RANDOM POLYTOPE WITH MANY FACETS
The zero cell of a hyperplane mosaic

Gilles Bonnet
gbonnet@uos.de

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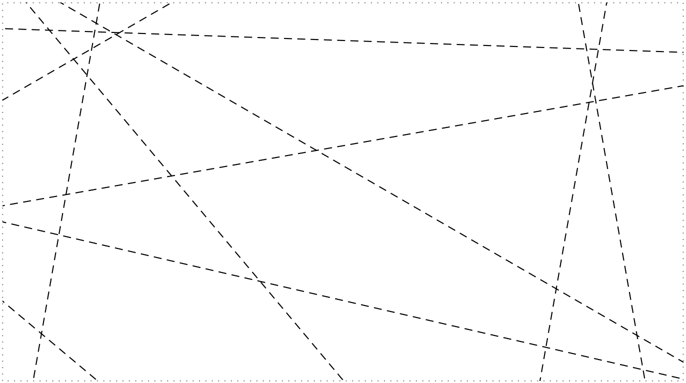
UNIVERSITÄT
OSNABRÜCK

Poisson hyperplane process

… in $\mathbb{R}^d$



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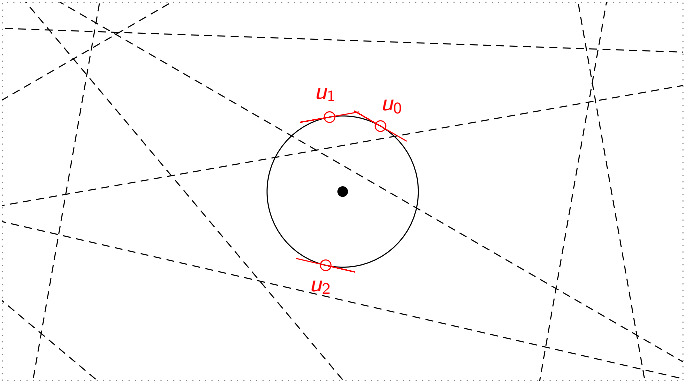

u_0, u_1,... sequence of i.i.d. points on $\mathbb{S}^{d-1}$ w.r.t. to an even measure φ.

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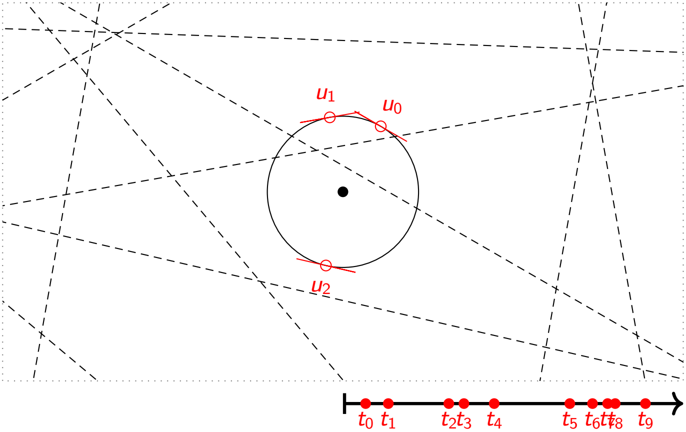

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t_0, t_1, … Poisson point process on $\mathbb{R}_+$

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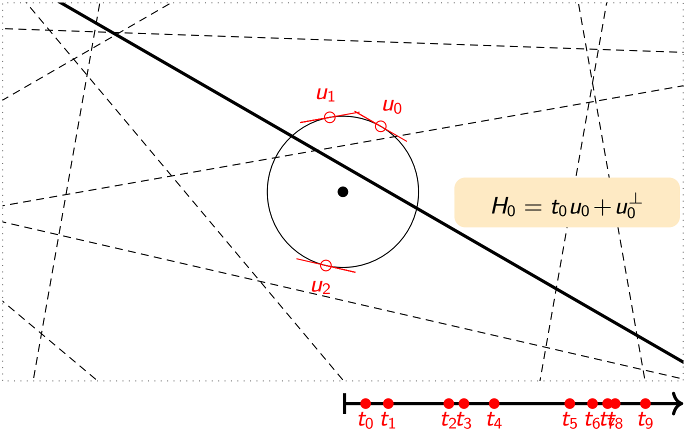

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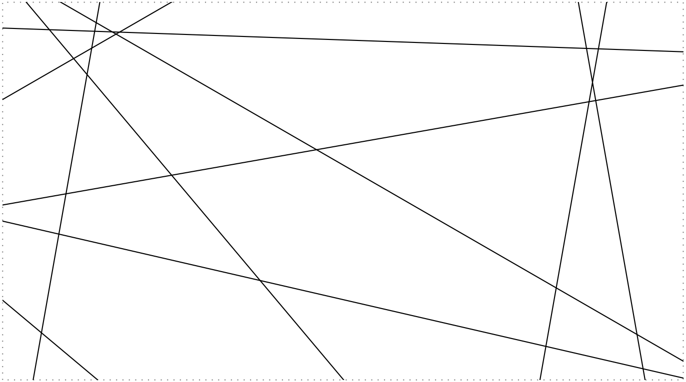
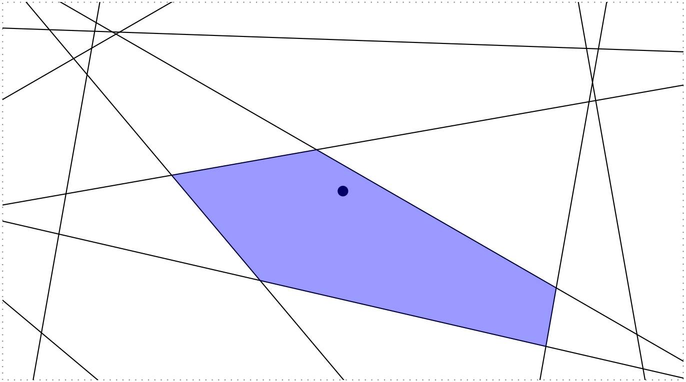


Hyperplane process $\eta = \{H_0, H_1, \ldots\}$

Poisson hyperplane process ⇒ **Zero cell Z_o**



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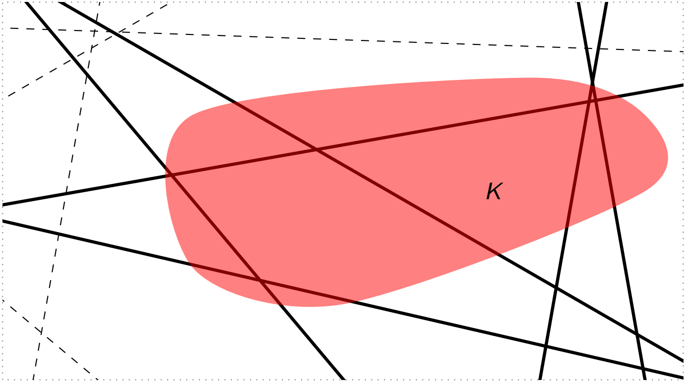
Φ-content



$\eta(K) = \#\{H \in \eta : H \cap K \neq\} = \pi(\Phi(K))$ with $\Phi(\cdot) =$ **?**

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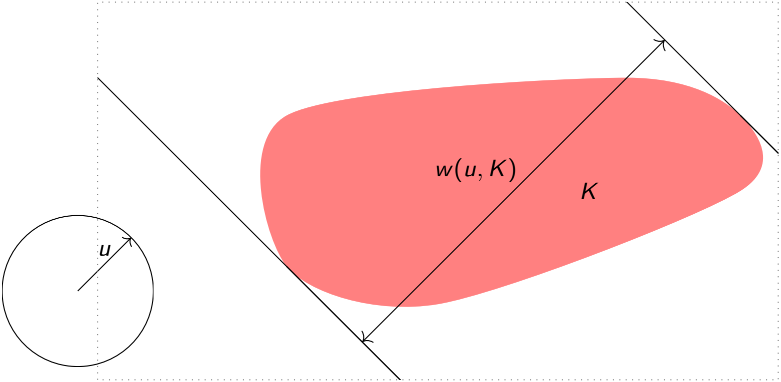

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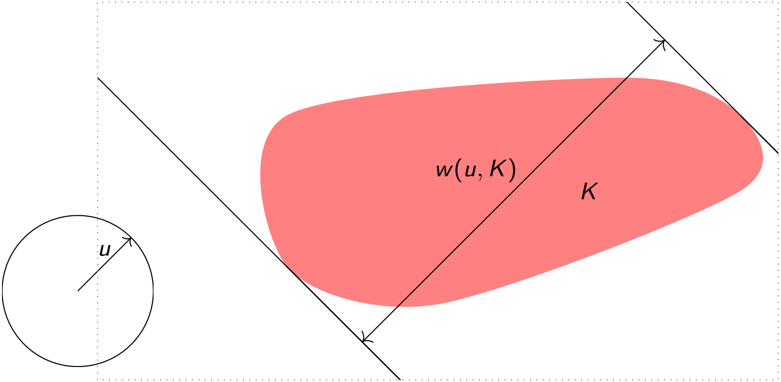

$$\eta(K) = \#\{H \in \eta : H \cap K \neq\} = \pi(\Phi(K)) \text{ with } \Phi(\cdot) = \int_{\mathbb{S}^{d-1}} w(u,K)\,\mathrm{d}\varphi(u)$$

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$Z_{\mathbf{o}} \in \mathcal{P}_{\mathbf{o}}$ ... simple polytope containing the origin.

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$$\mathcal{P}_{\mathbf{o}} \simeq \mathbb{R}_+ \times \mathcal{P}_{\mathbf{o},\Phi}$$
$$Z_{\mathbf{o}} \mapsto (\Phi(Z_{\mathbf{o}}), \mathfrak{s}(Z_{\mathbf{o}}))$$

Polytopes

Conditioning on $f_{d-1}(Z_{\mathbf{o}}) = n$.

$Z_{\mathbf{o}} \in \mathcal{P}_{n,\mathbf{o}}$ … simple polytope with n facets containing the origin.

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Measure on $\mathbb{R}_+$ ... $t^{n-1}\mathrm{d}t$

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Measure on the shape space ... $\mu_{n,\Phi}(S) = \mu_n([0,1] \times S)$ for $S \subset \mathcal{P}_{n,\mathbf{o},\Phi}$

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Complementary Theorem (for the zero cell)

Miles, Møller, Zuyev, Baumstark, Last,. . .

Theorem

Let $n \geq d+1$.

1.
$$\mathbb{P}\left(f_{d-1}(Z_{\mathbf{o}}) = n\right) = n! \int_{\mathcal{P}_{n,\mathbf{o}}} \mathbb{1}\left(\Phi(P) < 1\right) \mathrm{d}\mu_n(P).$$

2. **(Complementary Theorem)**
   Conditioning on $f_{d-1}(Z_{\mathbf{o}}) = n$
   1. $\Phi(Z_{\mathbf{o}})$ and $\mathfrak{s}(Z_{\mathbf{o}})$ are independent random variables,
   2. $\Phi(Z_{\mathbf{o}})$ is Γ_n distributed, and
   3. $\mathfrak{s}(Z_{\mathbf{o}})$ has probability distribution $\mu_{n,\Phi}(\cdot)/\mu_{n,\Phi}(\mathcal{P}_{n,\mathbf{o},\Phi})$.

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$\Rightarrow$ The distribution of the number of facets is central!

Distribution of $f_{d-1}(Z_{\mathbf{o}})$

Theorem (Upper bound)

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If φ is the rotation invariant (or well spread) we also have:

Theorem (Lower bound)

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Idea of proof: Lower bound

$$\begin{aligned}\mathbb{P}\left(f(Z_{\mathbf{o}}) = n\right) &= n! \int_{\mathcal{P}_{n,\mathbf{o}}} \mathbb{1}\left(\Phi(P) < 1\right) \mathrm{d}\mu_n(P) \\ &\geq n! \int_{\mathcal{P}_{n,\mathbf{o}}} \mathbb{1}\left(P \text{ is close to a regular } n\text{-tope}\right) \mathrm{d}\mu_n(P).\end{aligned}$$

Idea of proof: Upper bound

Lemma (Polytopal approximation)

Let $P_{[n]} = \cap_{i=1}^{n} H_i^-$,

$$d_H(P_{[n]}, P_{[n-1]}) < c_1 \Phi(P) n^{-\frac{2}{d-1}}.$$

… consequence of a result of Reisner, Schütt, Werner

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$\Downarrow \oplus$ Complementary Theorem

$$\mathbb{P}\left(f(Z_{\mathbf{o}}) = n\right) \leq c_1\, n^{-\frac{2}{d-1}}\, \mathbb{P}\left(f(Z_{\mathbf{o}}) = n-1\right)$$

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$\Downarrow$ iterations

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Size distribution

Theorem (Distribution of Φ)

For $a > 0$

$$\mathbb{P}(\Phi(Z) > a) < \exp\left(-a + c_1 a^{\frac{d+1}{d-1}}\right).$$

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Similar with other size measurements, e.g. the volume.

(with different constants and only for a big enough)

Kendall's problem: Shape of Big cells

Miles, Hug, Reitzner, Schneider, ...

Theorem

Conditioning on $V_d(Z_{\mathbf{o}}) > a$ with $a \to \infty$,

$$\mathfrak{s}(Z_{\mathbf{o}}) \simeq B = \text{Blaschke body of } \varphi.$$

More precisely, there exists c, $c' > 0$ such that

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- similar with $V_{d-1}, \ldots, V_2$, diameter, inradius, circumradius,
- counterexample for $\Phi > a$ if φ has finite support.

Kendall's problem: Shape of Big cells, $f_{d-1}(Z_{\mathbf{o}}) = n \to \infty$

Assume that φ is rotation invariant.

Conjecture

Conditioning on $f_{d-1}(Z_{\mathbf{o}}) = n \to \infty$,

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Partial results:

Theorem (Big cells are not elongated)

Let $1 \leq i < j \leq \lceil (d-1)/2 \rceil$. There exists a small $\epsilon > 0$ such that

$$\lim_{n\to\infty} \mathbb{P}\left(\frac{V_j(Z_{\mathbf{o}})^{\frac{1}{j}}}{V_i(Z_{\mathbf{o}})^{\frac{1}{i}}} < \epsilon \mid f_{d-1}(Z_{\mathbf{o}}) = n \right) = 0.$$

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Theorem

$$\mathbb{P}\left(d(Z_\mathbf{o}, B^d) < cn^{-\frac{2}{d-1}} \mid f_{d-1}(Z_\mathbf{o}) = n \right) > \left(\frac{c_2}{c_1} \right)^n.$$

THANK YOU!