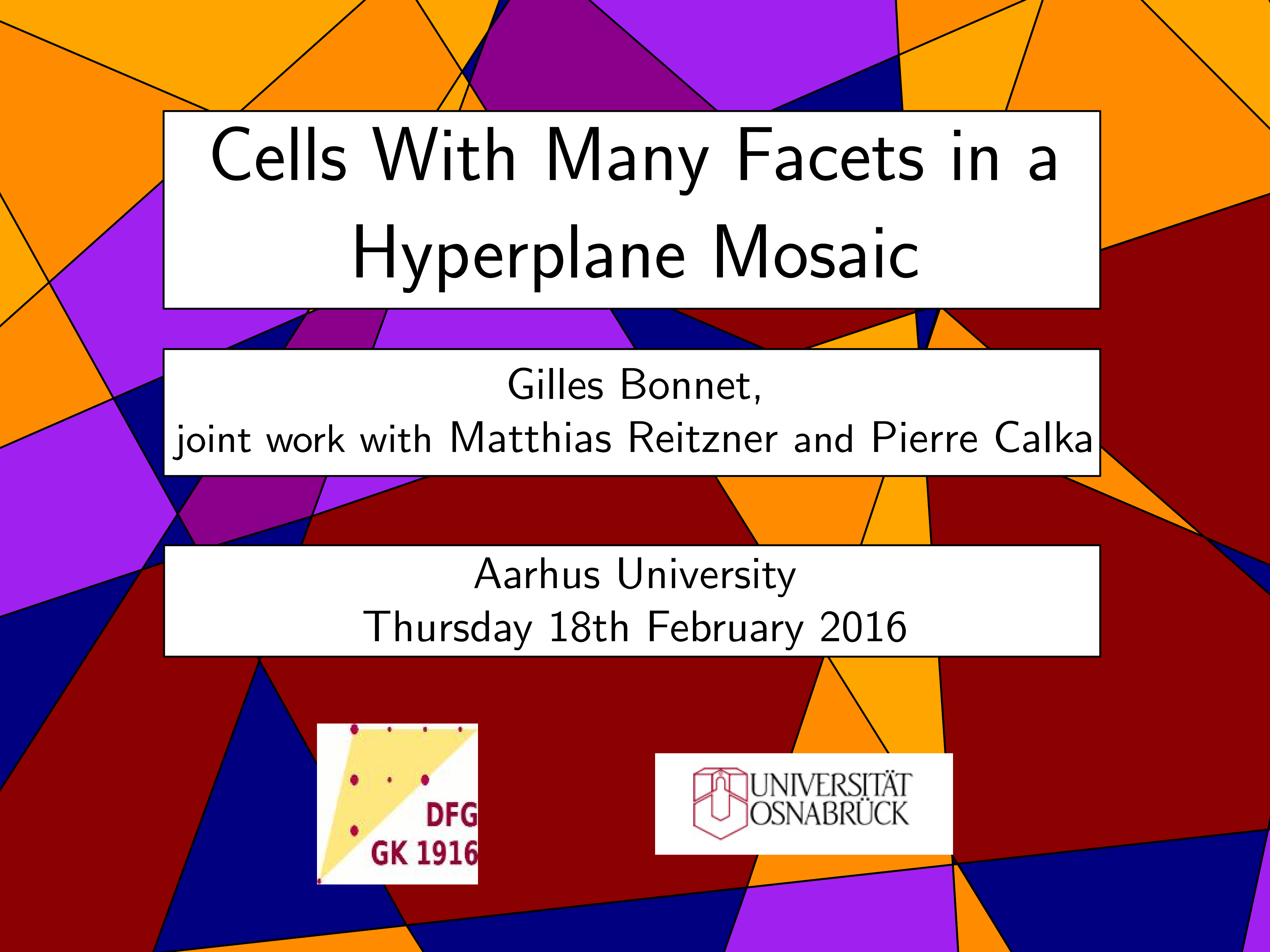

Cells With Many Facets in a Hyperplane Mosaic

Gilles Bonnet,
joint work with Matthias Reitzner and Pierre Calka

Aarhus University
Thursday 18th February 2016

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Stationary Poisson Hyperplane Mosaic in $\mathbb{R}^d$

η Poisson Hyperplane Process of **intensity measure** Θ

set of hyperplanes in $\mathbb{R}^d$

$\eta \subset \mathcal{H}$

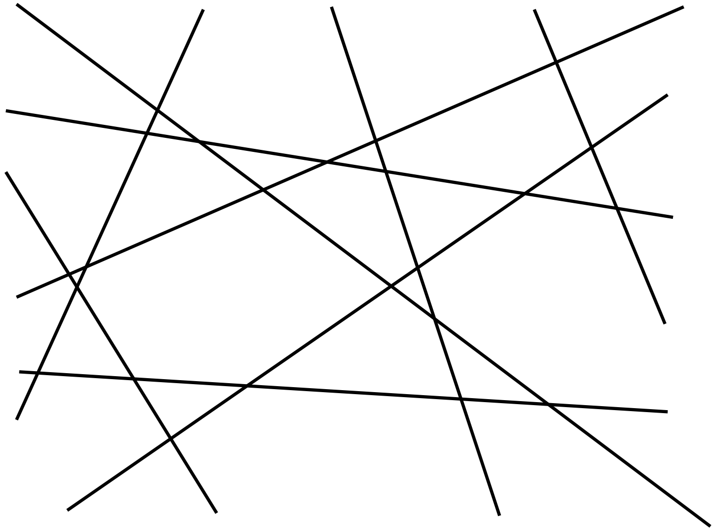
$A_1, \ldots, A_k \subset \mathcal{H}$ disjoint

$\Rightarrow \eta(A_1), \ldots, \eta(A_k)$ are **independent** random variables,

$\eta(A_i)$ is **Poisson distributed** of mean value $\Theta(A_i)$

$$\mathbb{P}(\eta(A) = n) = e^{-\Theta(A)} \frac{\Theta(A)^n}{n!}$$

number of hyperplanes of η in A_i



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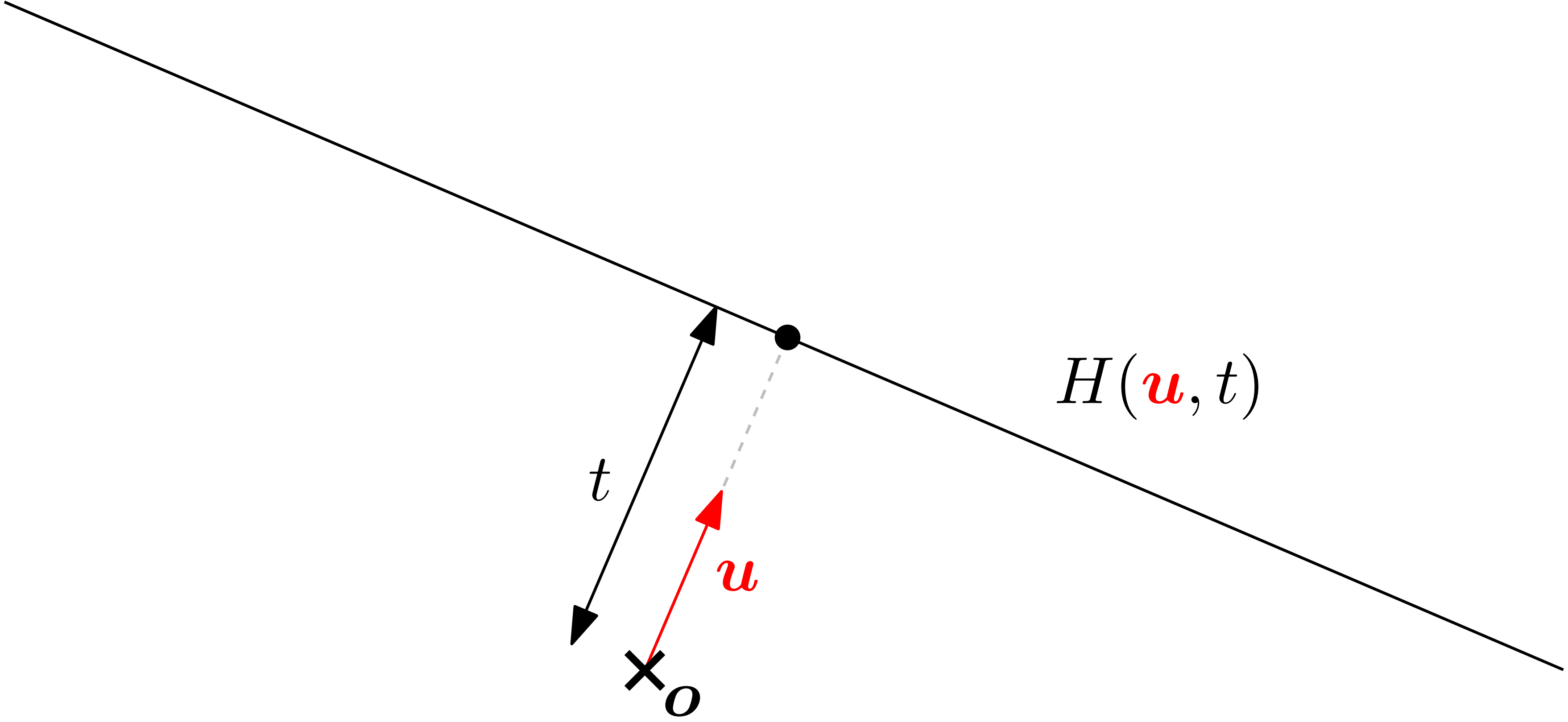
φ **directional distribution** (even measure on $\mathbb{S}^{d-1}$)

$$\Theta(\cdot) = \int_{\mathbb{S}^{d-1}} \int_0^\infty \mathbb{1}\{H(\boldsymbol{u}, t) \in \cdot\}\, \mathrm{d}t\, \varphi(\mathrm{d}\boldsymbol{u})$$



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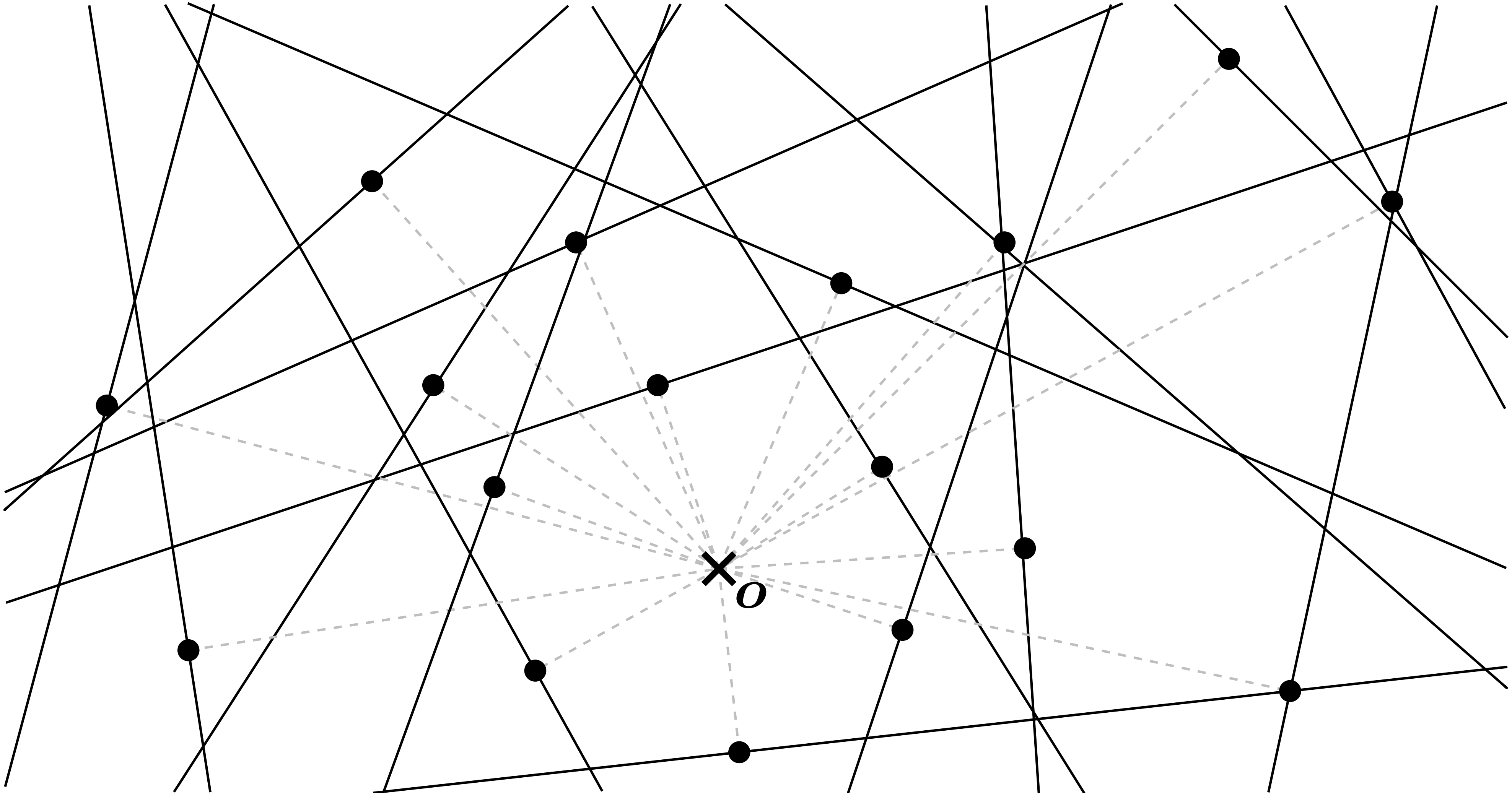

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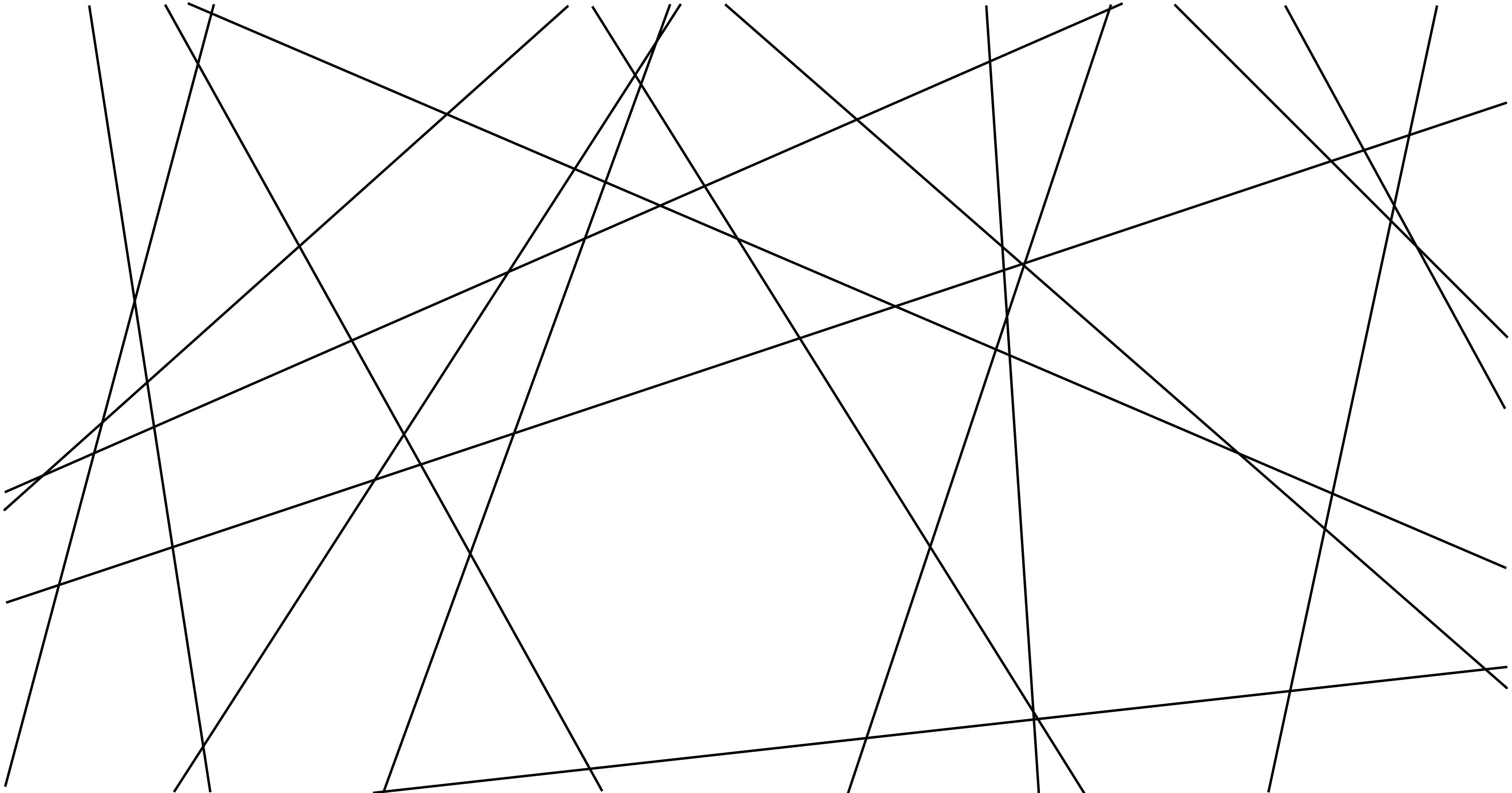

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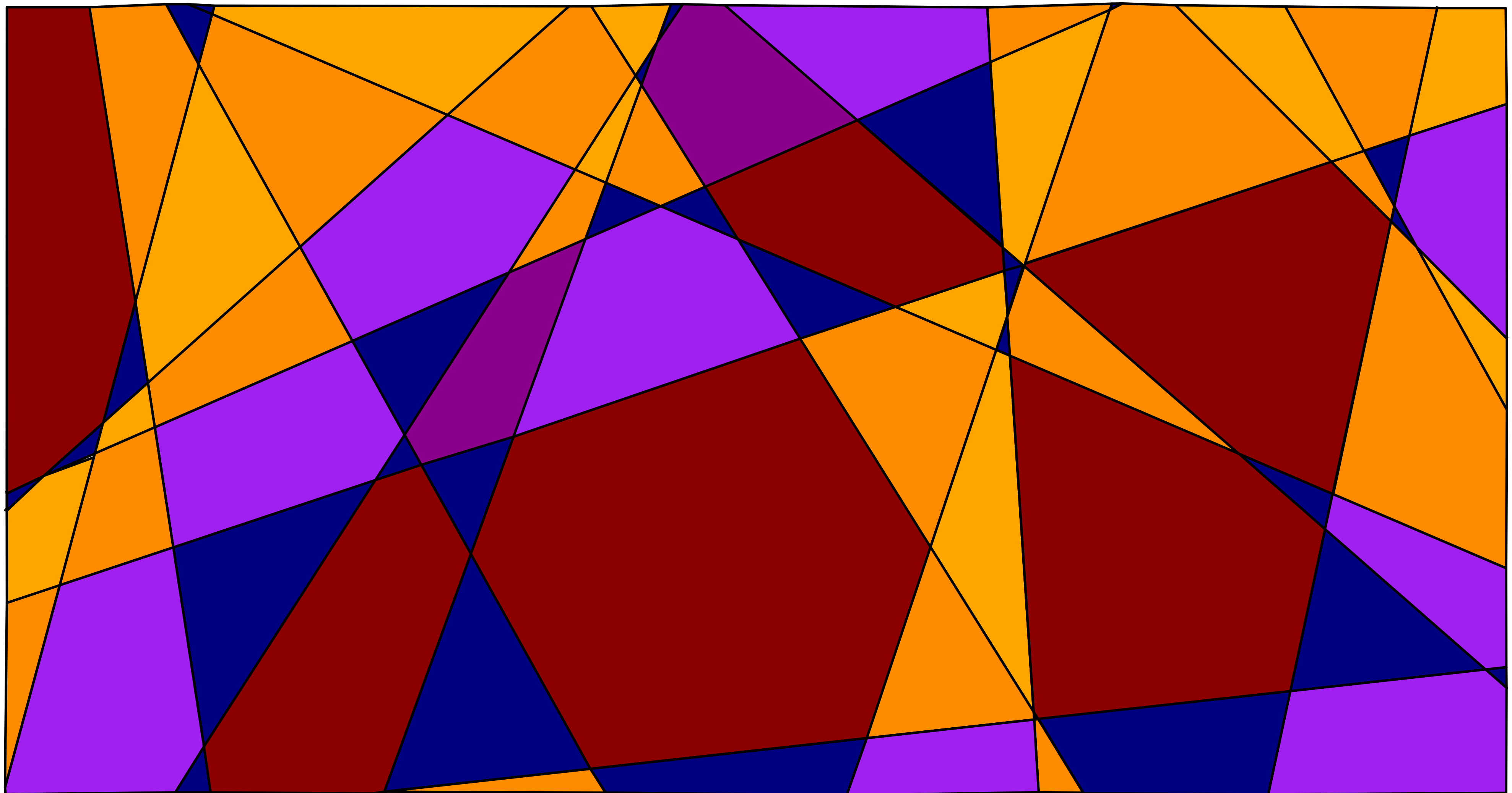
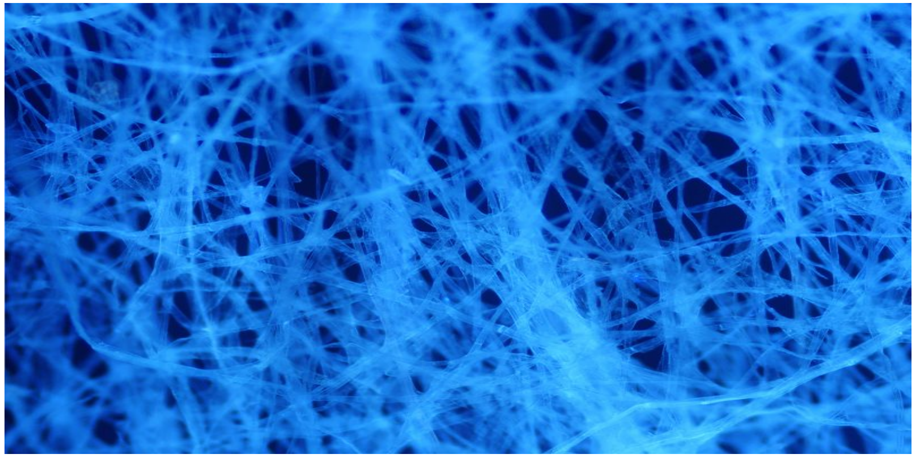
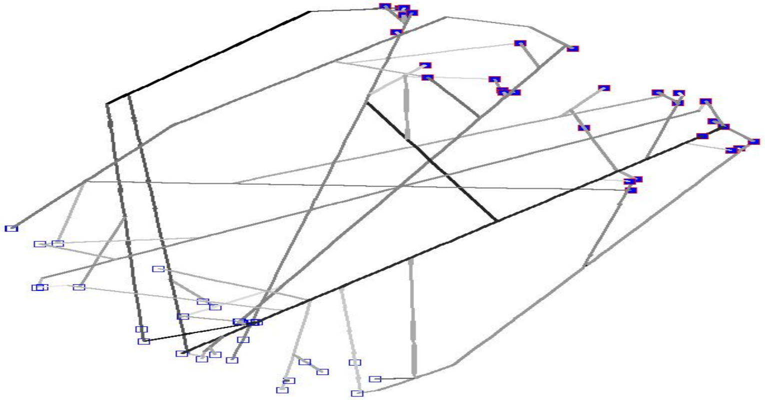


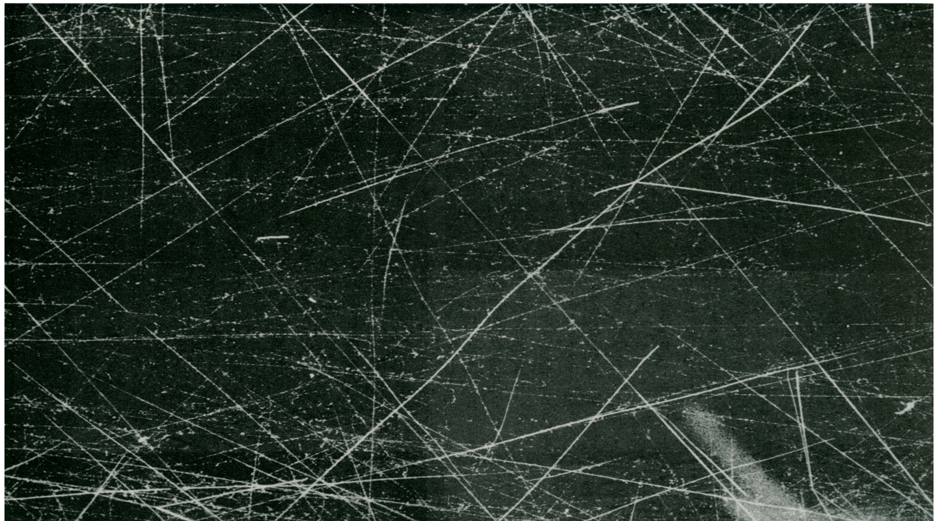
Applications



Microscopic structure of the paper



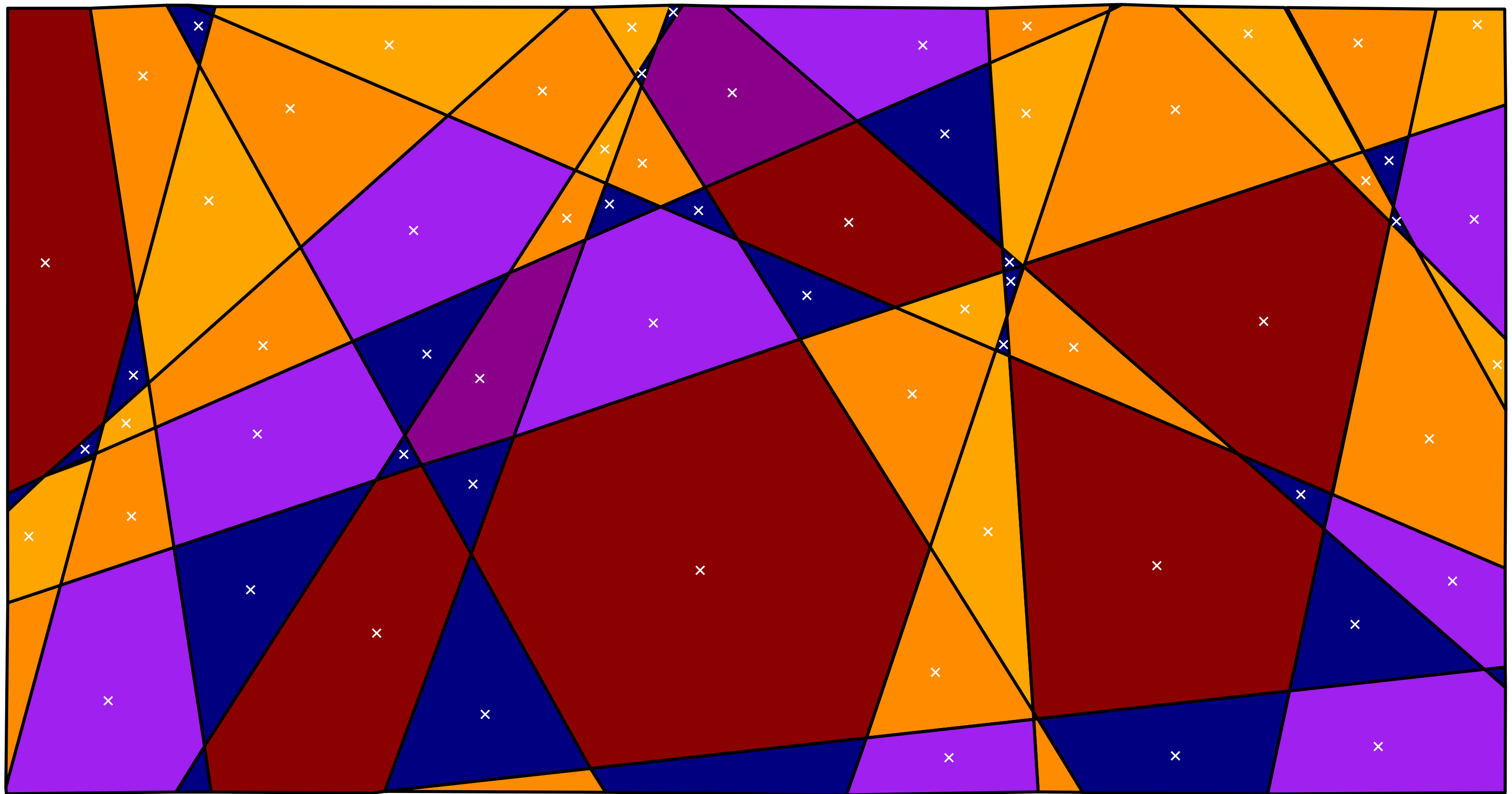
Random Networks



Cloud chamber

$\mathbb{P}(Z \text{ has } n \text{ facets})$

typical cell



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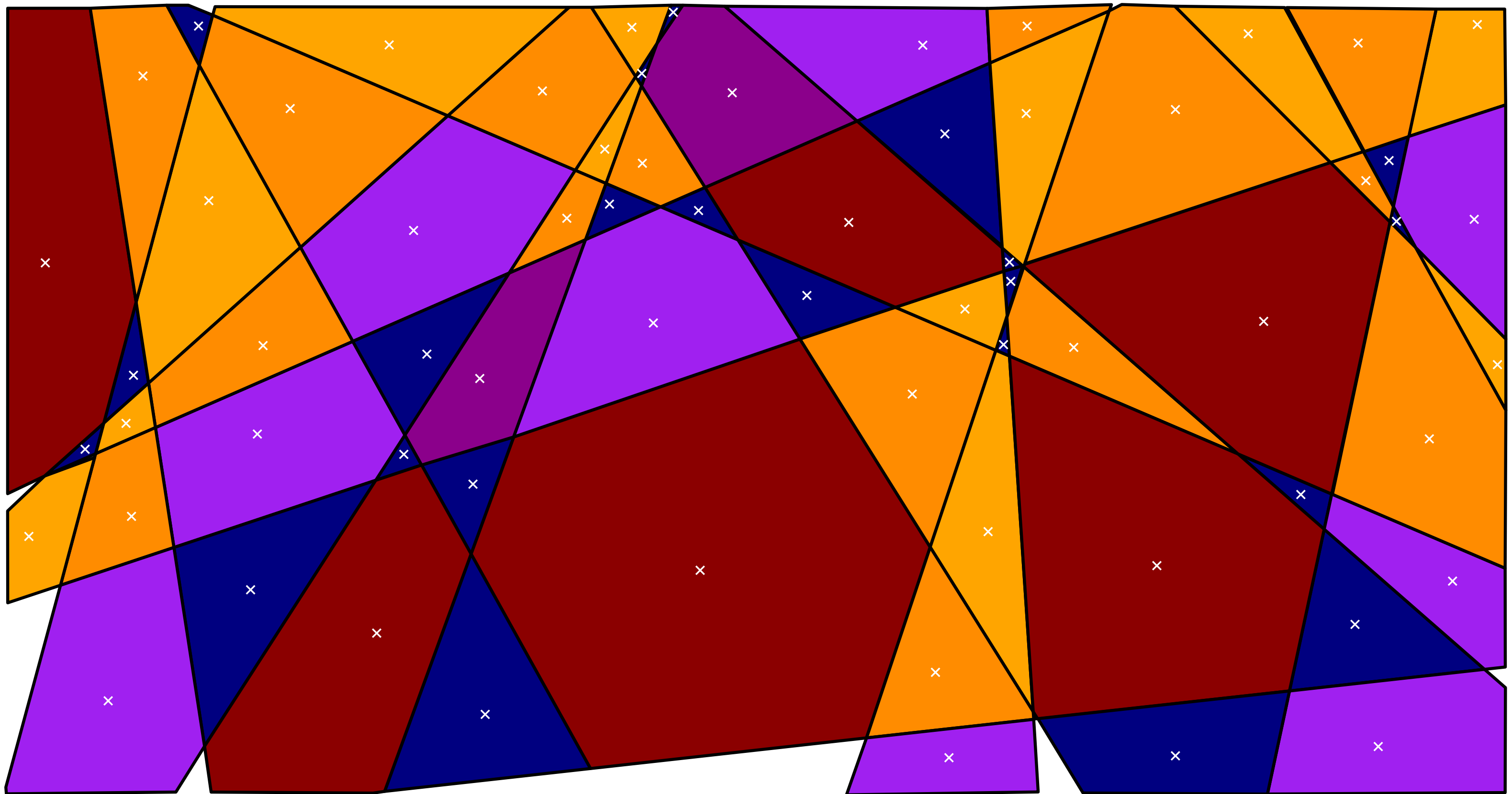
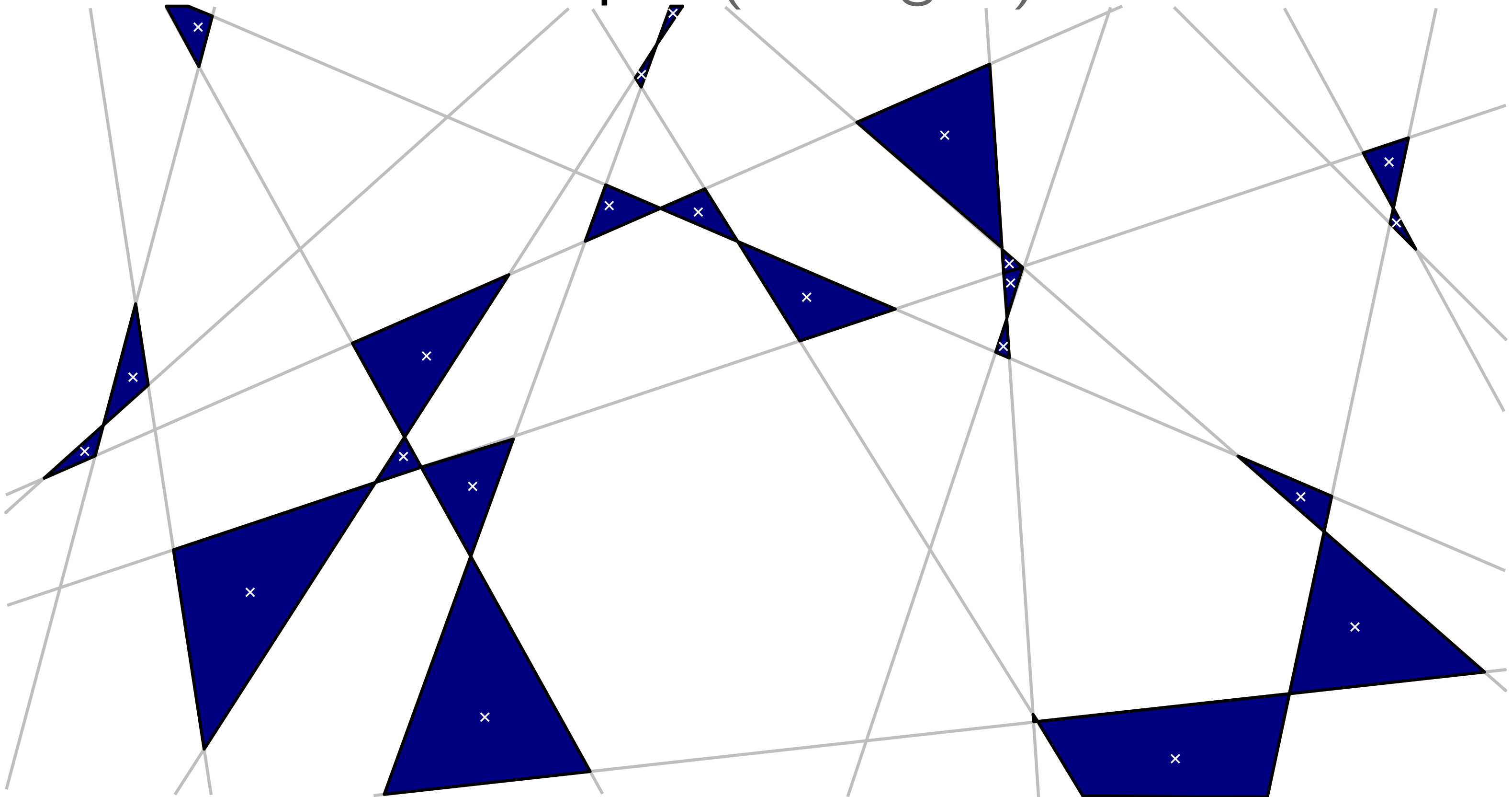


$\mathbb{P}(Z$ has n facets$)$

typical cell

23 3-topes

3-topes (Triangles)

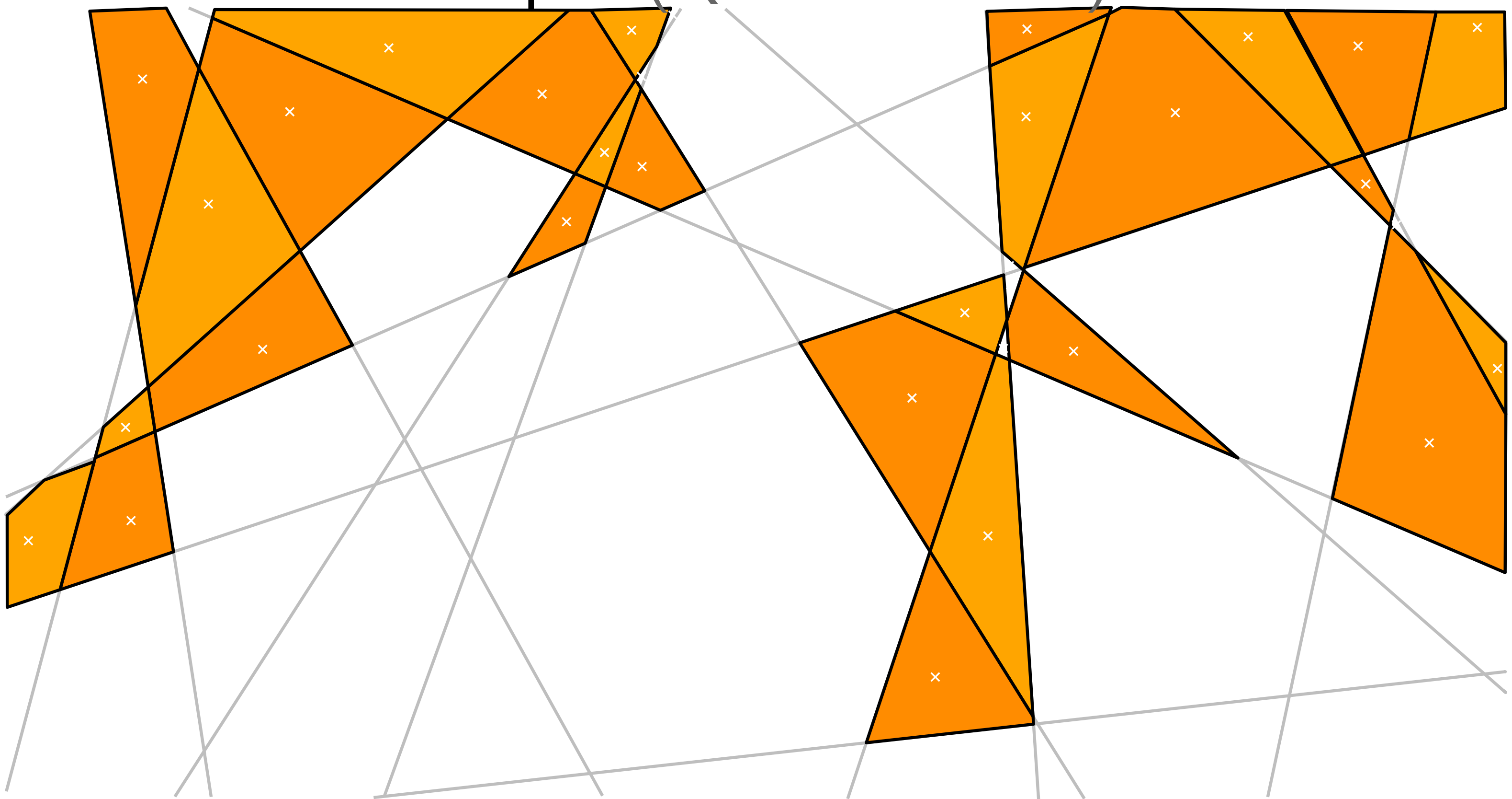


$\mathbb{P}(Z$ has n facets$)$

typical cell

23 3-topes
27 4-topes

4-topes (Quadrilaterals)

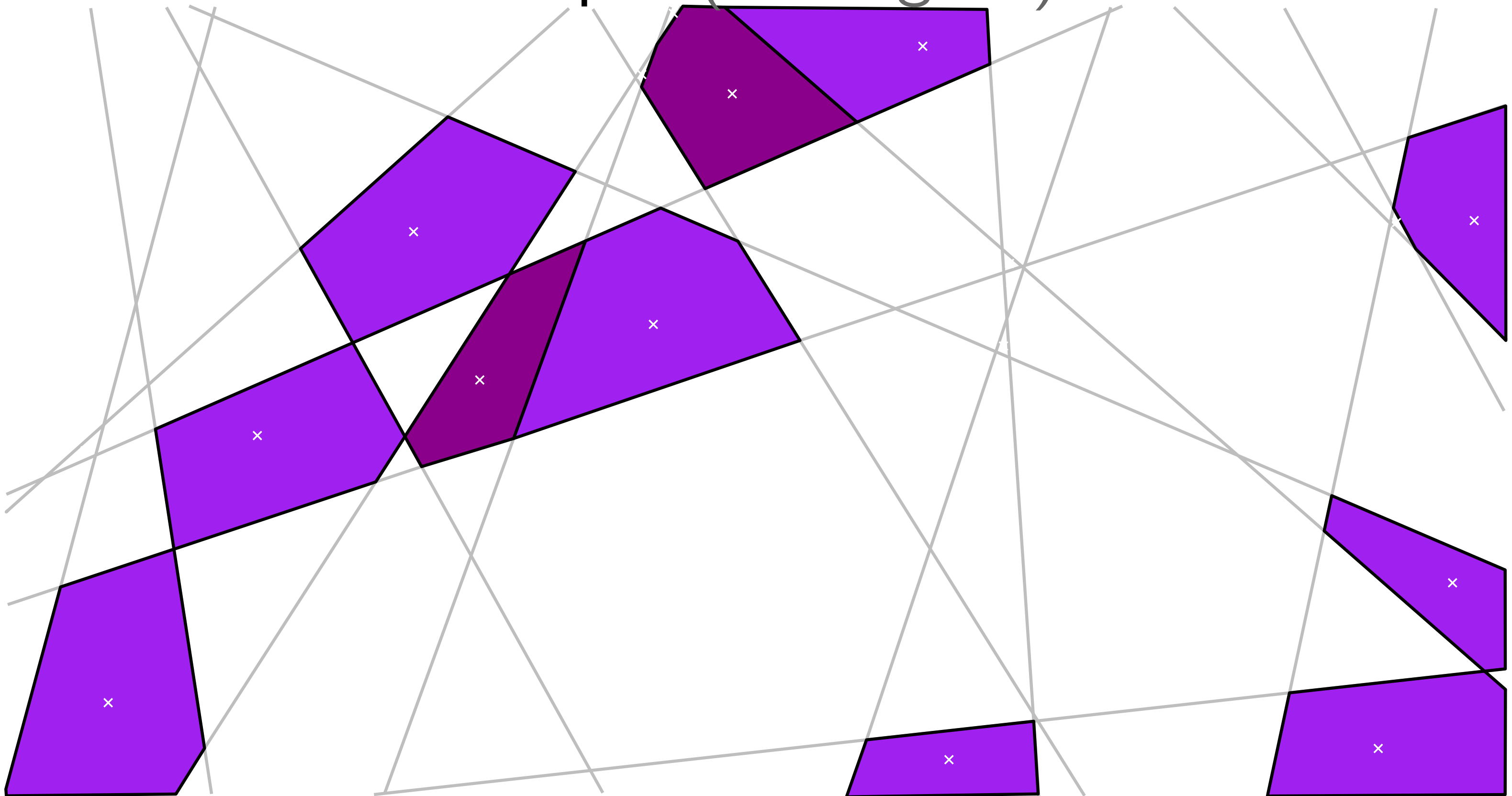


$\mathbb{P}(Z \text{ has } n \text{ facets})$

typical cell

23	3-topes
27	4-topes
11	5-topes

5-topes (Pentagons)

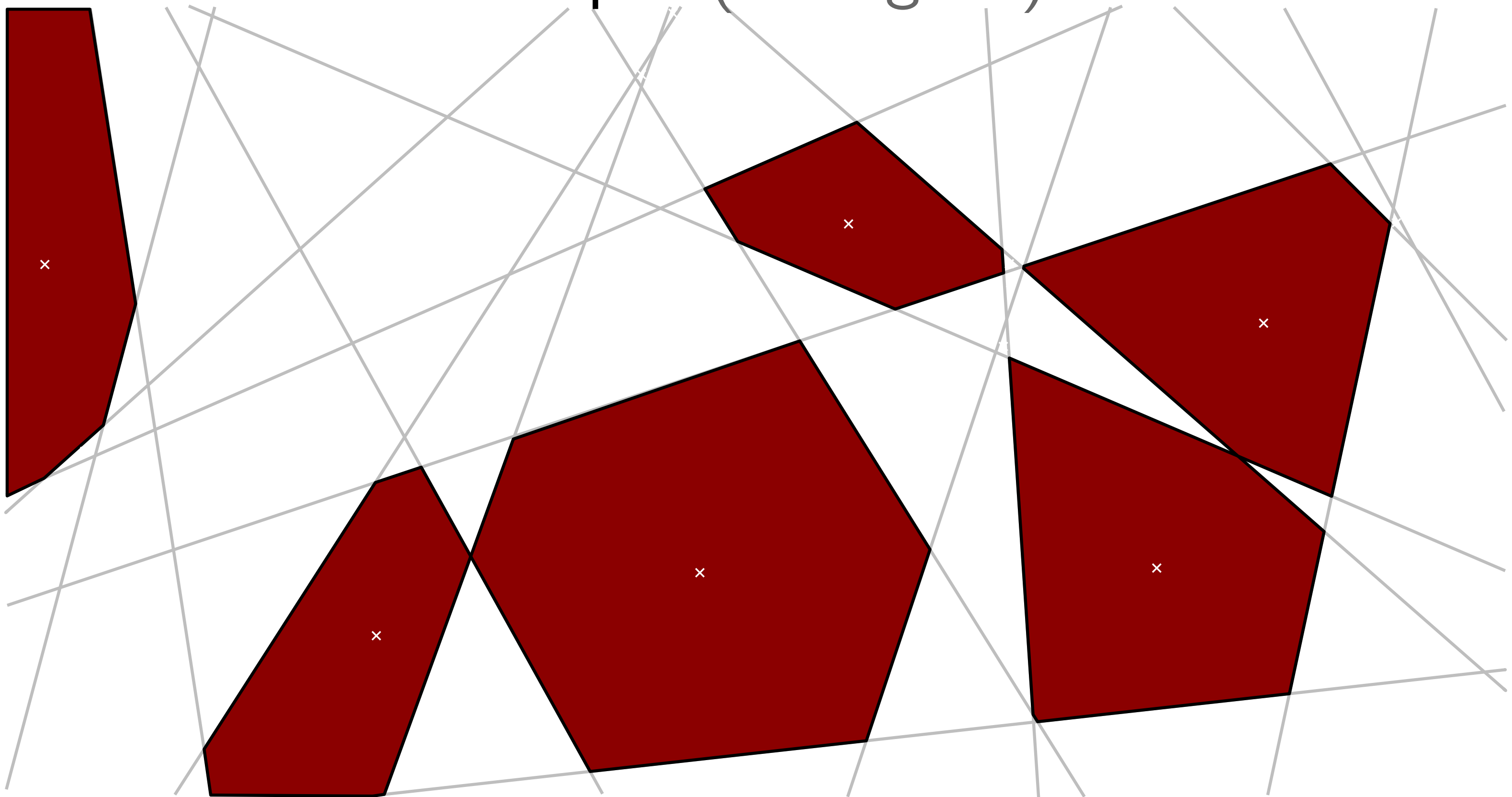


$\mathbb{P}(Z$ has n facets)

typical cell

23	3-topes
27	4-topes
11	5-topes
6	6-topes

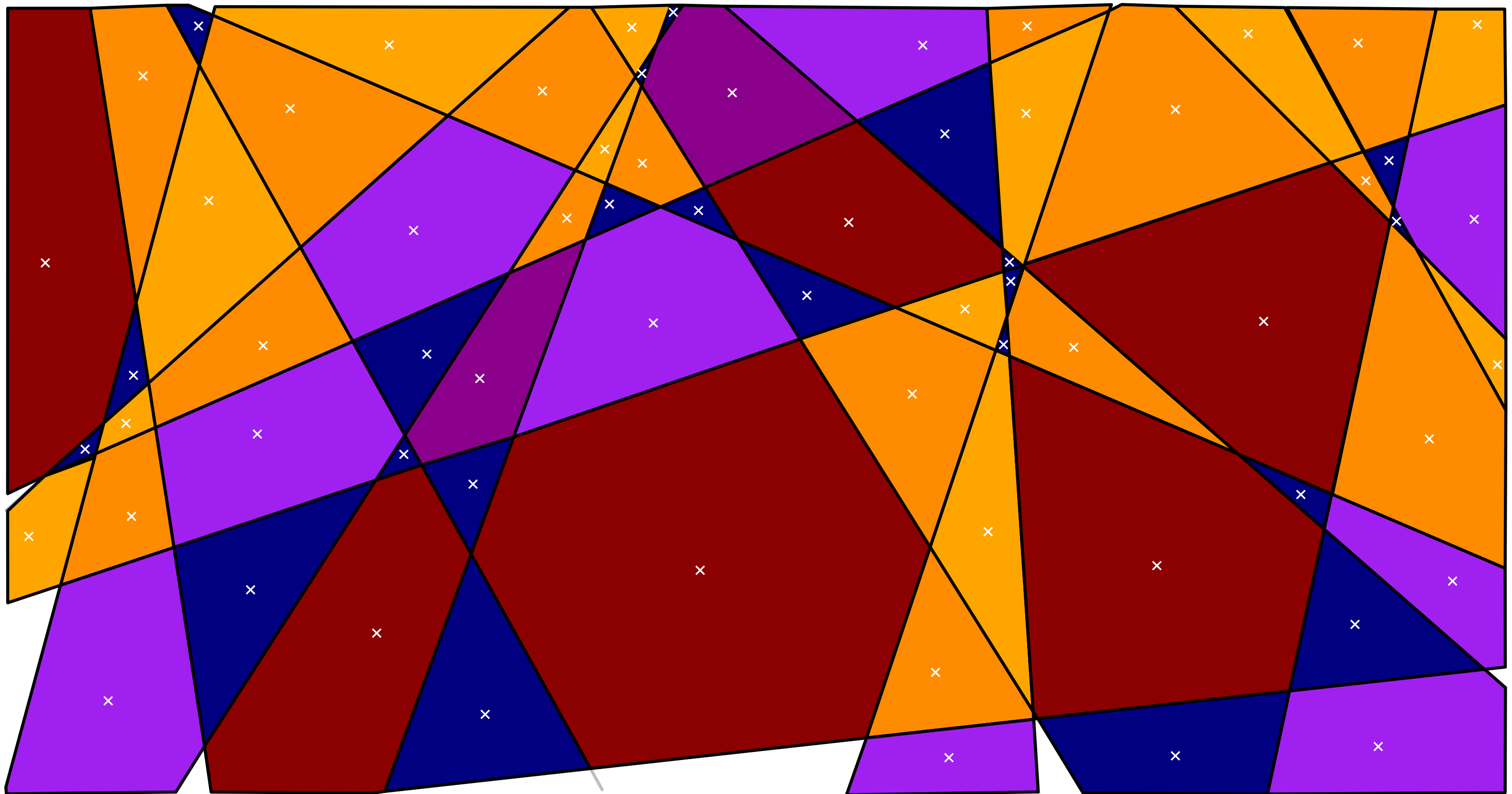
6-topes (Hexagons)



$\mathbb{P}(Z \text{ has } n \text{ facets})$

typical cell

23	3-topes
27	4-topes
11	5-topes
6	6-topes
67	cells with center in the window



$\mathbb{P}(\underbrace{Z}_{\text{typical cell}} \text{ has } n \text{ facets})$

23	3-topes	$23/67 = 0.34...$	$\simeq \mathbb{P}(Z \text{ has } 3 \text{ facets})$
27	4-topes	$27/67 = 0.40...$	$\simeq \mathbb{P}(Z \text{ has } 4 \text{ facets})$
11	5-topes	$11/67 = 0.16...$	$\simeq \mathbb{P}(Z \text{ has } 5 \text{ facets})$
6	6-topes	$6/67 = 0.09...$	$\simeq \mathbb{P}(Z \text{ has } 6 \text{ facets})$

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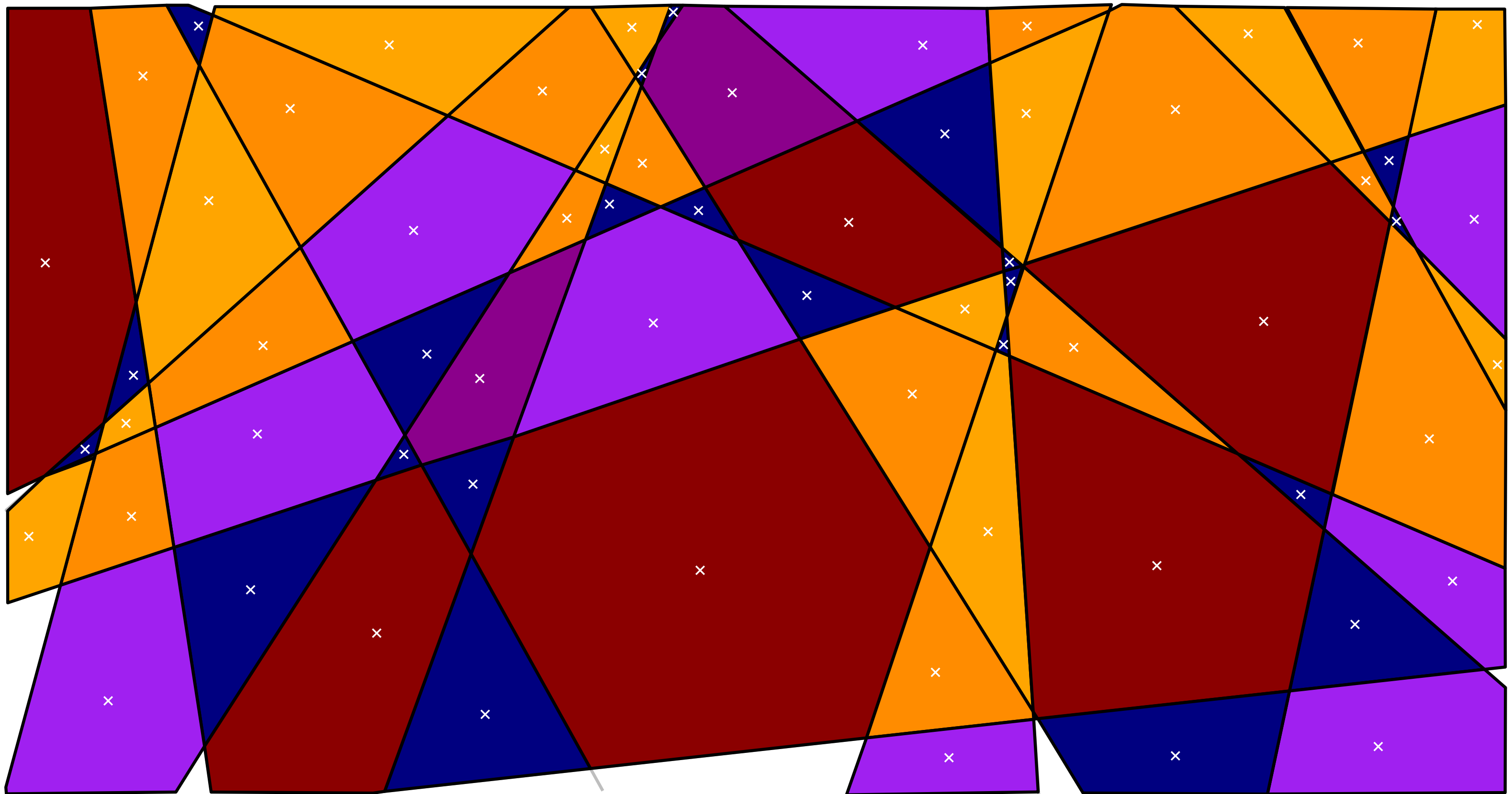

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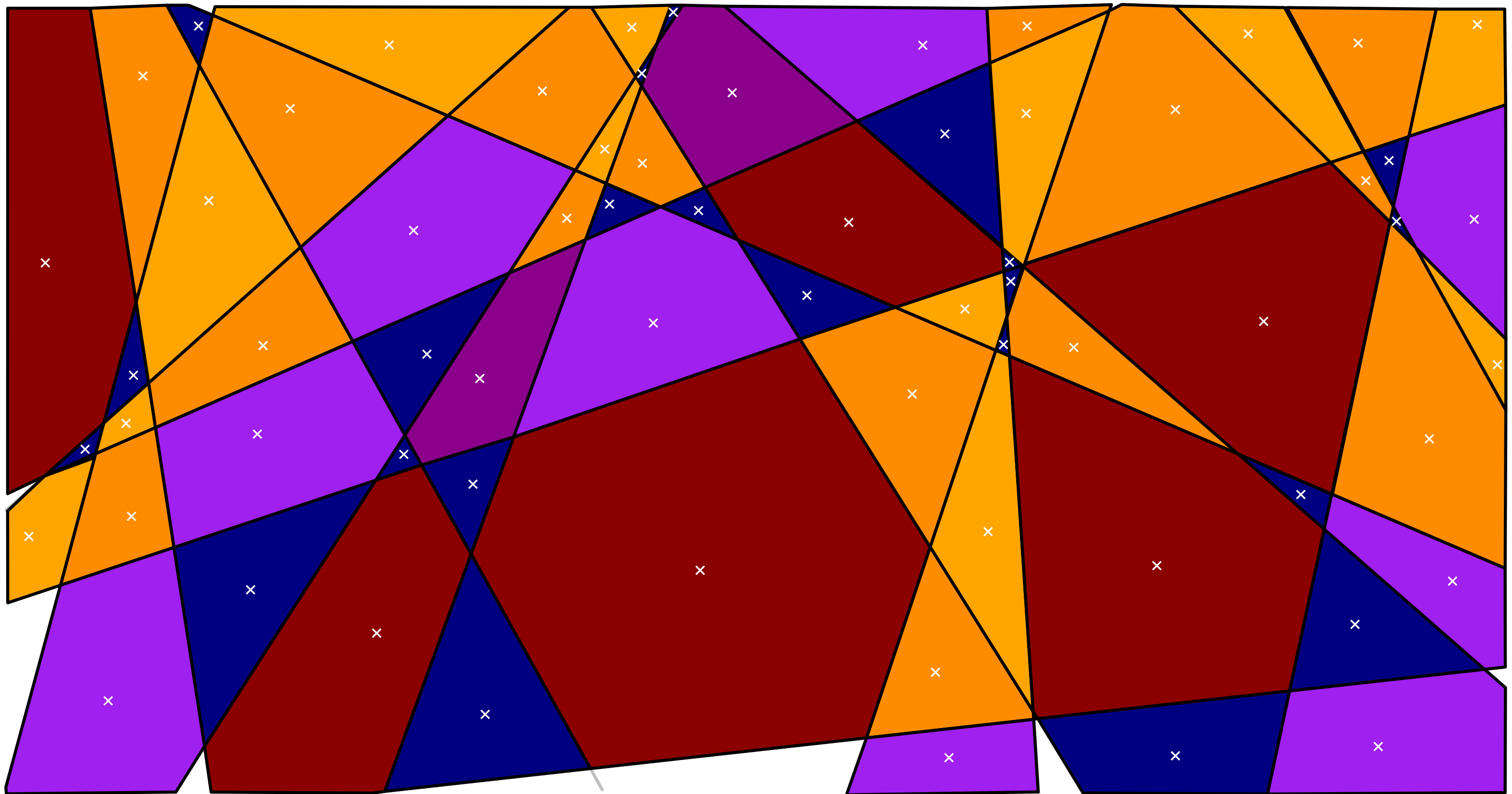
		$0/67 = 0$	$\simeq \mathbb{P}(Z$ has 7 or more facets$)$

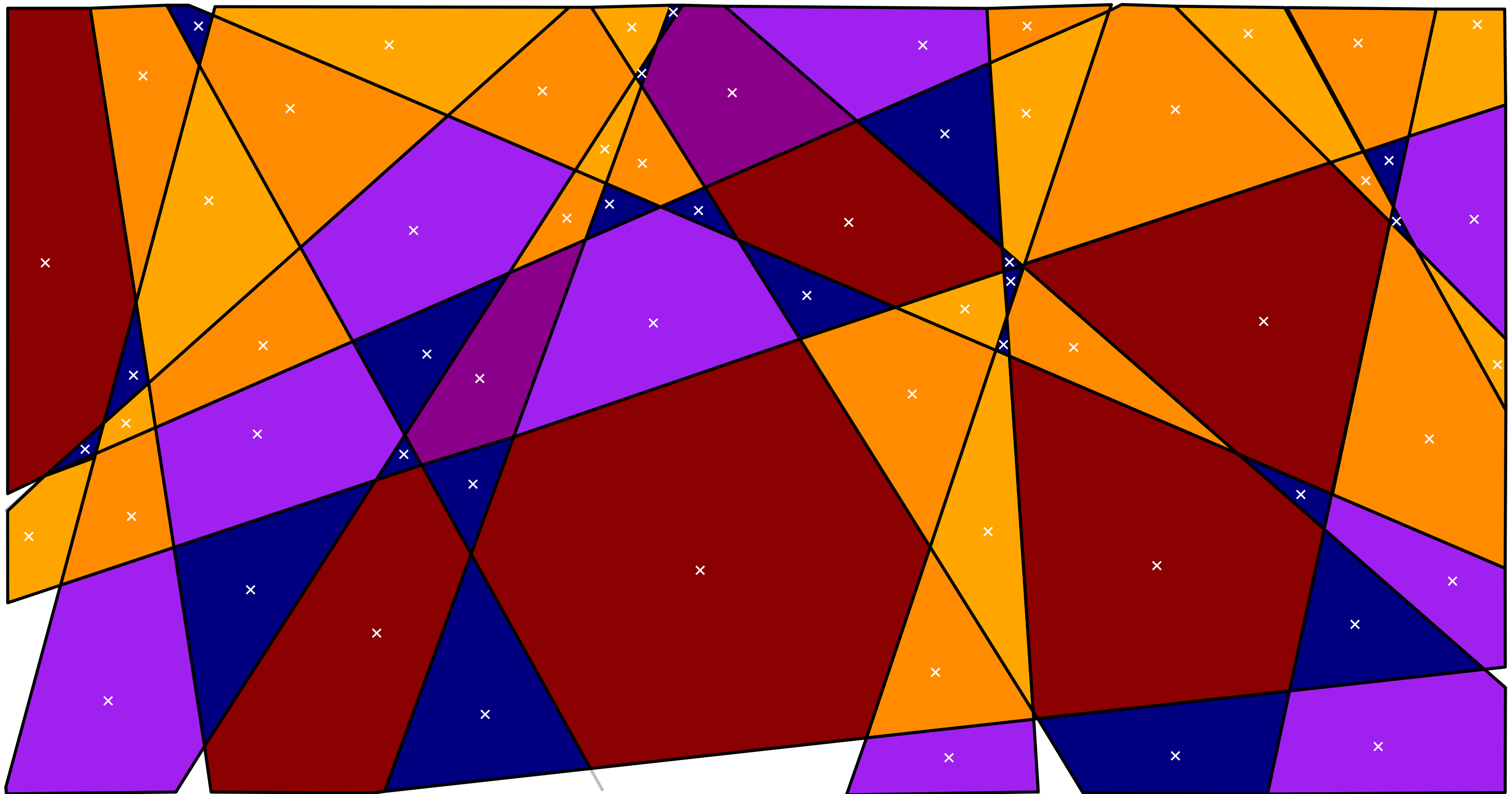


$\mathbb{P}(Z$ has n facets$)$

$d = 2$, isotropy

typical cell

$\mathbb{P}(Z$ has 3 facets$) = 2 - \pi^2/6 = 0.36\ldots$ [Miles 1964]



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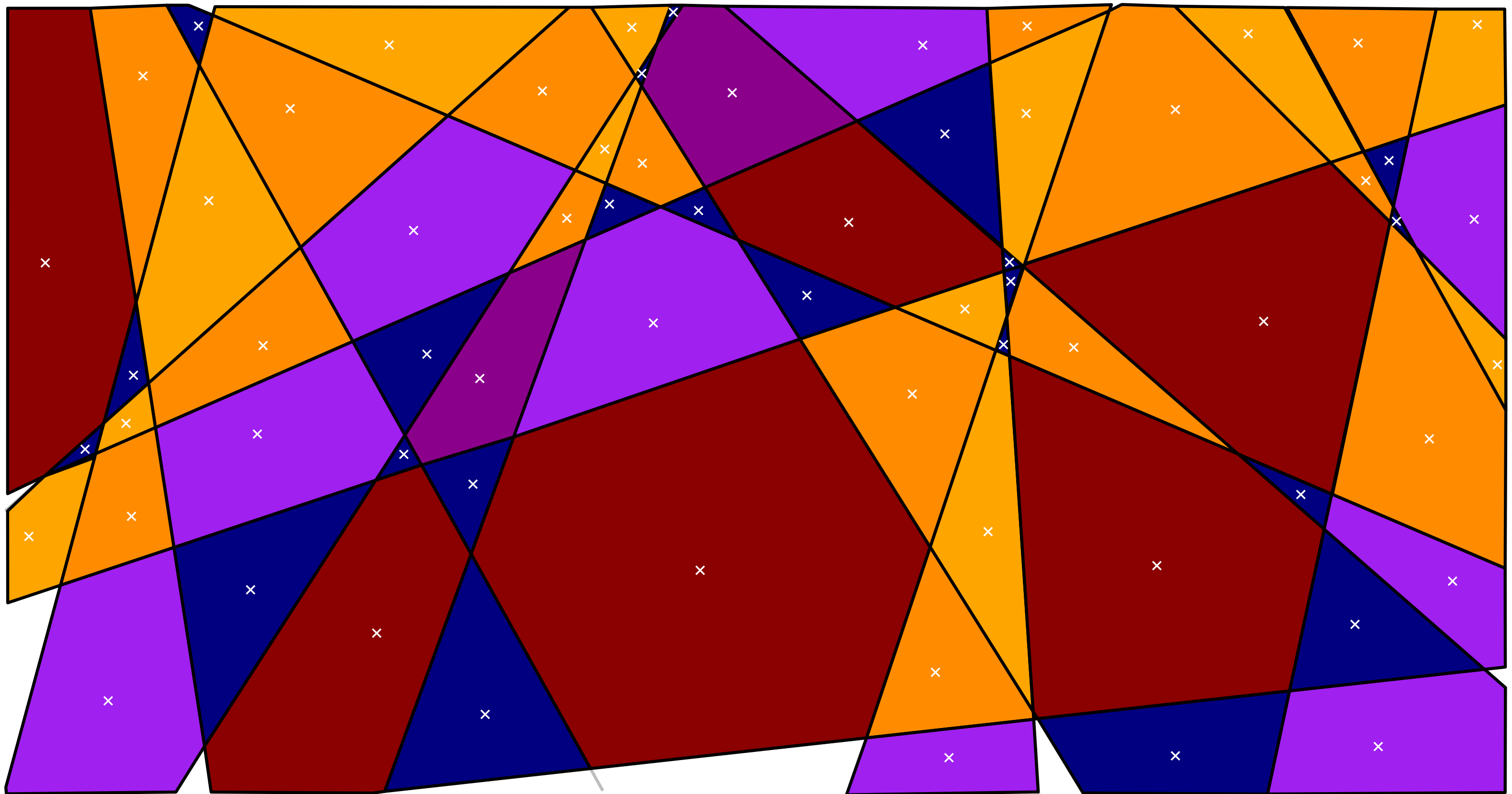
$\mathbb{P}(Z$ has 4 facets$) = \pi^2 \log 2 - \frac{1}{3} - \frac{7}{36}\pi^2 - \frac{7}{2}\zeta(3) = 0.38...$

[Tanner 1983]



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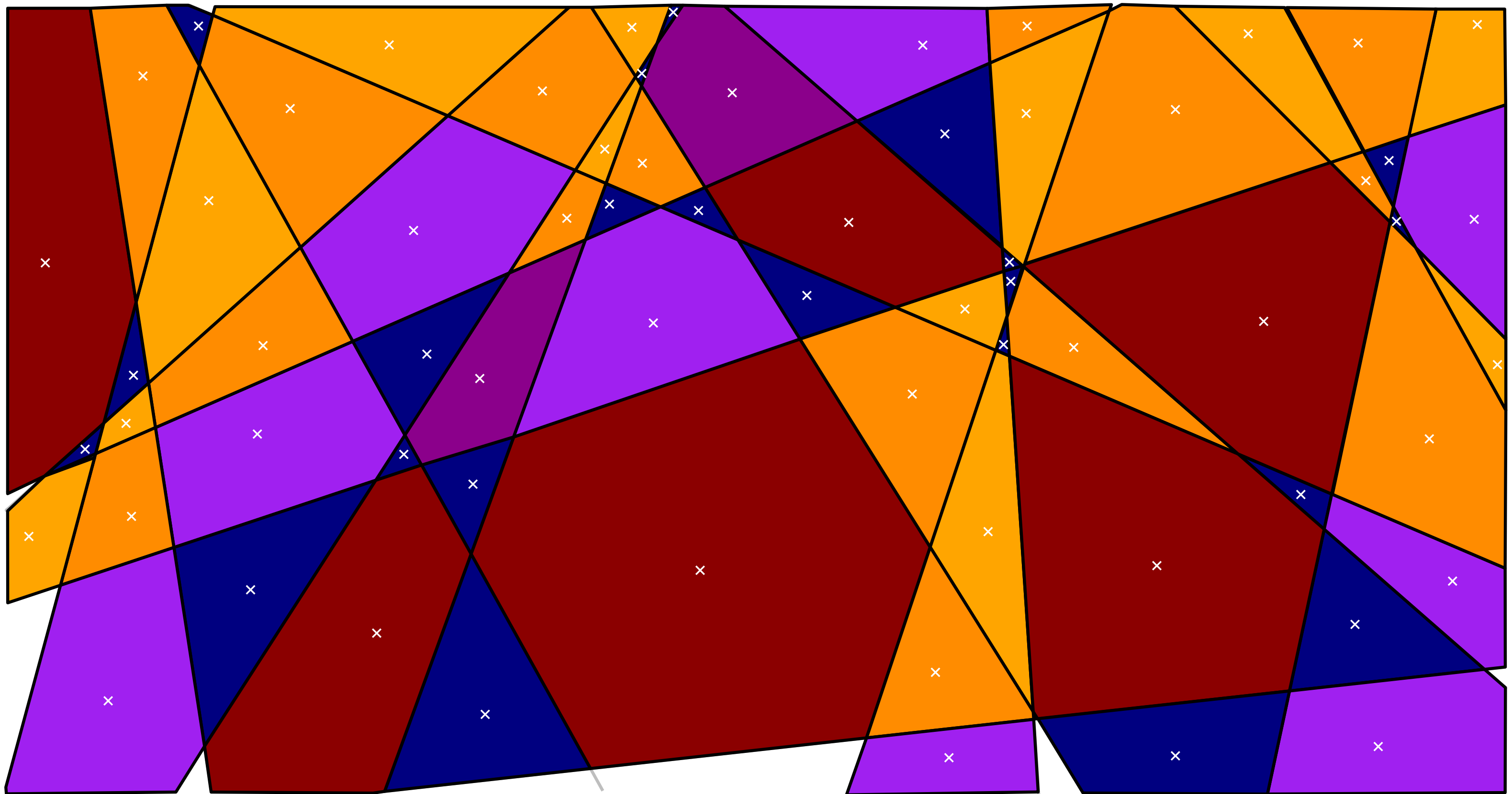
Approximation by Monte Carlo simulation for $n = 5, ..., 12$

[Crain and Miles 1976] [George 1987] [Michel and Paroux 2007]



Goal

$$\mathbb{P}(Z \text{ has } n \text{ facets})?$$

typical cell

when $n \to \infty$

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$$\mathbb{P}(Z \text{ has } n \text{ facets})?$$

when $n \to \infty$

typical cell

In a specific case it is already known:

Theorem [Calka and Hilhorst 2008]

In the 2-dimensional isotropic case we have that

$$\mathbb{P}(Z \text{ has } n \text{ facets}) \sim \alpha\boldsymbol{\beta^{n} n^{-2n}} n^{-3/2} \text{ when } n \to \infty$$

with $\alpha = 2/(3\pi^{5/2})$ and $\beta = \pi^2 e^2$

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$$\mathbb{P}(Z \text{ has } n \text{ facets})?$$

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We generalize this to any dimension and nice directional distribution:

Main Theorem

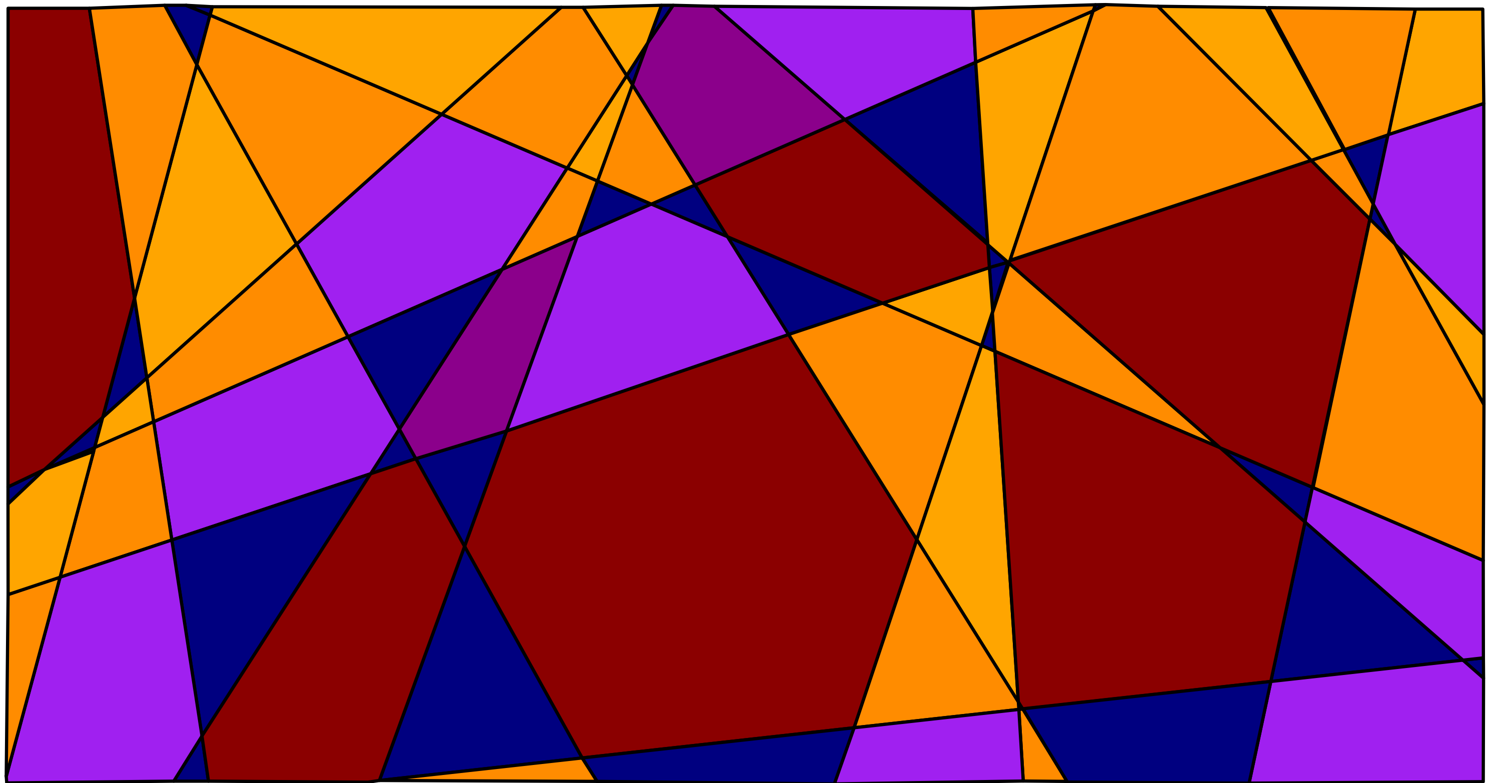
There exist constants c_1 and c_2 depending on d and φ such that for n big enough we have

$$c_1^n n^{-2n/(d-1)} < \mathbb{P}(Z \text{ has } n \text{ facets}) < c_2^n n^{-2n/(d-1)}$$

Typical Cell Z

$X = X_\eta$ . . . **Mosaic**: Point Process in $\mathcal{P}$

Set of polytopes



Typical Cell Z

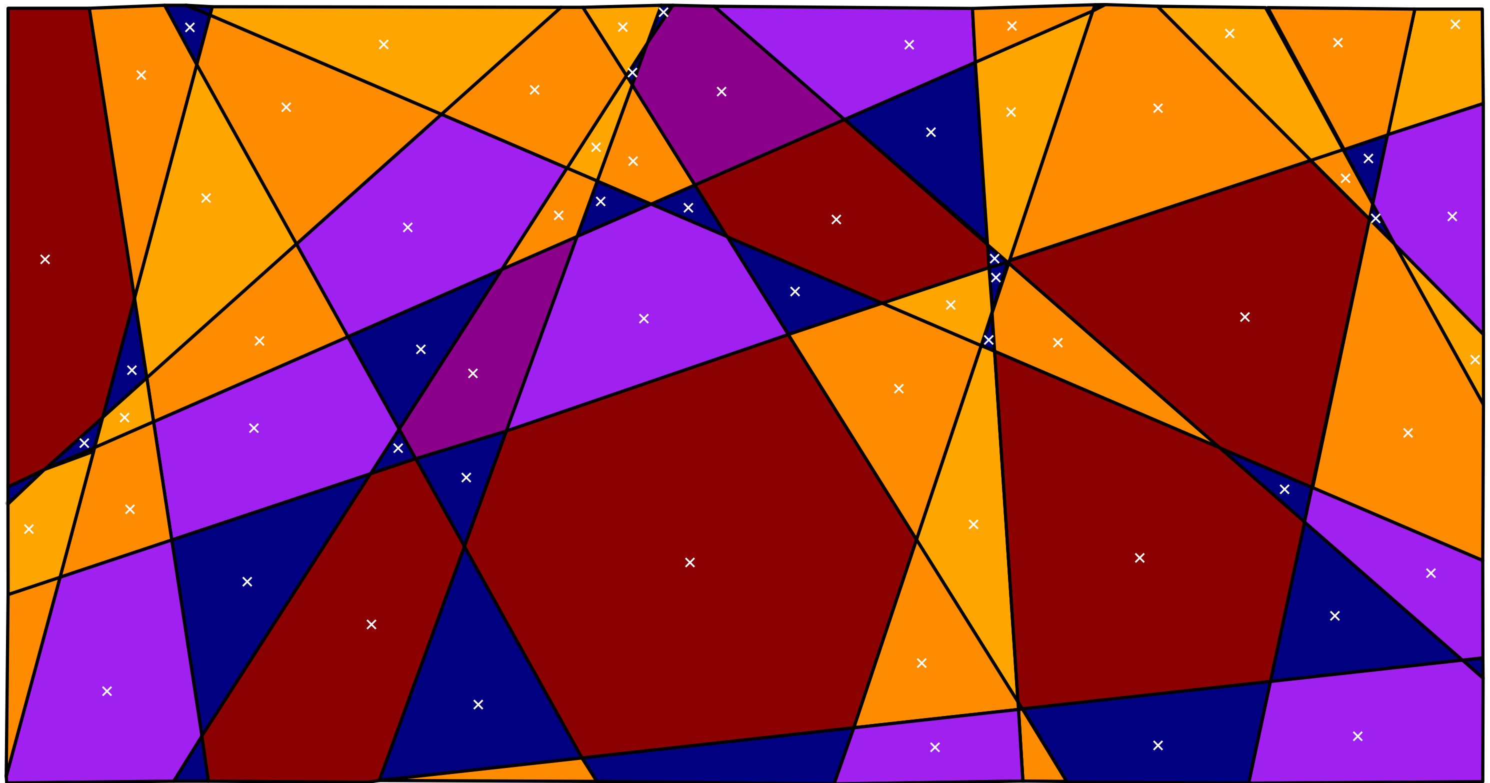
$X = X_\eta$ . . . **Mosaic**: Point Process in $\mathcal{P}$

Set of polytopes

$\mathfrak{c} : \mathcal{P} \to \mathbb{R}^d$ a **center function**

e.g. center of mass, center of the circumball...

$$\mathfrak{c}(tP + x) = t\mathfrak{c}(P) + x$$



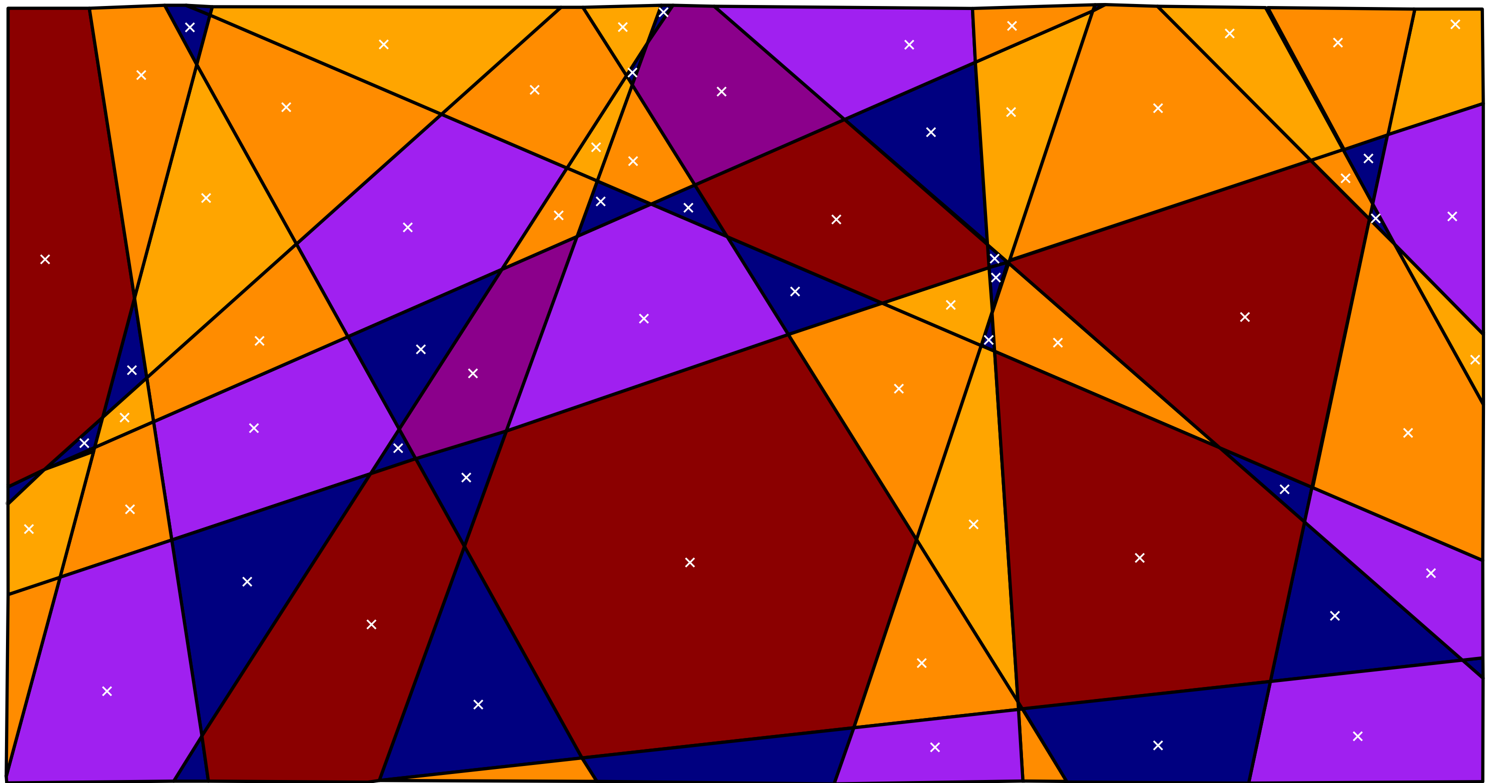
Typical Cell Z

$X = X_\eta$ ... **Mosaic**: Point Process in $\mathbb{R}^d \times \mathcal{P}_\mathfrak{c}$

Set of centred polytopes

$\mathfrak{c} : \mathcal{P} \to \mathbb{R}^d$ a **center function**

X is a germ-grain process with **germs** $\mathfrak{c}(P)$ and **grains** $P - \mathfrak{c}(P)$



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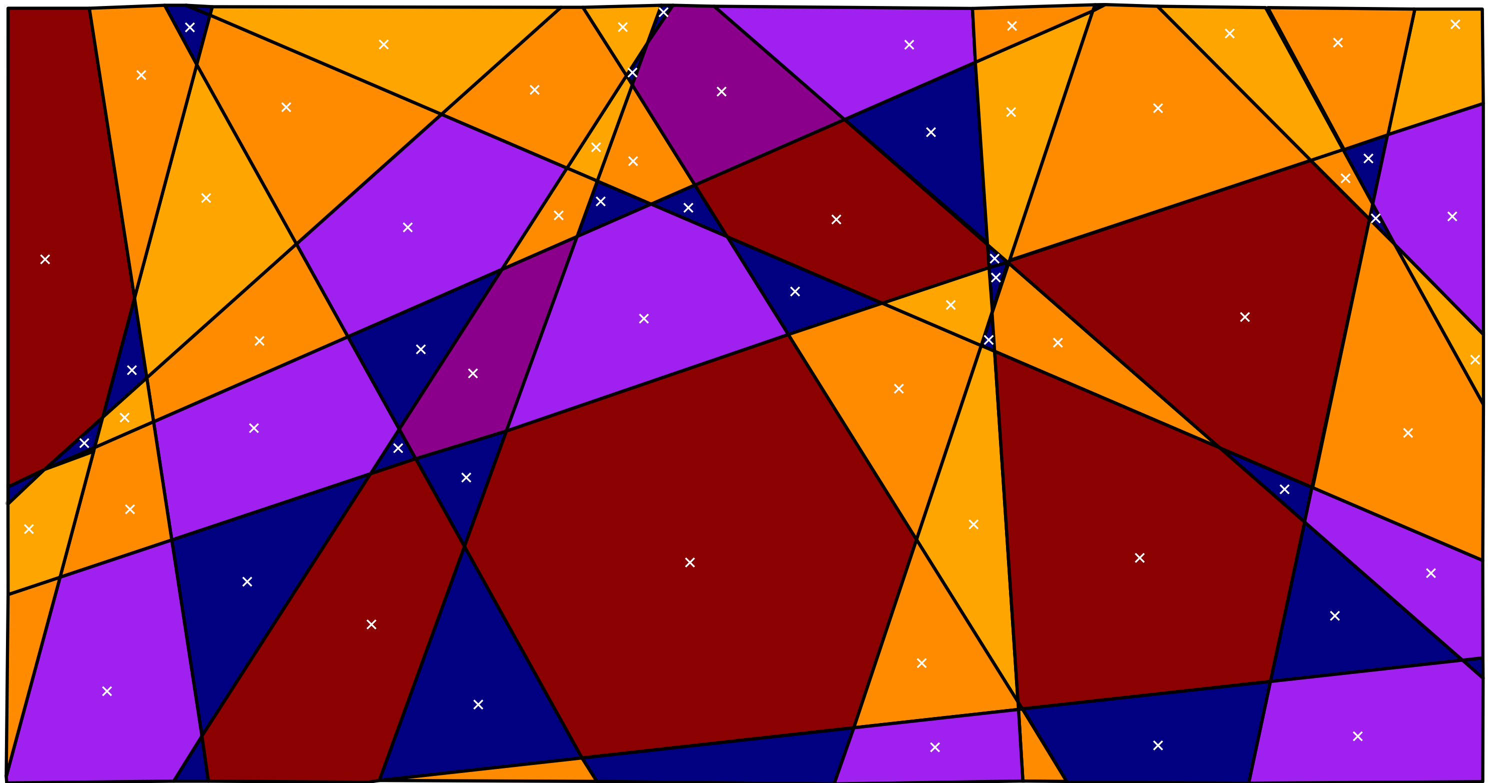
Its intensity measure has the form $\gamma\lambda_d \otimes \mathbb{Q}$

grain distribution

intensity

Lebesgue measure



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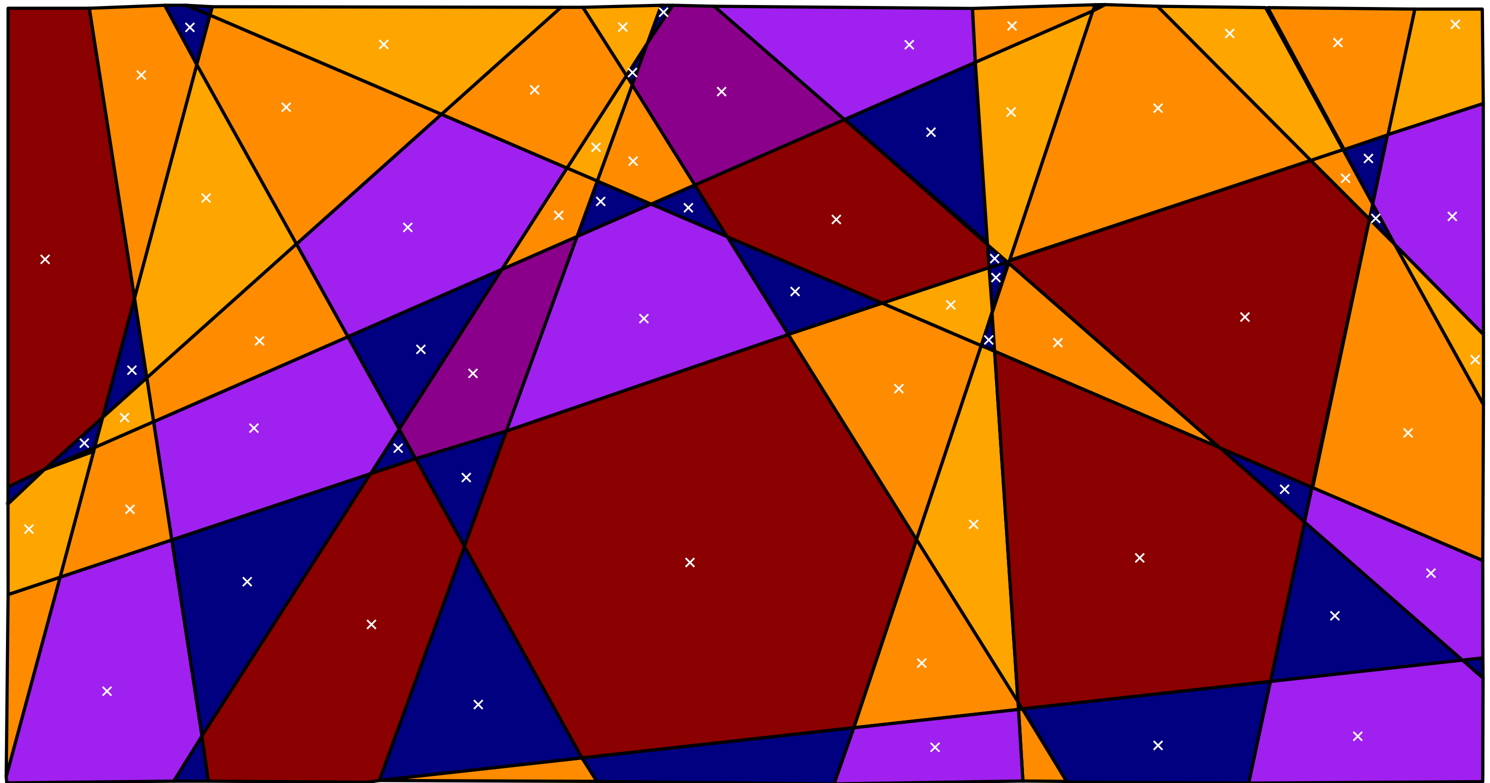
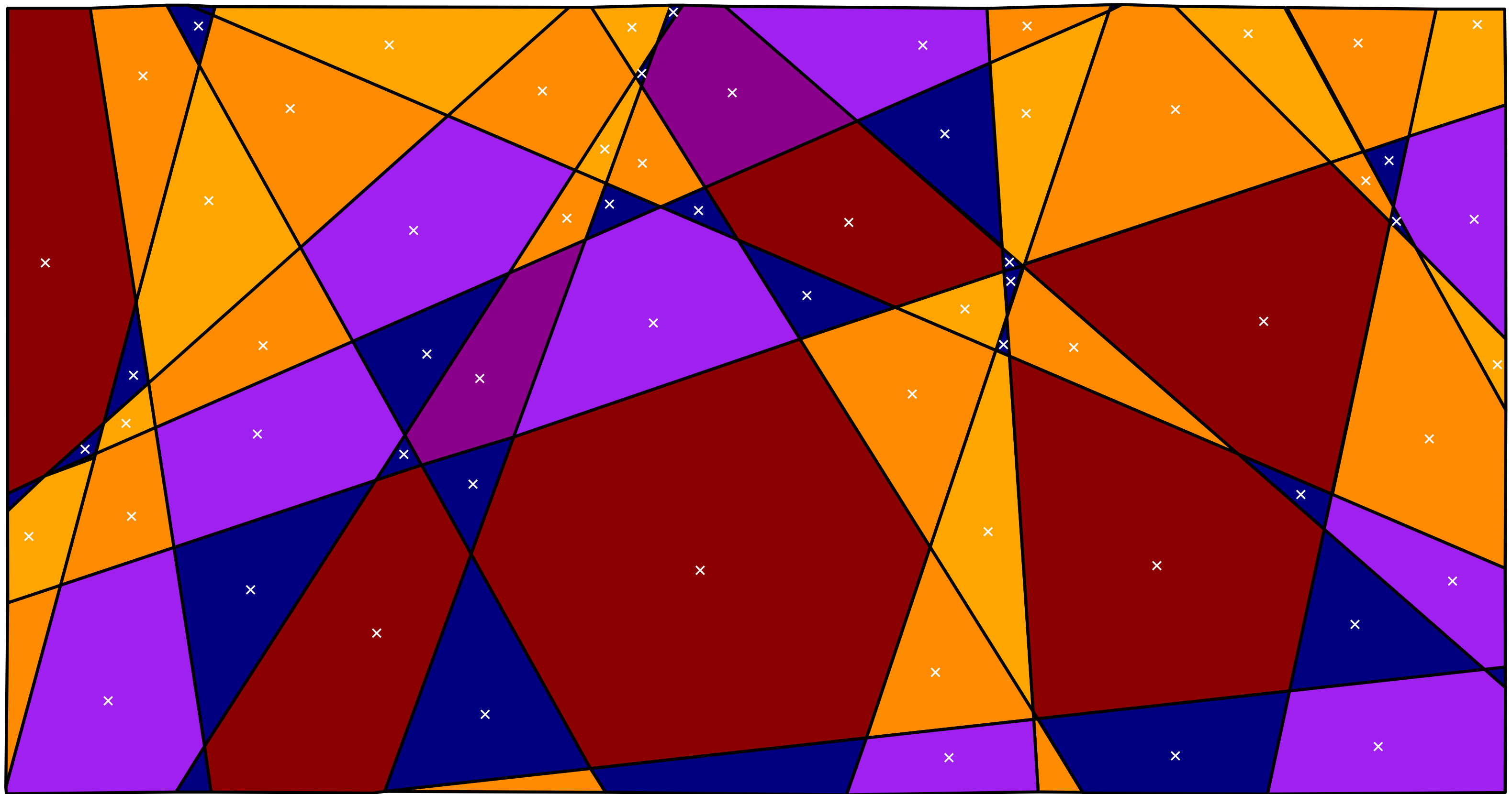
intensity

Lebesgue measure

Z . . . **Typical cell** $=$ random centred polytope with distribution $\mathbb{Q}$



Cells With n Facets

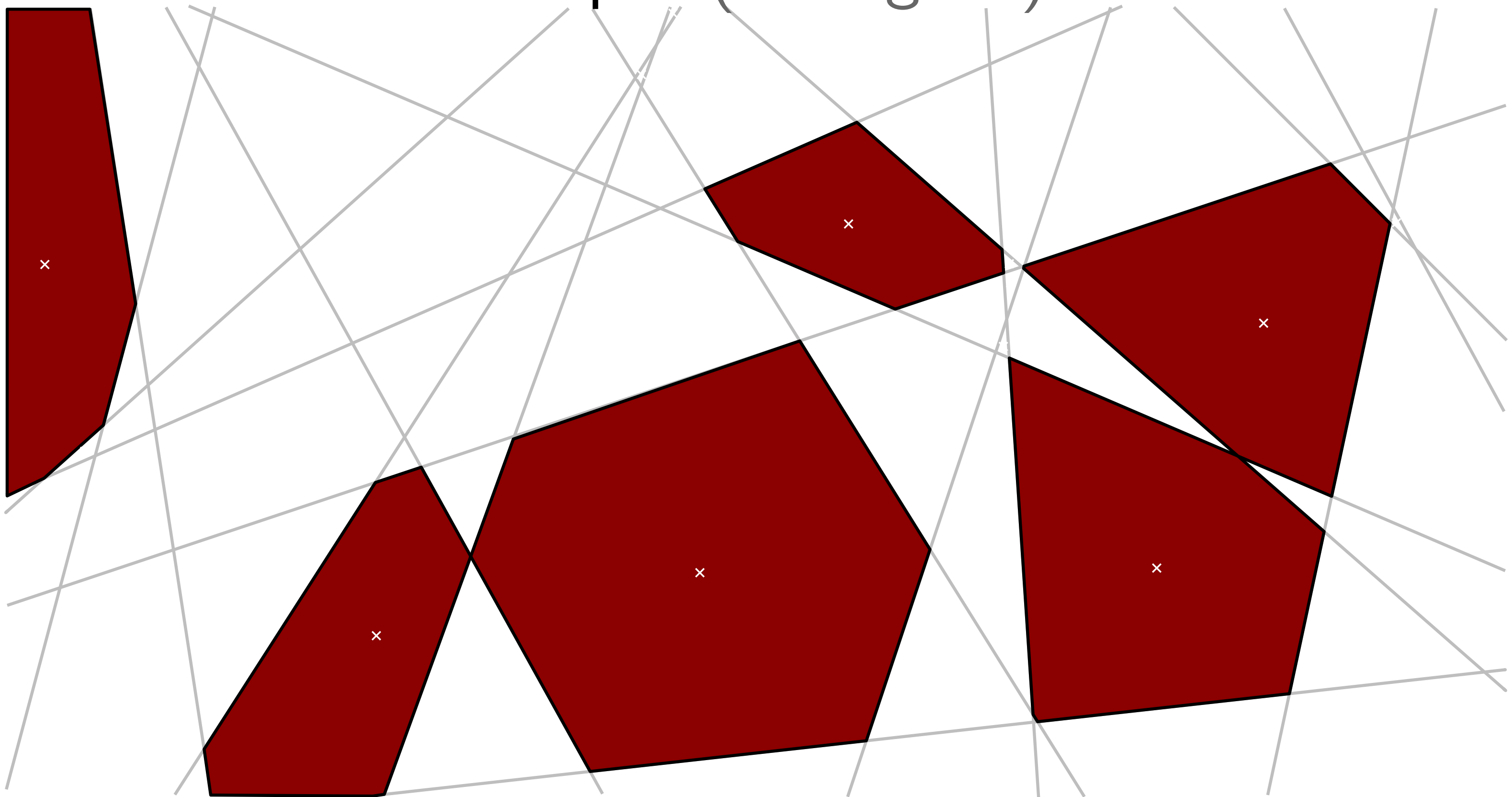


Cells With n Facets

Set of centred n-topes

$$\begin{aligned}\mathbb{P}(Z \text{ has } n \text{ facets}) &= \mathbb{Q}(\mathcal{P}_{n,\mathfrak{c}}) \\ &= \gamma^{-1}\gamma\lambda_d([0,1]^d)\mathbb{Q}(\mathcal{P}_{n,\mathfrak{c}}) \\ &= \gamma^{-1}\mathbb{E}\,\underbrace{X(\mathcal{P}_{n,[0,1]^d})}_{\text{number of } n\text{-topes of } X \text{ with center in } [0,1]^d}\end{aligned}$$

6-topes (Hexagons)

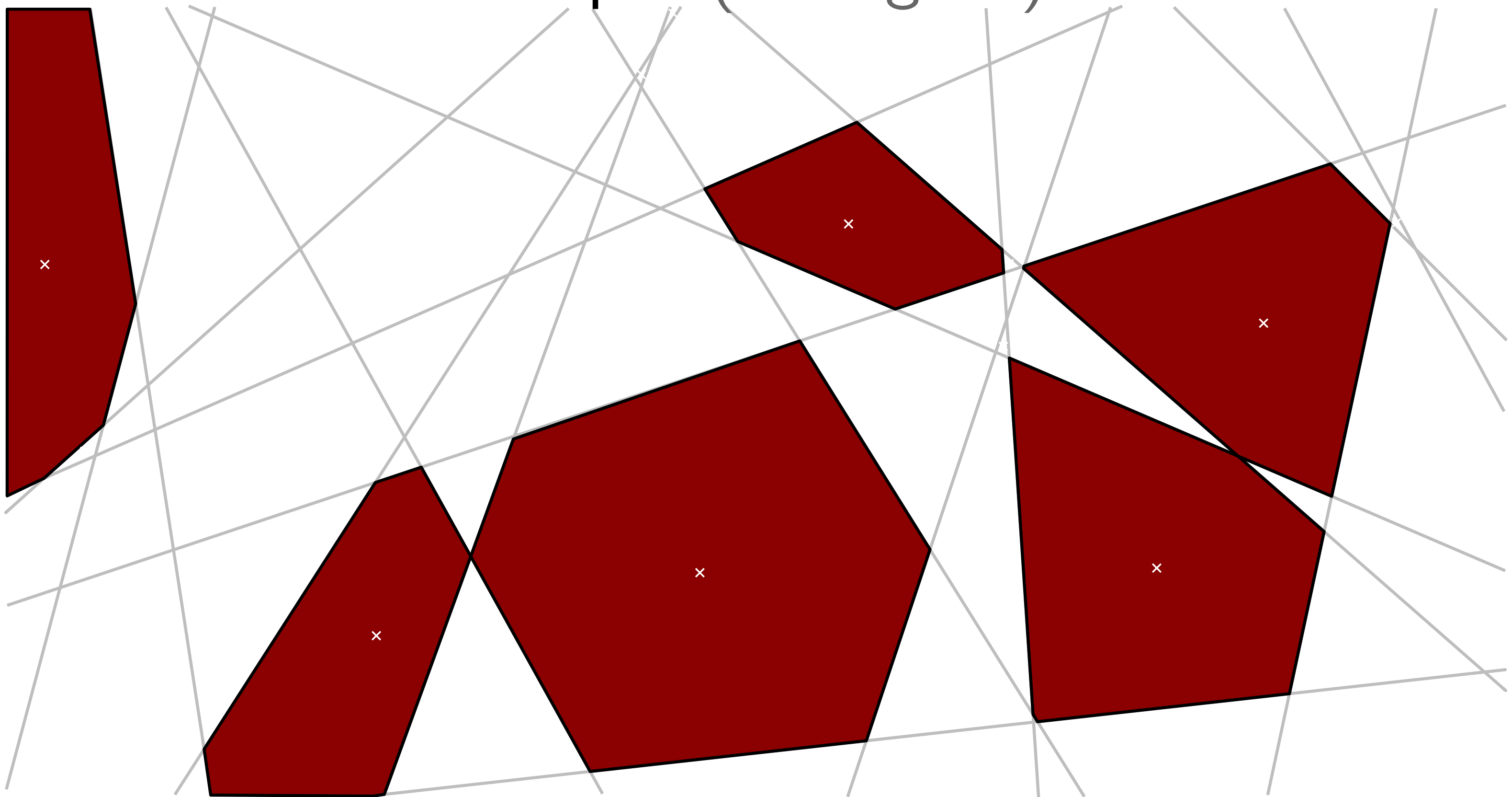


Cells With n Facets

Let $D \subset \mathcal{P}_n$. number of cells of X in D $P = \bigcap_{i=1}^{n} H_i^{\epsilon_i}$

$$X(D) = \frac{1}{n!} \sum_{P \in \eta^n_{\neq} \times \{\pm 1\}^n} \mathbf{1}(P \in D) \, \mathbf{1}(\eta \cap P = \emptyset) .$$

6-topes (Hexagons)



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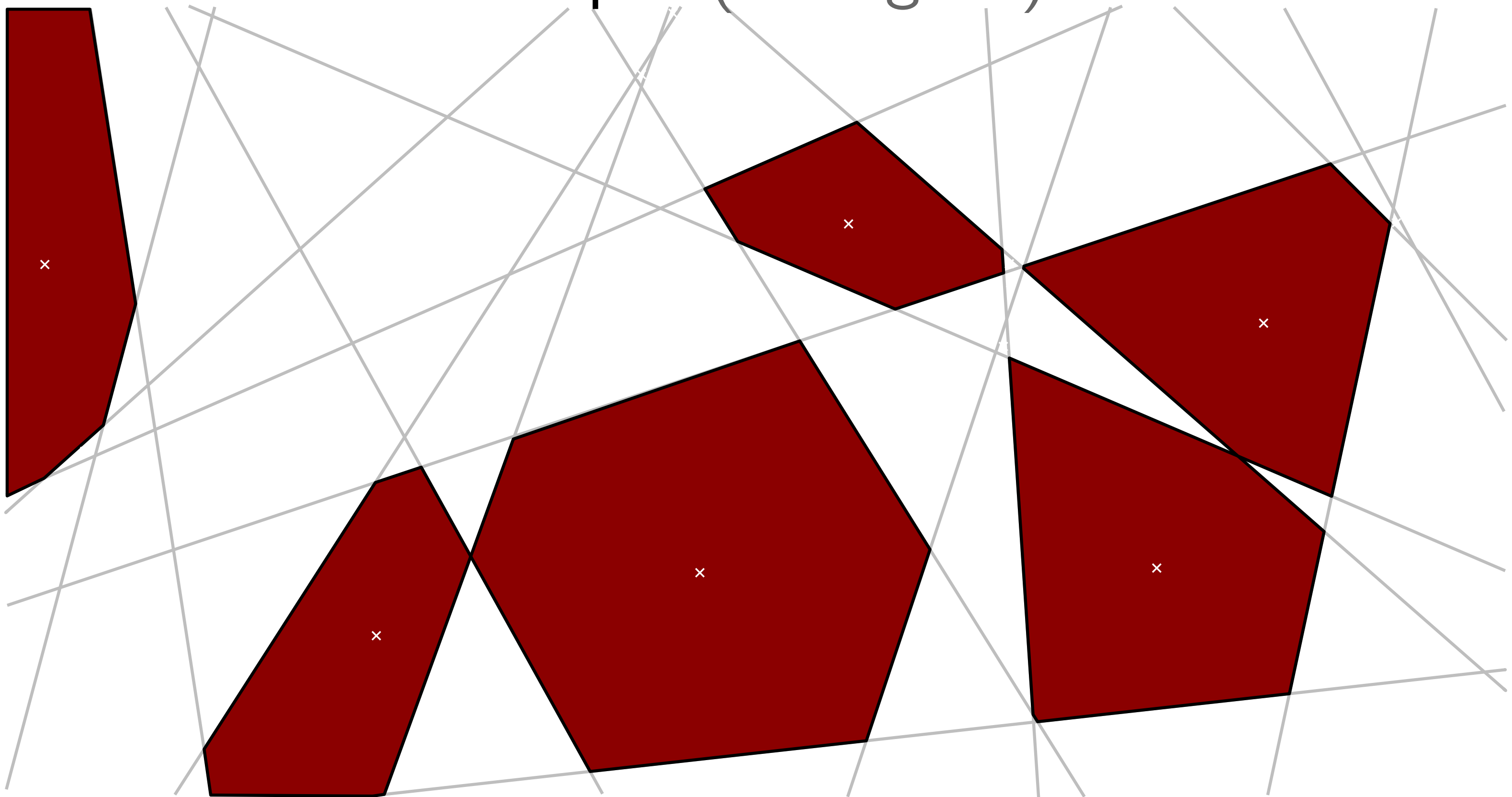
6-topes (Hexagons)



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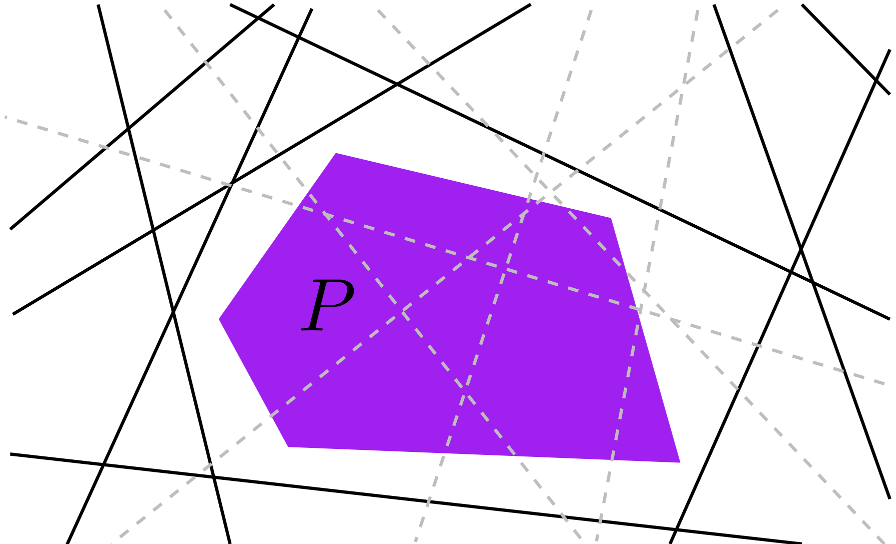
$$\mathbb{P}(\eta \cap P = \emptyset) = \mathbb{P}\left(\eta\left(\mathcal{H}_P\right) = 0\right)$$

$\mathcal{H}_P = \{H \text{ hyperplane} \mid H \cap P \neq \emptyset\}$



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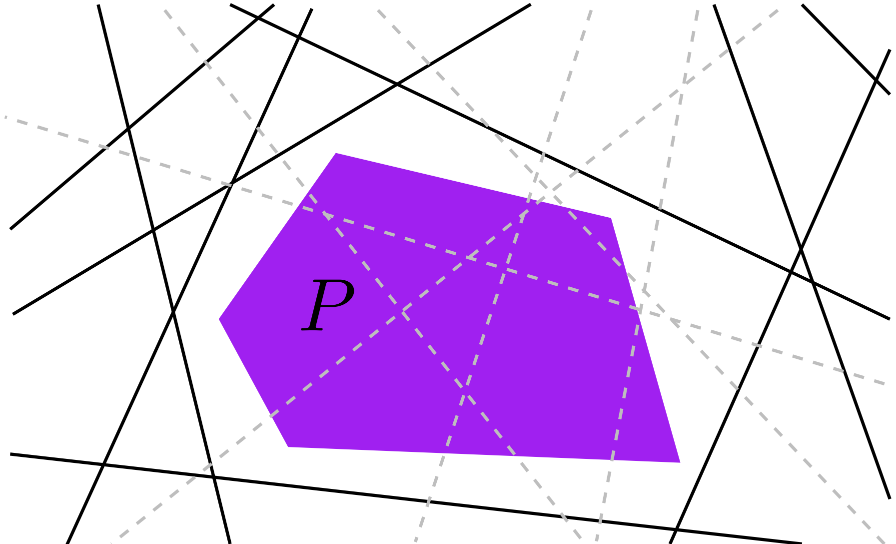

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$$\Phi(P) := \Theta(\mathcal{H}_P) = \int_{\mathbb{S}^{d-1}}\int_0^\infty \mathbb{1}\{H(\boldsymbol{u},t) \cap P \neq \emptyset\}\,\mathrm{d}t\,\varphi(\mathrm{d}\boldsymbol{u})$$

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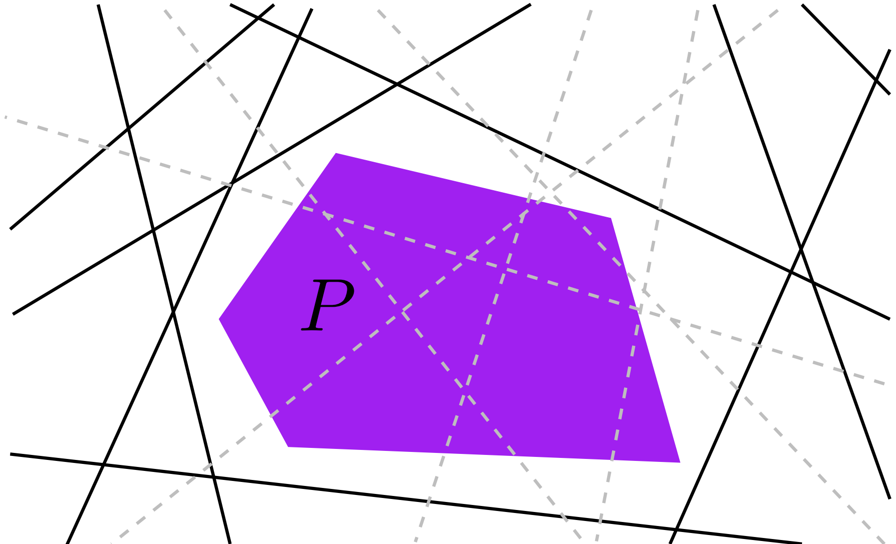

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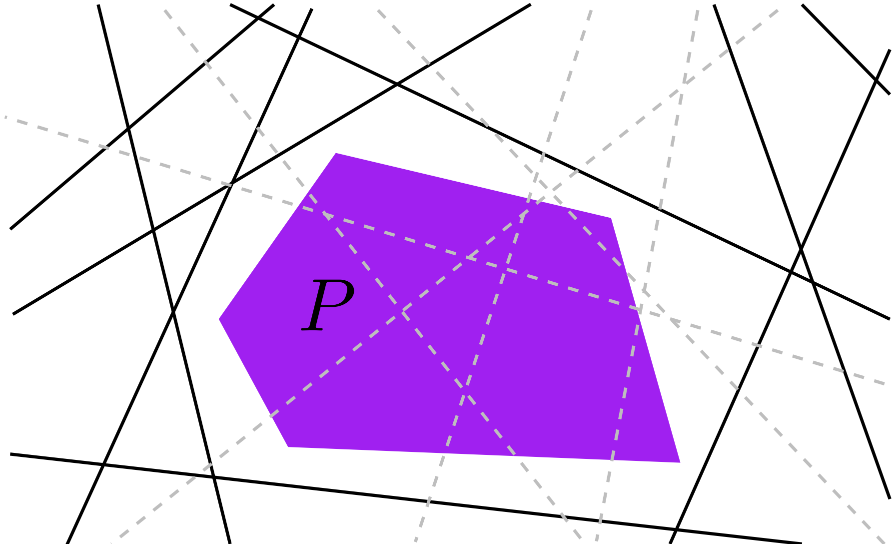

$$\Phi : \mathcal{P} \to (0,\infty)$$
$$\mathbb{P}(\eta \cap P = \emptyset) = \exp(-\Phi(P))$$
$$\Phi(tP + x) = t\Phi(P)$$

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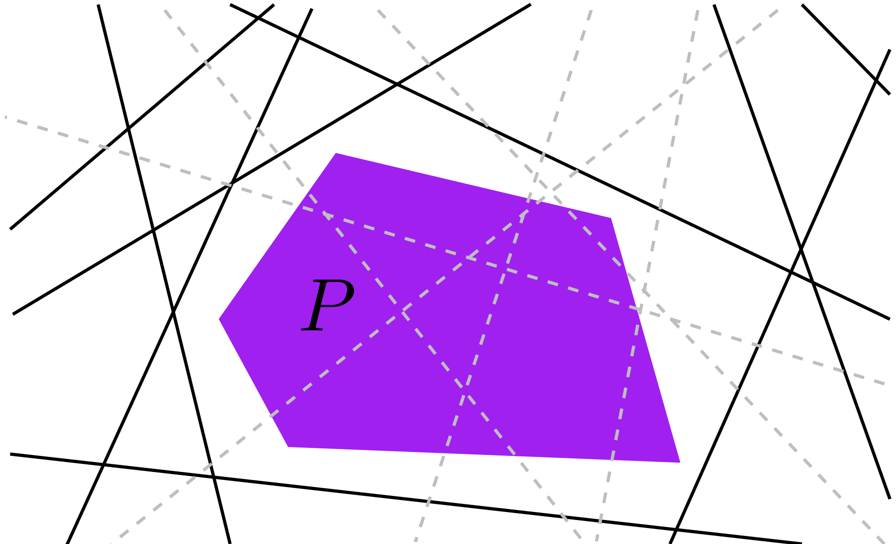

$$\Phi : \mathcal{P} \to (0, \infty)$$
$$\mathbb{P}(\eta \cap P = \emptyset) = \exp(-\Phi(P))$$
$$\Phi(tP + x) = t\Phi(P)$$

Cells With n Facets

Let $D \subset \mathcal{P}_n$.

number of cells of X in D

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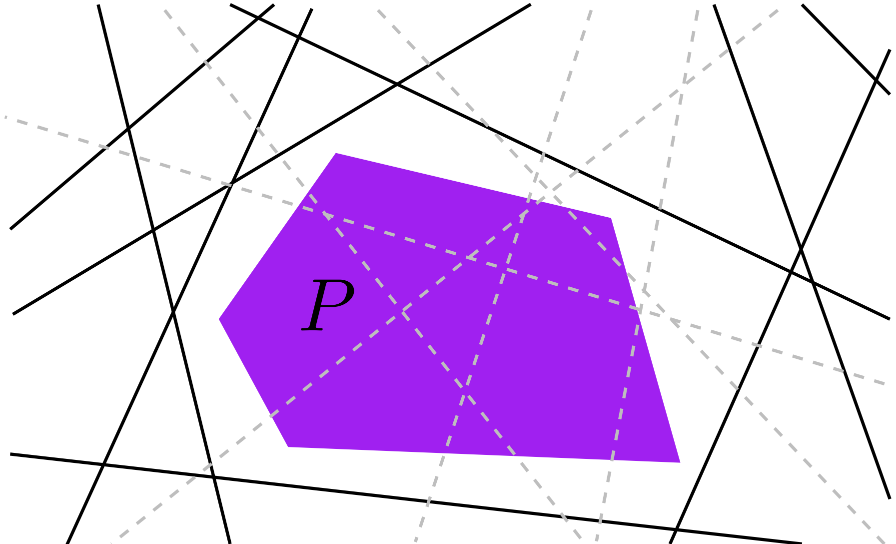

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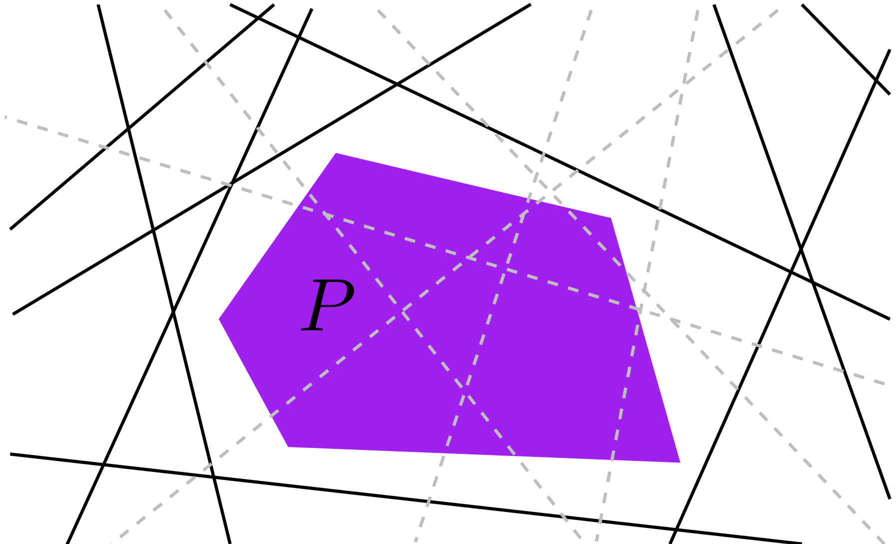
Cells With n Facets

Let $D \subset \mathcal{P}_n$. number of cells of X in D $P = \bigcap_{i=1}^{n} H_i^{\epsilon_i}$

$$\mathbb{E}X(D) = \int_{\mathcal{P}_n} \mathbf{1}\left(P \in D\right) \exp\left(-\Phi\left(P\right)\right) \Theta_n(\mathrm{d}P)$$

where $\Theta_n(\cdot) := \frac{1}{n!}\int \cdots \int_{\mathcal{H}^n} \sum_{\boldsymbol{\epsilon}\in\{\pm 1\}^n} \mathbf{1}\left(P \in \cdot\right) \Theta(\mathrm{d}H_n) \cdots \Theta(\mathrm{d}H_1)$



Θ_n measure on $\mathcal{P}_n$

$\Theta_n(t \cdot + x) = t^n \Theta_n(\cdot)$

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Decomposition of the Measure Θ_n

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Decomposition of the Measure Θ_n

Set of centred n-topes with Φ-content 1

$$\begin{aligned}\mathfrak{h}_n : \mathcal{P}_n &\overset{\sim}{\to} \mathbb{R}^d \times (0,\infty) \times \mathcal{P}^1_{n,\mathfrak{c}} \\ P &\mapsto \left(\mathfrak{c}(P), \Phi(P), \underbrace{\Phi(P)^{-1}(P-\mathfrak{c}(P))}_{\mathfrak{s}(P)}\right),\end{aligned}$$

Shape of P

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Shape of P

pushforward

Lebesgue measure

$$\boxed{\mathfrak{h}_n(\Theta_n) = \lambda_d \otimes \lambda_{1,n-d} \otimes \Theta^1_{n,\mathfrak{c}}}$$

$\lambda_{1,n-d}([0,a])=a^{n-d}$

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$$\mathbb{E}X(D) = \int_{\mathcal{P}^1_{n,\mathfrak{c}}}\int_{(0,\infty)}\int_{\mathbb{R}^d} \mathbf{1}\left(\mathfrak{h}_n^{-1}(c,t,s)\in D\right) e^{-t} t^{n-d-1}\mathrm{d}c\,\mathrm{d}t\,\Theta^1_{n,\mathfrak{c}}(\mathrm{d}P)$$

Decomposition of the Measure Θ_n

Set of centred n-topes with Φ-content 1

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Complementary Theorem (Miles 1971)

If we condition the typical cell Z to have n facets, then

- $\Phi(Z)$ and $\mathfrak{s}(Z)$ are independent
- $\Phi(Z)$ is Gamma distributed with parameter $n-d$

$\mathbb{P}(Z \text{ has } n \text{ facets})$

$$\gamma\mathbb{P}(Z \text{ has } n \text{ facets}) = \mathbb{E}X(\mathcal{P}_{n,[0,1]^d})$$

$$= \int_{\mathcal{P}^1_{n,\mathfrak{c}}}\int_{(0,\infty)}\int_{[0,1]^d} e^{-t}t^{n-d-1}\mathrm{d}c\,\mathrm{d}t\,\Theta^1_{n,\mathfrak{c}}(\mathrm{d}P)$$

$$= (n-d-1)!\Theta^1_{n,\mathfrak{c}}(\mathcal{P}^1_{n,\mathfrak{c}})$$

$$= \frac{(n-d-1)!}{n!}\int\cdots\int_{\mathcal{H}^n}\sum_{\boldsymbol{\epsilon}\in\{\pm1\}^n}\mathbb{1}\left(\mathfrak{c}(P)\in[0,1]^d\right)\mathbb{1}\left(\Phi(P)<1\right)\mathbb{1}\left(P\in\mathcal{P}_n\right)\,\Theta(\mathrm{d}H_n)\cdots\Theta(\mathrm{d}H_1)$$

Lower Bound

$$\gamma\frac{n!}{(n-d-1)!}\mathbb{P}(Z \text{ has } n \text{ facets})$$

$$= \int\cdots\int_{\mathcal{H}^n} \sum_{\epsilon\in\{\pm 1\}^n} \mathbb{1}\left(\mathfrak{c}(P) \in [0,1]^d\right)\mathbb{1}\left(\Phi(P) < 1\right)\mathbb{1}\left(P \in \mathcal{P}_n\right)\ \Theta(\mathrm{d}H_n)\cdots\Theta(\mathrm{d}H_1)$$

$$> n!\int\cdots\int_{\mathcal{H}^n} \mathbb{1}(H_1 \in S_1)\cdots\mathbb{1}(H_n \in S_n)\ \Theta(\mathrm{d}H_n)\cdots\Theta(\mathrm{d}H_1)$$

$$= n!\Theta(S_1)^n$$

$$> n!\left(cn^{-(d-1)/(d+1)}\right)^n$$

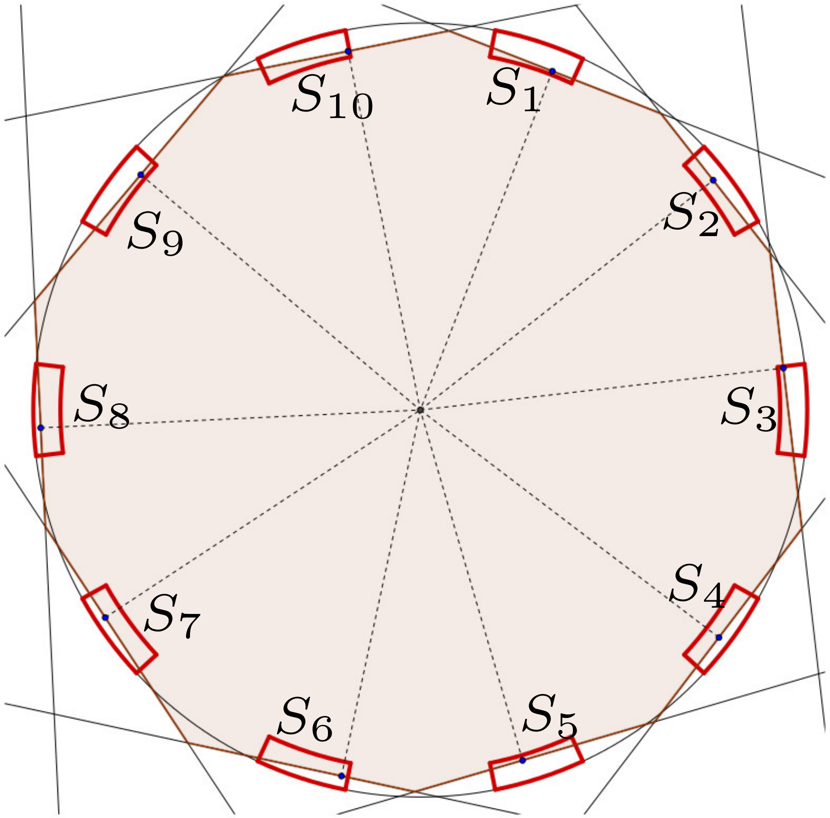
$$\sim c^n n^{-2n/(d-1)}$$

Stirling

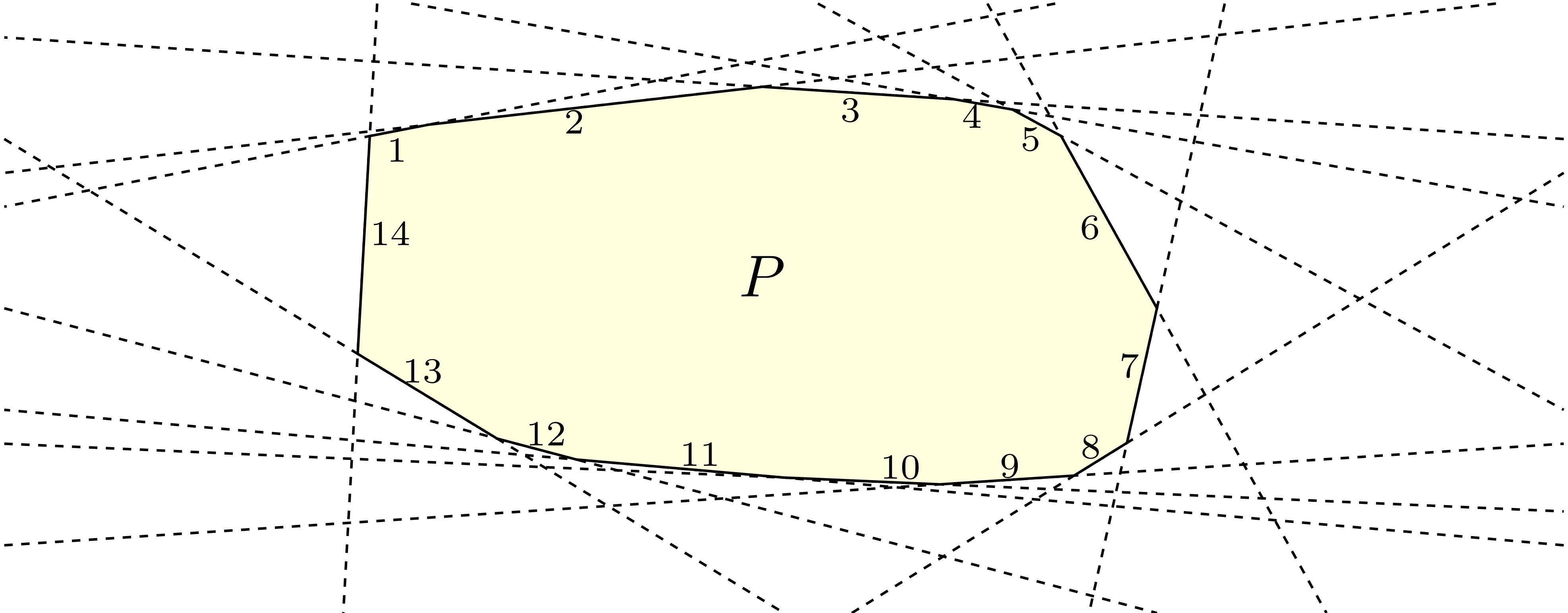
Theorem: Lower bound

There exists a constant c_1 depending on d and φ such that for n big enough we have

$$\mathbb{P}(Z \text{ has } n \text{ facets}) > c_1^n n^{-2n/(d-1)}$$



Approximation by Deleting One Facet



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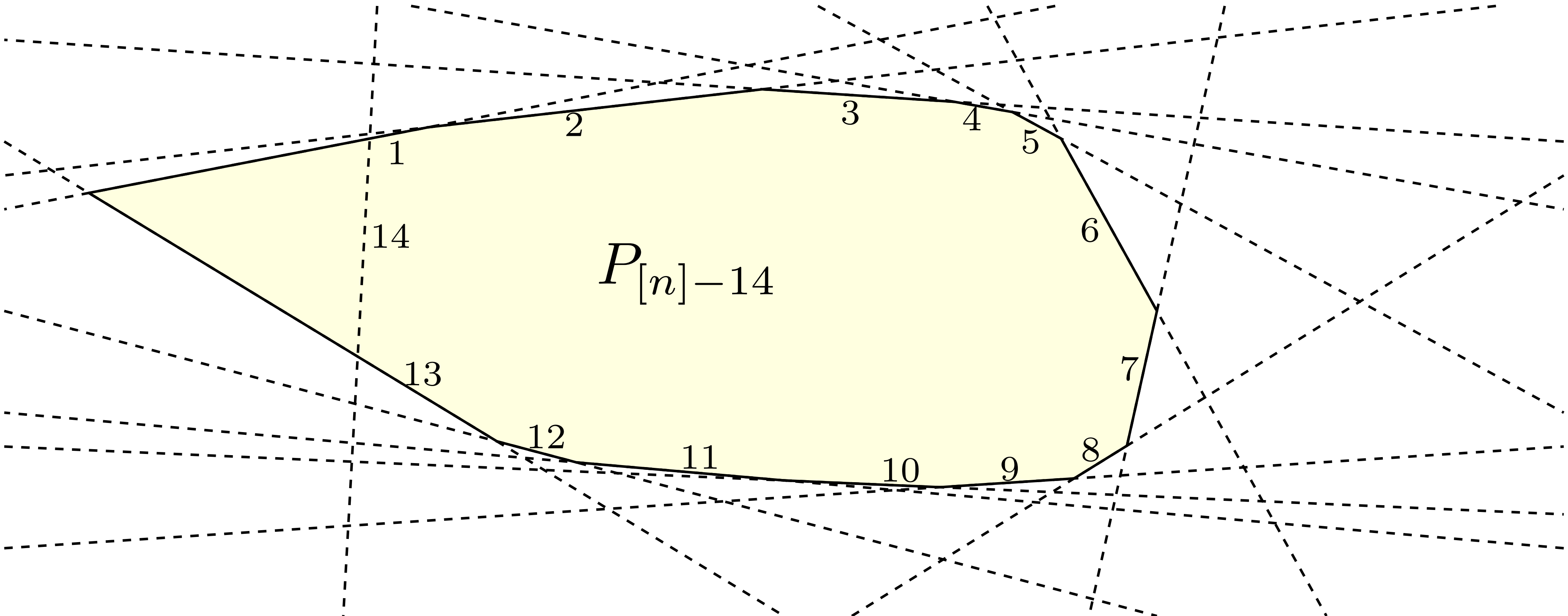

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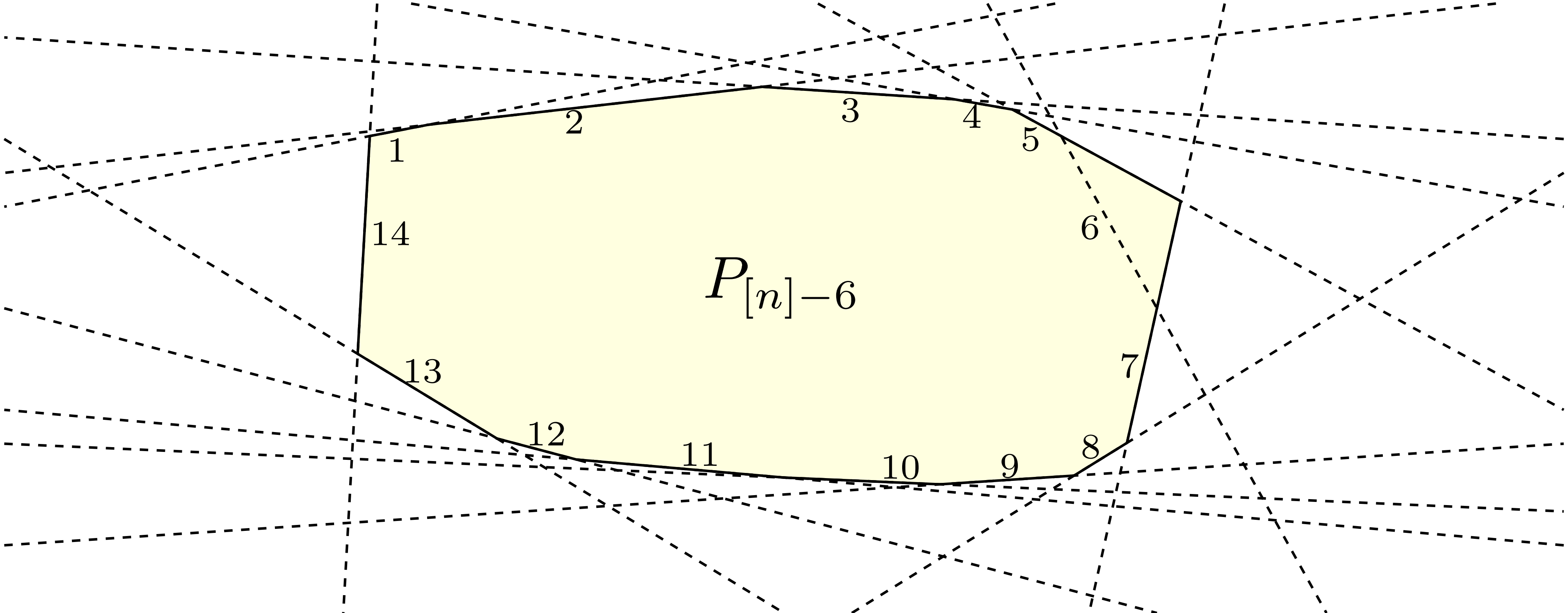

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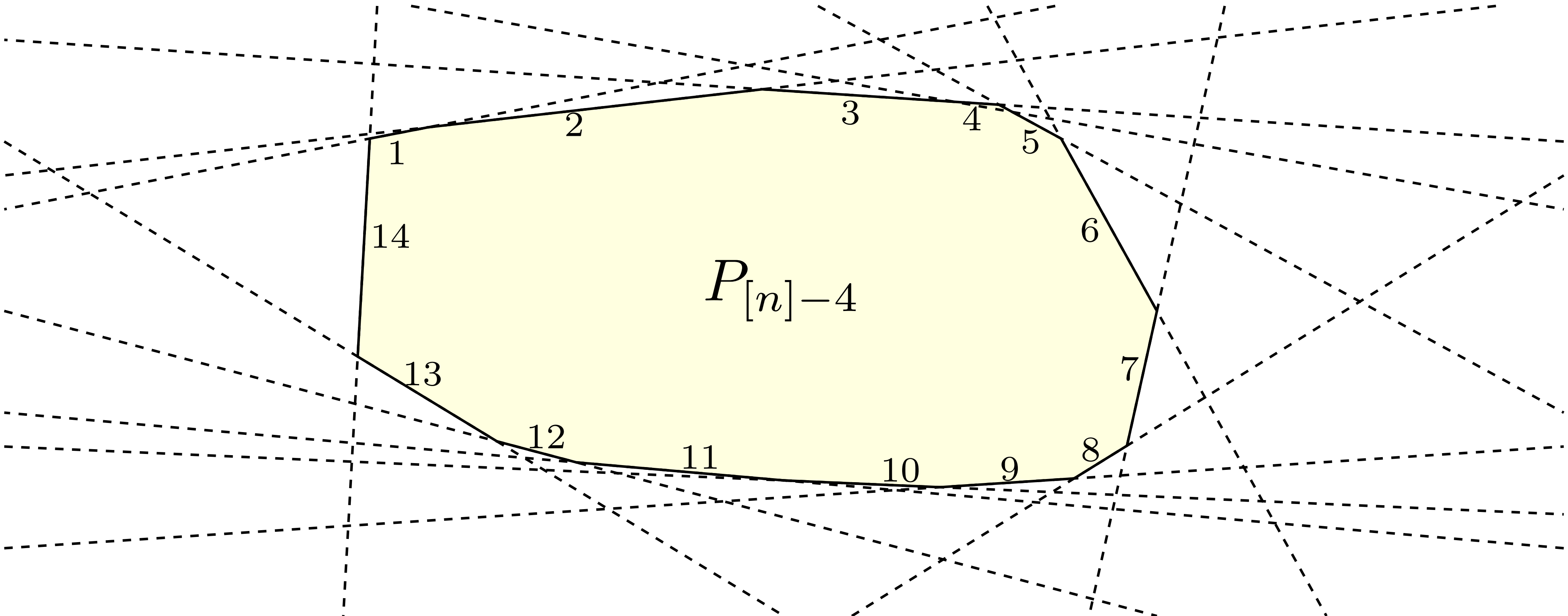


Approximation by Deleting One Facet

There exists a constant c_0 such that:

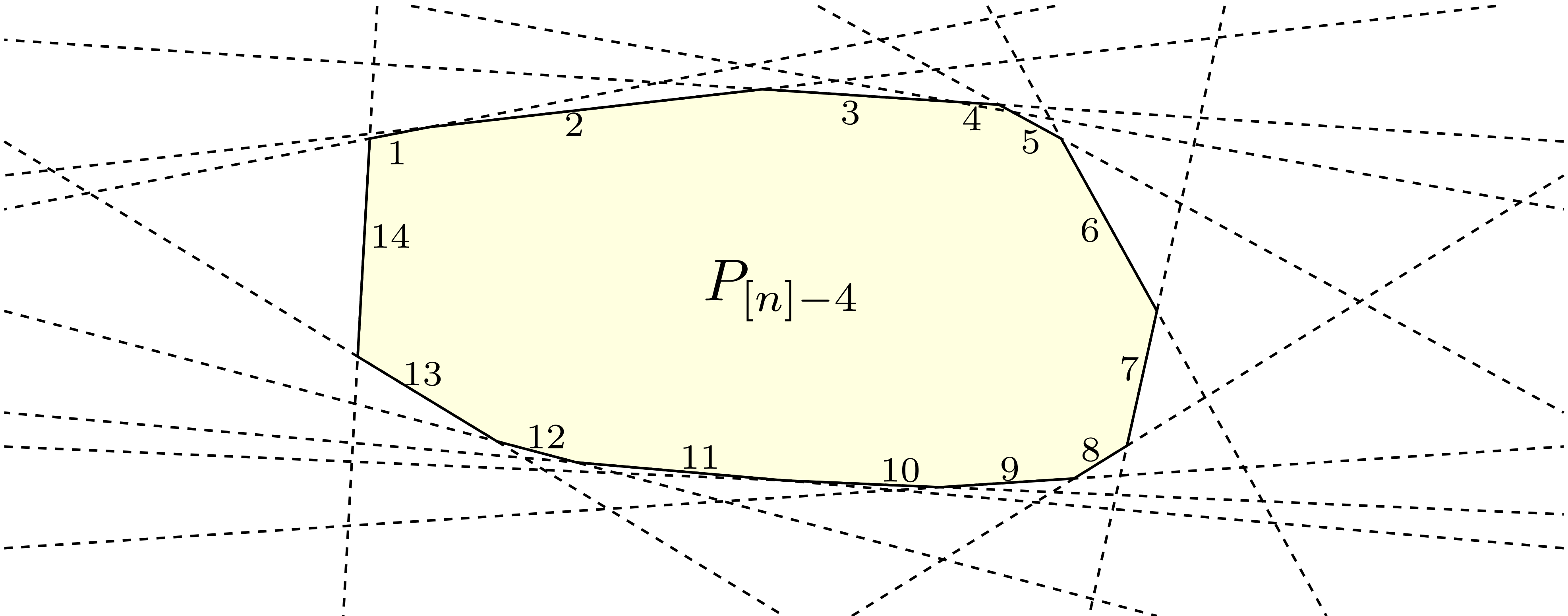
Theorem

Let $P = \cap_{i=1}^{n} H_i^-$ be a simple polytope with n big enough.
There exists a subset $J \subset [n]$ of cardinality at least $n/4$ such that for any $j \in J$ we have

$$d_H\left(P, P_{[n]-j}\right) < c_0 n^{-2/(d-1)} \Phi\left(P\right)$$

and

$$\Phi\left(P_{[n]-j}\right) < \exp\left(c_0 n^{-1-2/(d-1)}\right) \Phi(P).$$



Upper Bound

$$\gamma\frac{n!}{(n-d-1)!}\mathbb{P}(Z \text{ has } n \text{ facets})$$
$$=\int\cdots\int_{\mathcal{H}^n}\sum_{\boldsymbol{\epsilon}\in\{\pm1\}^n}\mathbb{1}\left(\mathfrak{c}(P)\in[0,1]^d\right)\mathbb{1}\left(\Phi(P)<1\right)\mathbb{1}\left(P\in\mathcal{P}_n\right)\,\Theta(\mathrm{d}H_n)\cdots\Theta(\mathrm{d}H_1)$$

Upper Bound

$$\frac{\gamma}{4}\frac{n!}{(n-d-1)!}\mathbb{P}(Z \text{ has } n \text{ facets})$$

$$< \int \cdots \int_{\mathcal{H}^n} \sum_{\epsilon \in \{\pm 1\}^n} \mathbb{1}\left(\mathfrak{c}(P) \in [0,1]^d\right) \mathbb{1}\left(\Phi(P) < 1\right) \mathbb{1}\left(P \in \mathcal{P}_n\right)$$

$$\mathbb{1}\left(d_H\left(P, P_{[n-1]}\right) < c_0 n^{-2/(d-1)}\right)$$

$$\mathbb{1}\left(\Phi\left(P_{[n-1]}\right) < \exp\left(c_0 n^{-1-2/(d-1)}\right)\right) \; \Theta(\mathrm{d}H_n) \cdots \Theta(\mathrm{d}H_1)$$

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$$\mathbb{1}\left(\Phi\left(P_{[n-1]}\right)<\exp\left(c_0 n^{-1-2/(d-1)}\right)\right)\ \Theta(\mathrm{d}H_n)\cdots\Theta(\mathrm{d}H_1)$$

$$<\int\cdots\int_{\mathcal{H}^{n-1}}\sum_{\boldsymbol{\epsilon}\in\{\pm1\}^{n-1}}\mathbb{1}\left(\mathfrak{c}(P_{[n-1]})\in[0,2]^d\right)\mathbb{1}\left(\Phi\left(P_{[n-1]}\right)<\exp\left(c_0 n^{-1-2/(d-1)}\right)\right)$$

$$\mathbb{1}\left(P_{[n-1]}\in\mathcal{P}_{n-1}\right)$$

$$\left(\int_{\mathcal{H}}\sum_{\epsilon_n=\pm1}\mathbb{1}\left(d_H\left(P,P_{[n-1]}\right)<c_0 n^{-2/(d-1)}\right)\Theta(\mathrm{d}H_n)\right)\Theta(\mathrm{d}H_1)\cdots\Theta(\mathrm{d}H_n)$$

Upper Bound

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Upper Bound

$$
\begin{aligned}
&\gamma\frac{n!}{(n-d-1)!}\mathbb{P}(Z \text{ has } n \text{ facets})\\
<&\int\cdots\int_{\mathcal{H}^{n-1}}\sum_{\boldsymbol{\epsilon}\in\{\pm1\}^{n-1}} \mathbb{1}\left(\mathfrak{c}(P_{[n-1]})\in[0,2]^d\right)\mathbb{1}\left(\Phi\left(P_{[n-1]}\right)<\exp\left(c_0 n^{-1-2/(d-1)}\right)\right)\\
&\quad\mathbb{1}\left(P_{[n-1]}\in\mathcal{P}_{n-1}\right)\\
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<&2^d\exp\left(c_0 n^{-1-2/(d-1)}\right)^{n-d-1} c_0 n^{-2/(d-1)}\\
&\quad\int\cdots\int_{\mathcal{H}^{n-1}}\sum_{\boldsymbol{\epsilon}\in\{\pm1\}^{n-1}}\mathbb{1}\left(\mathfrak{c}(P_{[n-1]})\in[0,1]^d\right)\mathbb{1}\left(\Phi\left(P_{[n-1]}\right)<1\right)\mathbb{1}\left(P_{[n-1]}\in\mathcal{P}_{n-1}\right)\\
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\end{aligned}
$$

Upper Bound

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$$<2^d\exp\left(c_0 n^{-1-2/(d-1)}\right)^{n-d-1}c_0 n^{-2/(d-1)}$$

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$$\Theta(\mathrm{d}H_{n-1})\cdots\Theta(\mathrm{d}H_1)$$

$$<2^d\exp\left(c_0 n^{-1-2/(d-1)}\right)^{n-d-1}c_0 n^{-2/(d-1)}\gamma\frac{(n-1)!}{(n-d-2)!}\mathbb{P}(Z \text{ has } n-1 \text{ facets})$$

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Theorem: Upper bound

There exists a constant c_2 depending on d and φ such that for n big enough we have

$$\mathbb{P}(Z \text{ has } n \text{ facets}) < c_2^n n^{-2n/(d-1)}$$

Shape of a Cell With Many Facets

Hug, Reitzner and Schneider [2004,2007] studied big cells. For example in the *isotropic case* they show that for any $j = 2, \cdots, d$ and any $\varepsilon > 0$ we have

j-th intrinsic volume

$$\boxed{\mathbb{P}\left(d_H\left(\mathfrak{s}(Z), \mathbb{B}^d\right) > \varepsilon \mid V_j(Z) > a\right) \to 0 \text{ when } a \to \infty}$$

‘*Big*’ cells are almost spherical.

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Conjecture

The result above remains true if you change $V_j(Z)$ by $V_1(Z)$ or $\mathrm{NumberOfFaces(Z)}$.

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'*Big*' cells are almost spherical.

Conjecture

The result above remains true if you change $V_j(Z)$ by $V_1(Z)$ or $\mathrm{NumberOfFaces(Z)}$.

We did a small step in the direction of this conjecture:

Theorem

There exists $\epsilon > 0$ such that for any $j \leq \lceil (d-1)/2 \rceil$ we have

$$\mathbb{P}\left(\frac{V_j(Z)}{V_1(Z)^j} < \epsilon \mid Z \text{ has } n \text{ facets}\right) \to 0 \text{ when } n \to \infty.$$

Cell with many facets are not '*too flat*'.

Take Home Message and Perspectives

Main Theorem

There exist constants c_1 and c_2 depending on d and φ such that for n big enough we have

$$c_1^n n^{-2n/(d-1)} < \mathbb{P}(Z \text{ has } n \text{ facets}) < c_2^n n^{-2n/(d-1)}$$

$\Rightarrow$ Generalization : Other kind of mosaics, the zero cell...

Theorem

There exists $\epsilon > 0$ such that for any $j \leq \lceil (d-1)/2 \rceil$ we have

$$\mathbb{P}\left(\frac{V_j(Z)}{V_1(Z)^j} < \epsilon \,\middle|\, Z \text{ has } n \text{ facets}\right) \to 0 \text{ when } n \to \infty.$$

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Take Home Message and Perspectives

Main Theorem

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THANK YOU!