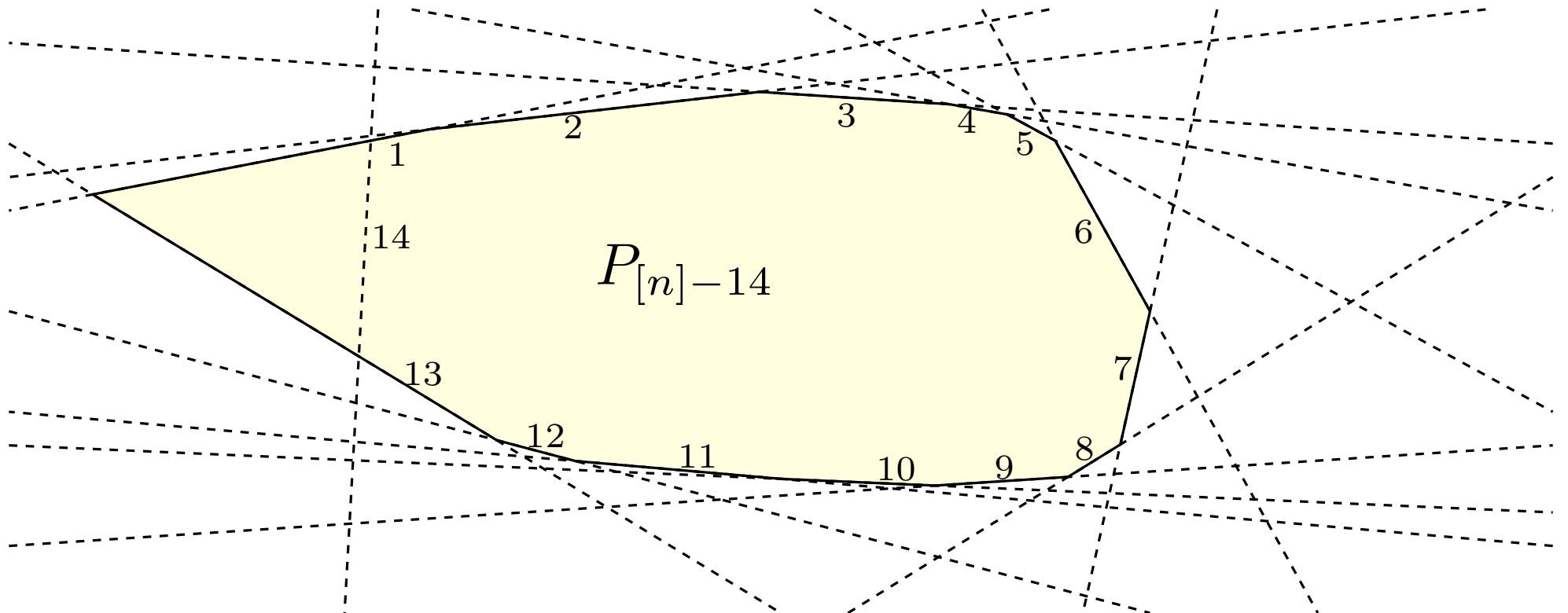
$$\mathbb{P}(Z \text{ has } n \text{ facets}) > c_1^n n^{-2n/(d-1)}$$



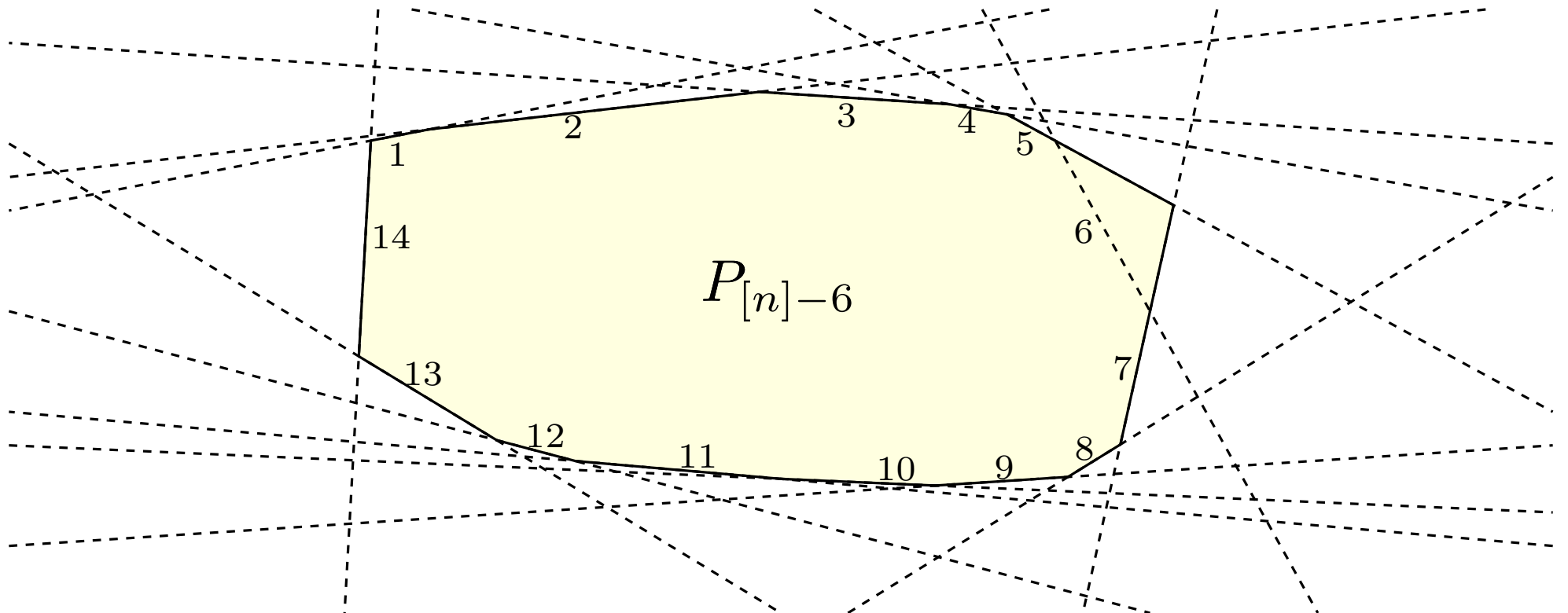
Approximation by Deleting One Facet



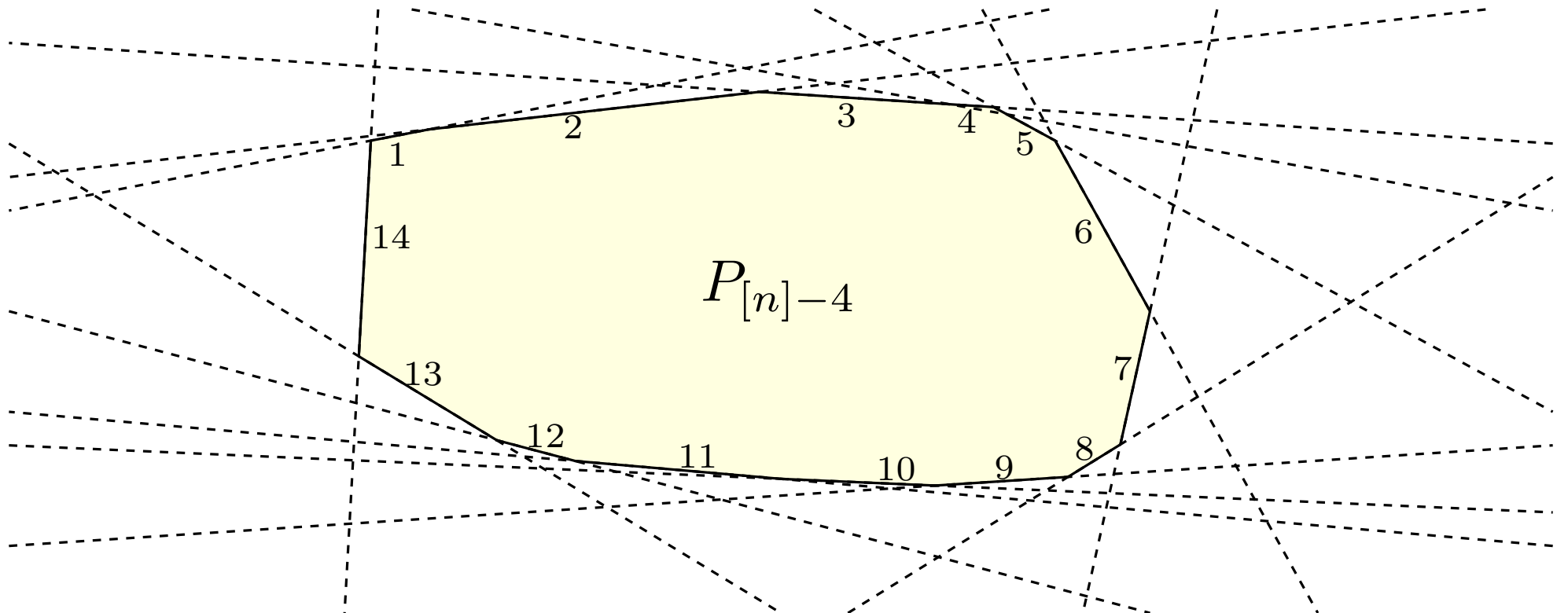
Approximation by Deleting One Facet



Approximation by Deleting One Facet



Approximation by Deleting One Facet



Approximation by Deleting One Facet

There exists a constant c_0 such that:

Theorem

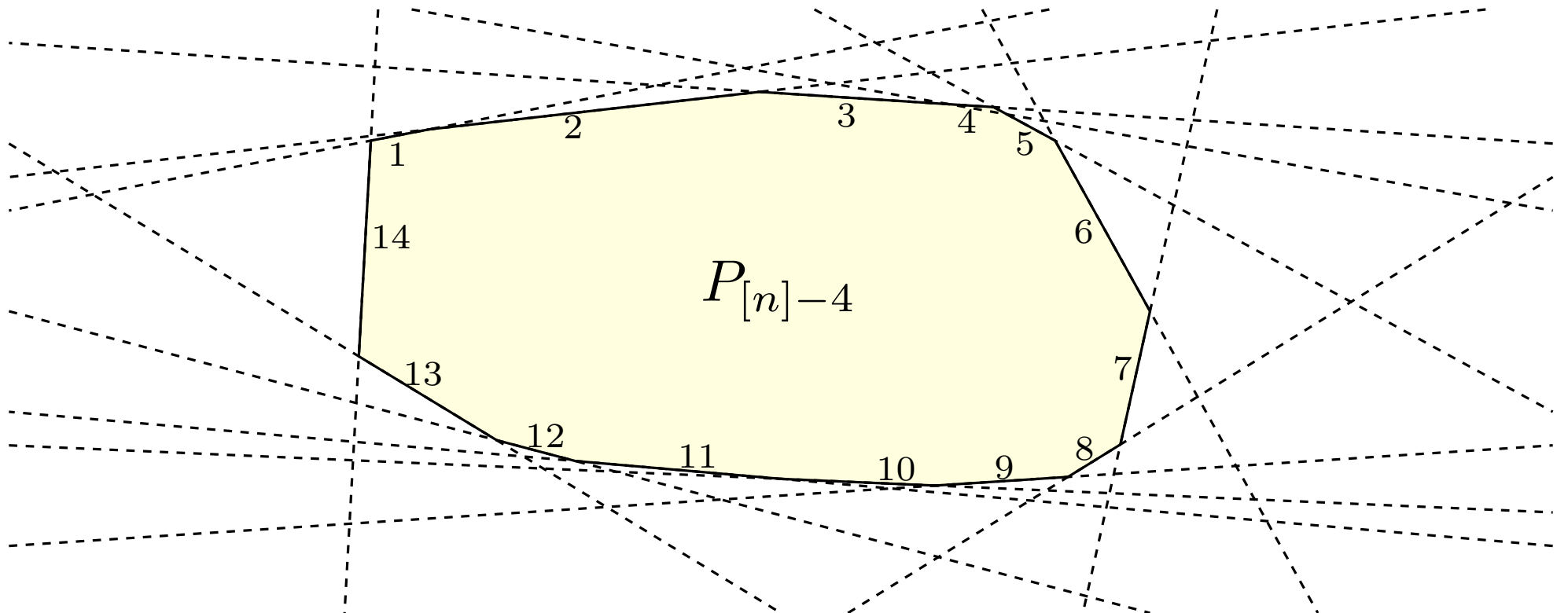
Let $P = \bigcap_{i=1}^n H_i^-$ be a simple polytope with n big enough.

There exists a subset $J \subset [n]$ of cardinality at least $n/4$ such that for any $j \in J$ we have

$$d_H(P, P_{[n]-j}) < c_0 n^{-2/(d-1)} \Phi(P)$$

and

$$\Phi(P_{[n]-j}) < \exp\left(c_0 n^{-1-2/(d-1)}\right) \Phi(P).$$



Upper Bound

$$\begin{aligned} & \gamma \frac{n!}{(n-d-1)!} \mathbb{P}(Z \text{ has } n \text{ facets}) \\ &= \int \cdots \int_{\mathcal{H}^n} \sum_{\epsilon \in \{\pm 1\}^n} \mathbf{1}(\mathbf{c}(P) \in [0, 1]^d) \mathbf{1}(\Phi(P) < 1) \mathbf{1}(P \in \mathcal{P}_n) \Theta(dH_n) \cdots \Theta(dH_1) \end{aligned}$$

Upper Bound

$$\frac{\gamma}{4} \frac{n!}{(n-d-1)!} \mathbb{P}(Z \text{ has } n \text{ facets})$$

$$< \int \cdots \int_{\mathcal{H}^n} \sum_{\epsilon \in \{\pm 1\}^n} \mathbb{1}(\mathbf{c}(P) \in [0, 1]^d) \mathbb{1}(\Phi(P) < 1) \mathbb{1}(P \in \mathcal{P}_n)$$

$$\mathbb{1}\left(d_H(P, P_{[n-1]}) < c_0 n^{-2/(d-1)}\right)$$

$$\mathbb{1}\left(\Phi(P_{[n-1]}) < \exp\left(c_0 n^{-1-2/(d-1)}\right)\right) \Theta(dH_n) \cdots \Theta(dH_1)$$

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$$\mathbb{1}(P_{[n-1]} \in \mathcal{P}_{n-1})$$

$$\left(\int_{\mathcal{H}} \sum_{\epsilon_n = \pm 1} \mathbb{1}\left(d_H(P, P_{[n-1]}) < c_0 n^{-2/(d-1)}\right) \Theta(dH_n) \right) \Theta(dH_1) \cdots \Theta(dH_n)$$

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& \gamma \frac{n!}{(n-d-1)!} \mathbb{P}(Z \text{ has } n \text{ facets}) \\
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Theorem: Upper bound

There exists a constant c_2 depending on d and φ such that for n big enough we have

$$\mathbb{P}(Z \text{ has } n \text{ facets}) < c_2^n n^{-2n/(d-1)}$$

Shape of a Cell With Many Facets

Hug, Reitzner and Schneider [2004,2007] studied big cells. For example in the *isotropic case* they show that for any $j = 2, \dots, d$ and any $\varepsilon > 0$ we have

j -th intrinsic volume

$$\mathbb{P} \left(d_H \left(\mathfrak{s}(Z), \mathbb{B}^d \right) > \varepsilon \mid V_j(Z) > a \right) \rightarrow 0 \text{ when } a \rightarrow \infty$$

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The result above remains true if you change $V_j(Z)$ by $V_1(Z)$ or $\text{NumberOfFaces}(Z)$.

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We did a small step in the direction of this conjecture:

Theorem

There exists $\epsilon > 0$ such that for any $j \leq \lceil (d-1)/2 \rceil$ we have

$$\mathbb{P} \left(\frac{V_j(Z)}{V_1(Z)^j} < \epsilon \mid Z \text{ has } n \text{ facets} \right) \rightarrow 0 \text{ when } n \rightarrow \infty.$$

Cell with many facets are not '*too flat*'.

Take Home Message and Perspectives

Main Theorem

There exist constants c_1 and c_2 depending on d and φ such that for n big enough we have

$$c_1^n n^{-2n/(d-1)} < \mathbb{P}(Z \text{ has } n \text{ facets}) < c_2^n n^{-2n/(d-1)}$$

- \Rightarrow Explicit the dependence to d in the constants c_1 and c_2 .
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THANK YOU!