Poisson hyperplane tessellation
Asymptotic probabilities of the zero and typical cells

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Graduate school Combinatorial Structures in Geometry

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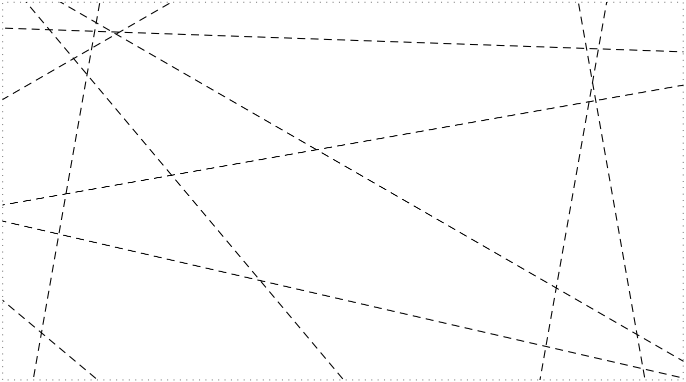
Poisson hyperplane process

... in $\mathbb{R}^d$



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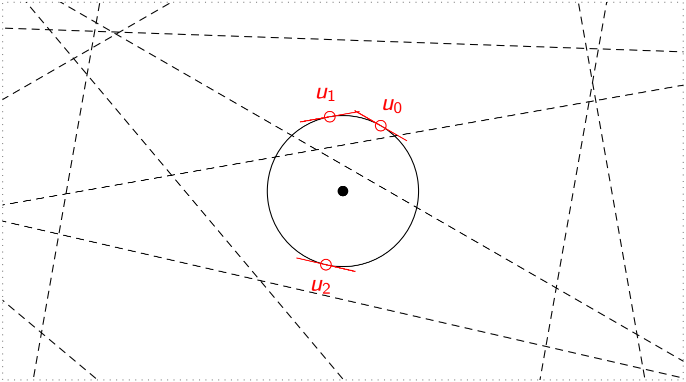

u_0, u_1,... sequence of i.i.d. points on $\mathbb{S}^{d-1}$ w.r.t. to a measure φ.

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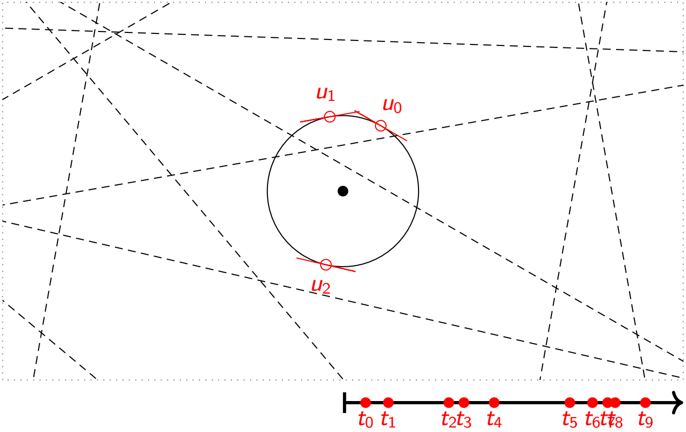

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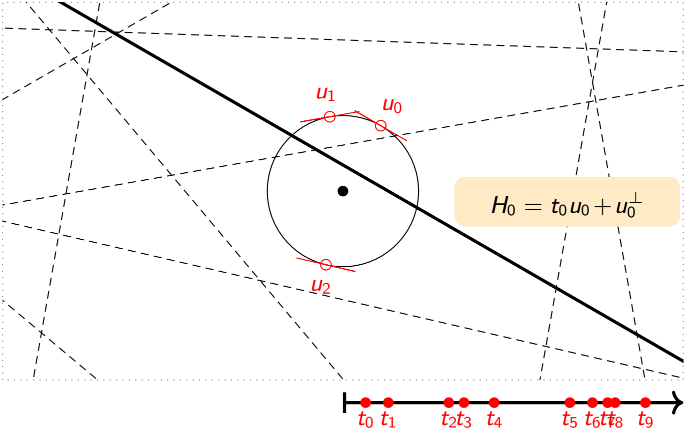

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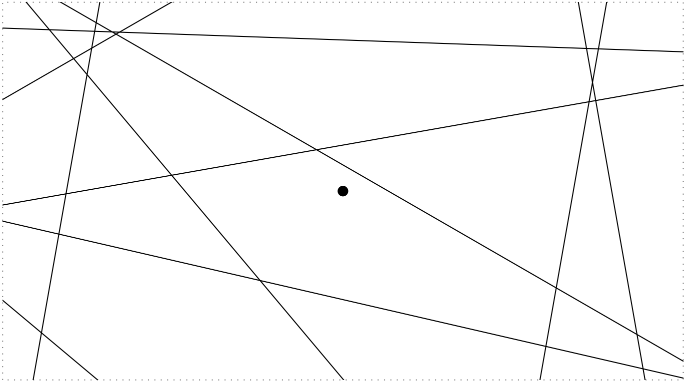


Hyperplane process $\eta = \{H_0, H_1, \ldots\}$

Poisson hyperplane process

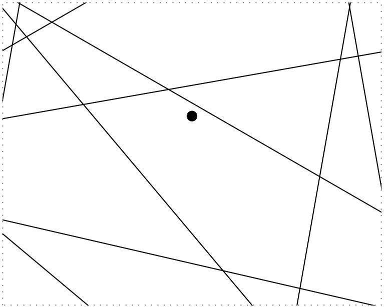
Intensity measure

$$\gamma\mu(\cdot) = \gamma \int_{\mathbb{S}^{d-1}} \int_0^\infty \mathbb{1}\left(H(u,t) \in \cdot\right) t^{r-1} \mathrm{d}t \, \mathrm{d}\varphi(u),$$

$\gamma > 0 \ldots$ intensity

$\varphi \ldots$ directional distribution

$r \geq 1 \ldots$ distance exponent



Poisson hyperplane process ⇒ Random polytopes

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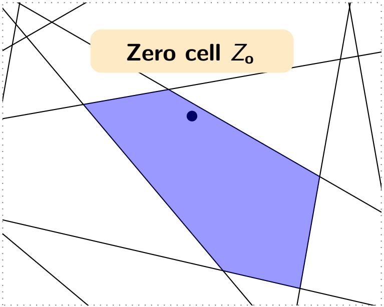

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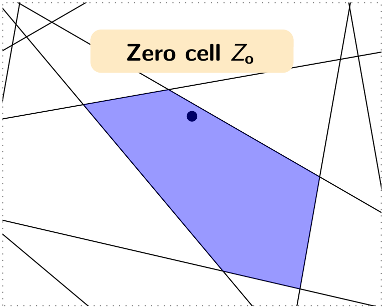

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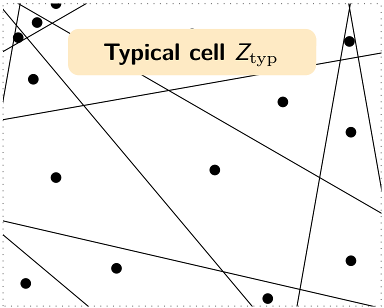

Require stationarity:
$r = 1$ and φ even.

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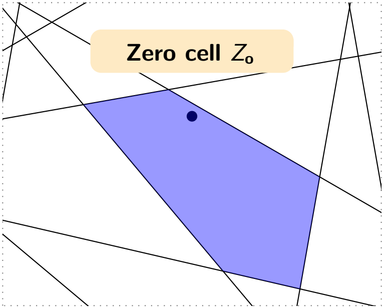

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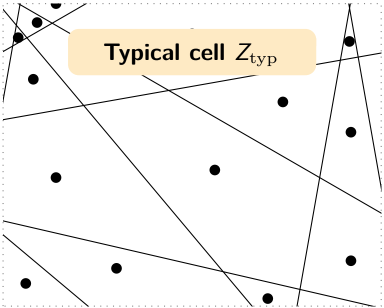

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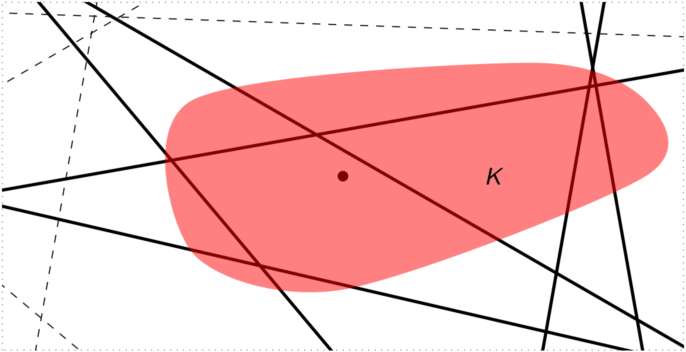
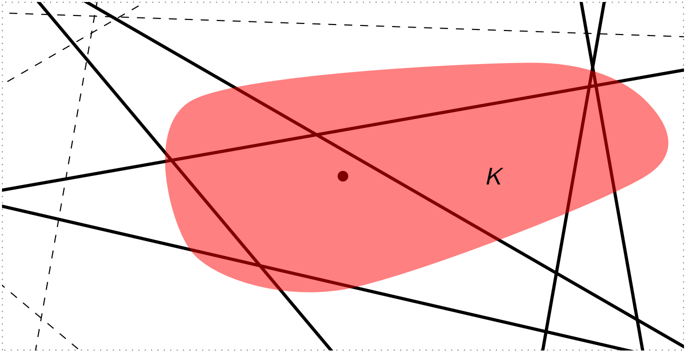


$\mathbb{P}(K \subset Z_{\mathbf{o}}) = ?$

Φ-content



$$\mathbb{P}(K \subset Z_{\mathbf{o}}) = \mathbb{P}(\#\{H \in \eta : H \cap K \neq \emptyset\} = 0)$$

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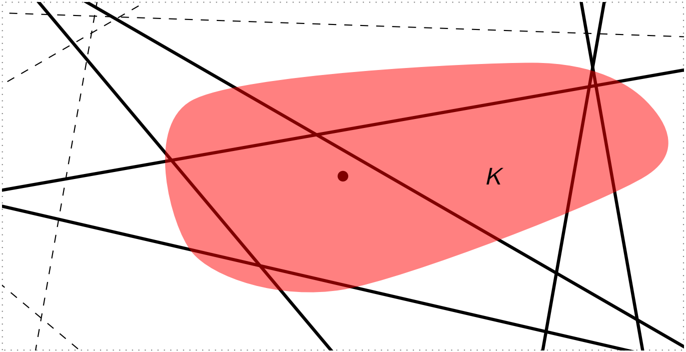

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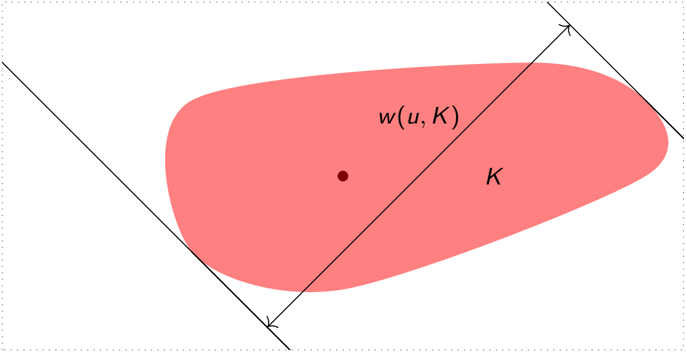

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Theorem

Let $n \geq d + 1$.

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   Conditionally on $f_{d-1}(Z_\mathbf{o}) = n$
   1. $\Phi(Z_\mathbf{o})$ and $\mathfrak{s}(Z_\mathbf{o})$ are independent random variables,
   2. $\Phi(Z_\mathbf{o})$ is Γ_n distributed.

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$$\times\, \mathrm{d}t_1 \varphi(\mathrm{d}u_1) \cdots \mathrm{d}t_n \varphi(\mathrm{d}u_n).$$

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$\Rightarrow$ The distribution of the number of facets is essential!

Distribution of $f_{d-1}(Z_{\mathbf{o}})$

Theorem (Upper bound)

$$\mathbb{P}(f(Z_{\mathbf{o}}) = n) \leq c_1^n n^{-\frac{2n}{d-1}}$$

$+$ there exists n_φ such that $\mathbb{P}(f(Z_{\mathbf{o}}) = n)$ is either vanishing or decreasing for $n > n_\varphi$.

If φ is the rotation invariant (or *well spread*) we also have:

Theorem (Lower bound)

$$\mathbb{P}(f(Z_{\mathbf{o}}) = n) \geq c_2^n n^{-\frac{2n}{d-1}}$$

Polytopal approximation

$d_H(\cdot,\cdot)\ldots$ Hausdorff distance
$\mathcal{P}_n = \{$polytopes with n facets $\}$
$d_H(K,\mathcal{P}_n) = \min_{P\in\mathcal{P}_n} d_H(K,P)$

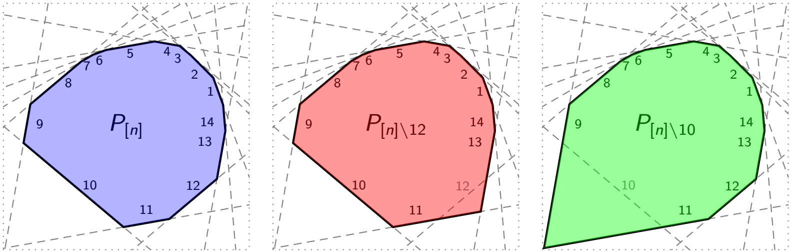
Classical result

Let $K \subset B^d$ be a convex body.

$$d_H(K,\mathcal{P}_n) < cn^{-\frac{2}{d-1}}$$

... where c is independent from K and n.

Polytopal approximation



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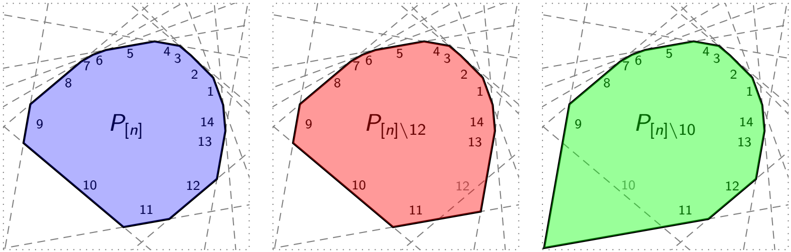

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Lemma (based on ideas from Reisner, Schütt and Werner '01)

There exists $I \subset [n]$ with $|I| \geq \frac{n}{4}$ such that, for any $i \in I$,

$$d_H(P_{[n]}, P_{[n]\setminus i}) < c\Phi(P_{[n]}) n^{-\frac{2}{d-1}},$$

and

$$\Phi(P_{[n]\setminus i}) < \exp\left(cn^{-\frac{d+1}{d-1}} \right) \Phi(P_{[n]}).$$

... where c is independent from n and $P_{[n]}$.

Idea of proof: Upper bound

$\Downarrow$ **Complementary theorem**

$$\mathbb{P}\left(f(Z_o) = n\right) = \int\limits_{(\mathbb{S}^{d-1}\times\mathbb{R})^n} \mathbb{1}\left(P_{[n]} \in \mathcal{P}_n\right) \mathbb{1}\left(\Phi\left(P_{[n]}\right) < 1\right) \mathrm{d}t_1 \varphi(\mathrm{d}u_1) \cdots \mathrm{d}t_n \varphi(\mathrm{d}u_n)$$

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$\Downarrow$ **Iterations**

$$\mathbb{P}\left(f(Z_o)=n\right)\leq c^2\left(n(n-1)\right)^{-\frac{2}{d-1}}\,\mathbb{P}\left(f(Z_o)=n-2\right)$$

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Idea of proof: Lower bound (isotropic case)

$$\mathbb{P}\left(f\left(Z_{\mathbf{o}}\right)=n\right)=\int_{(\mathbb{S}^{d-1}\times\mathbb{R})^n}\mathbb{1}\left(P_{[n]}\in\mathcal{P}_n\right)\mathbb{1}\left(\Phi\left(P_{[n]}\right)<1\right)\mathrm{d}t_1\varphi(\mathrm{d}u_1)\cdots\mathrm{d}t_n\varphi(\mathrm{d}u_n)$$

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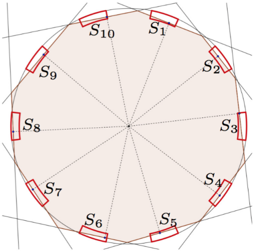

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\begin{aligned}
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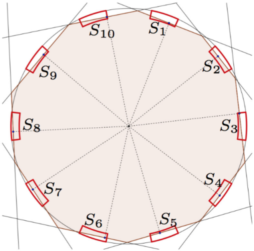
&= n!\left(\int_{\mathbb{S}^{d-1}\times\mathbb{R})} \mathbb{1}\left((u_1,t_1)\in S_1\right)\mathrm{d}t_1\varphi(\mathrm{d}u_1)\right)^n \\
&\geq n!\left(cn^{-\frac{d+1}{d-1}}\right)^n \geq c^n n^{-\frac{2n}{d-1}}
\end{aligned}
$$



D.G. Kendall's problem: Shape of Big cells

Conjecture (D.G. Kendall, 1987)

(planar, stationary and isotropic case)

'[. . .] the conditional law of the shape of $Z_{\mathbf{o}}$, given the area $V_2(Z_{\mathbf{o}})$, converges weakly, as $V_2(Z_{\mathbf{o}}) \to \infty$, to the degenerated law concentrated at the circular shape.'

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$$\tau := \min_K \frac{\Phi(K)}{V_d(K)^{\frac{1}{d}}} \overset{(isotropy)}{=} \frac{\Phi(B^d)}{V_d(B^d)^{\frac{1}{d}}}$$

Theorem (Hug, Reitzner, Schneider '04), (Hug, Schneider '07)

There exists c, $c' > 0$ such that, for any $\varepsilon > 0$ and $a > 0$,

$$\mathbb{P}\left(\frac{\Phi(Z_{\mathbf{o}})}{V_d(Z_{\mathbf{o}})^{\frac{1}{d}}} > \tau + \varepsilon \mid V_d(Z_{\mathbf{o}}) > a\right) \leq c \exp\left(-c'\varepsilon a^{\frac{1}{d}}\right).$$

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$$\mathbb{P}\left(\frac{\Phi(Z_\mathbf{o})}{V_d(Z_\mathbf{o})^{\frac{1}{d}}} > \tau + \varepsilon \mid V_d(Z_\mathbf{o}) > a\right) \leq c \exp\left(-c'\varepsilon a^{\frac{1}{d}}\right).$$

$$\Leftrightarrow \limsup_{a\to\infty} -a^{-\frac{1}{d}} \ln\left(\mathbb{P}\left(\frac{\Phi(Z_\mathbf{o})}{V_d(Z_\mathbf{o})^{\frac{1}{d}}} > \tau + \varepsilon \mid V_d(Z_\mathbf{o}) > a\right)\right) \leq c'\varepsilon.$$

D.G. Kendall's problem: Shape of Big cells

$$\tau := \min_K \frac{\Phi(K)}{V_d(K)^{\frac{1}{d}}} \overset{(isotropy)}{=} \frac{\Phi(B^d)}{V_d(B^d)^{\frac{1}{d}}}$$

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Theorem

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Shape of cells with many facets

Assume that φ is rotation invariant.

Conjecture

Conditionally on $f_{d-1}(Z_{\mathbf{o}}) = n \to \infty$,

$$\mathfrak{s}(Z_{\mathbf{o}}) \simeq B^d.$$

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$d_H(K, \mathcal{P}_n) = \min_{P \in \mathcal{P}_n} d_H(K, P)$
$d_H(K, \mathcal{K}_i) = \min_{K_i \in \mathcal{K}_i} d_H(K, K_i)$ where $\mathcal{K}_i \ldots$ convex bodies of dimension i

Theorem (Polytopal approximation of elongated convex bodies)

Let $1 < i < \frac{d-1}{2}$. Let $K \subset B^d$ be a convex body. For any $\varepsilon > 0$ there exists $\delta > 0$ such that, for $n > N(\varepsilon)$,

$$d_H(K, \mathcal{K}_i) < \delta \Rightarrow d_H(K, \mathcal{P}_n) < \varepsilon n^{-\frac{2}{d-1}}$$

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Theorem (Big cells are not elongated)

Let $1 < i < \frac{d-1}{2}$. There exists $\delta > 0$ such that

$$\lim_{n \to \infty} \mathbb{P}\left(d_H(Z_{\mathbf{o}}, \mathcal{K}_i) < \delta \mid f_{d-1}(Z_{\mathbf{o}}) = n\right) = 0.$$

Size distribution

Theorem (Distribution of Φ)

For $a > 0$

$$\mathbb{P}(\Phi(Z) > a) < \exp\left(-a + c_1 a^{\frac{d+1}{d-1}}\right).$$

If φ is rotation invariant (or well spread), then for $a > c_3$

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Similar with other size measurements, e.g. the volume.

Small typical cells

$\Sigma : \mathcal{K} \to \mathbb{R}_+ \ldots$ size measurement, e.g. Φ, V_d, ...

Theorem

Assume that φ is absolutely continuous.

$$\lim_{a \to 0} \mathbb{P}(f_{d-1}(Z_{\mathrm{typ}}) > d + 1 | \Sigma(Z_{\mathrm{typ}}) < a) = 0.$$

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Theorem

Assume that φ is absolutely continuous.

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$$\lim_{a \to 0} \mathbb{P}(f_{d-1}(Z_{\mathrm{typ}}) = d + 1 \, , \, \mathfrak{s}(Z_{\mathrm{typ}}) \in S | \Sigma(Z_{\mathrm{typ}}) < a) = \frac{c_\varphi(S)}{c_\varphi(\mathcal{P}_{d+1,\mathfrak{c},\Phi})}.$$

where $\frac{c_\varphi(\cdot)}{c_\varphi(\mathcal{P}_{d+1,\mathfrak{c},\Phi})}$ is a probability measure on
$\mathcal{P}_{d+1,\mathfrak{c},\Phi} = \{\text{simplices } P \ : \ \Phi(P) = 1, \mathfrak{c}(P) = \mathbf{o}\}$.

Articles

- Cells with many facets in a Poisson hyperplane tessellation.
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THANK YOU!