# Poisson hyperplane tessellation
# Asymptotic probabilities of the zero and typical cells

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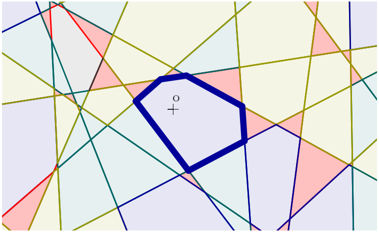

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# Poisson process

A *Poisson process* of *intensity measure* $\Theta$ on a state space $X$, is a random discrete subset $\eta \subset X$ such that for any pairwise disjoint subsets $A_1, \dots, A_n$ we have:

- For any $A \subset X$ and $k \in \mathbb{N}$, we have

$$\mathbb{P}\left(\#\eta \cap A = k\right) = e^{-\Theta(A)} \frac{\Theta(A)^k}{k!}.$$

- For any pairwise disjoint subsets $A_1, \dots, A_n$ we have that

$$\#\eta \cap A_1, \dots, \#\eta \cap A_n \text{ are independent random variables.}$$

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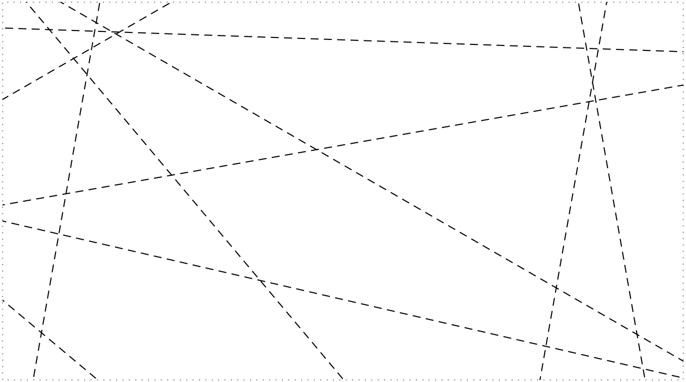
$\eta$ is identified with the corresponding counting measure, so we can write $\eta(A) = \#\eta \cap A$.

# Poisson hyperplane process



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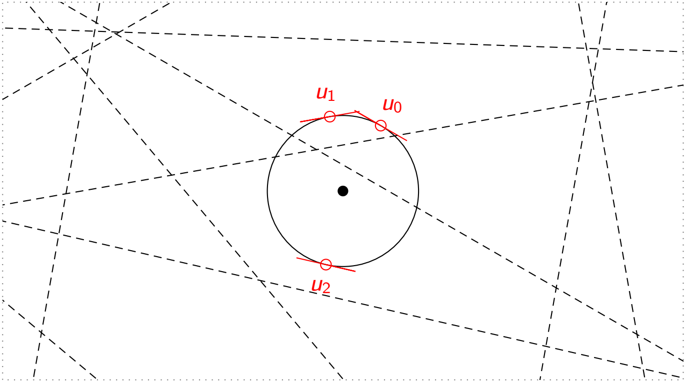

$u_0,\ u_1, \ldots$ sequence of i.i.d. points on $\mathbb{S}^{d-1}$ w.r.t. to a measure $\varphi$.

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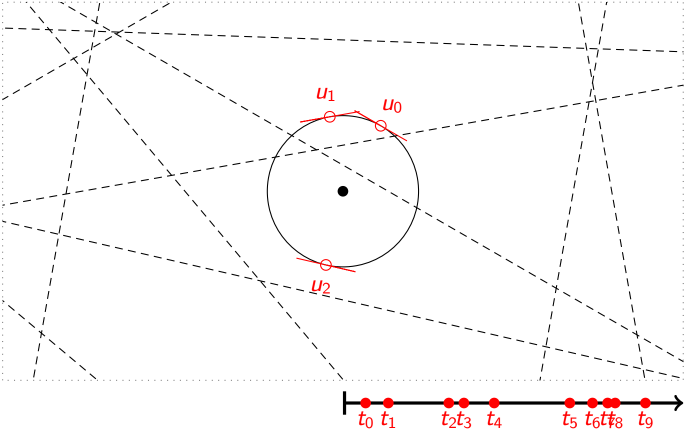

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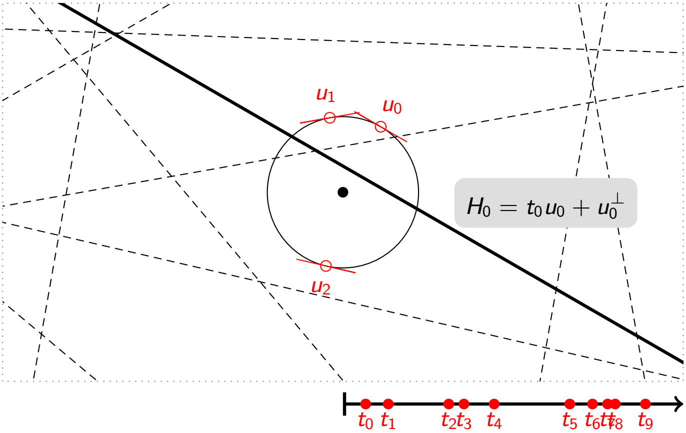

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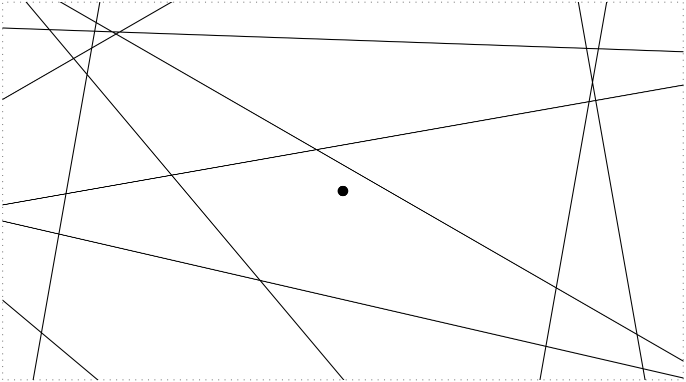


… in $\mathbb{R}^d$

Hyperplane process $\eta = \{H_0, H_1, \ldots\}$

# Poisson hyperplane process

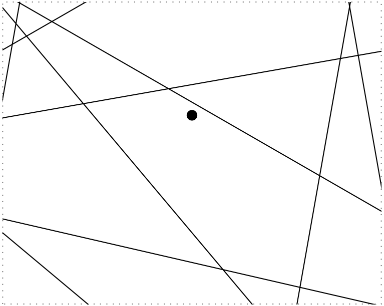
**Intensity measure**

$$\gamma\mu(\cdot) = \gamma \int_{\mathbb{S}^{d-1}} \int_0^\infty \mathbb{1}\left(H(u,t) \in \cdot\right) t^{r-1} \mathrm{d}t \, \mathrm{d}\varphi(u),$$

$\gamma > 0 \ldots$ intensity

$\varphi \ldots$ directional distribution

$r \geq 1 \ldots$ distance exponent



# Poisson hyperplane process ⇒ Random polytopes

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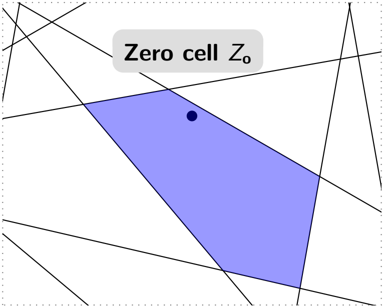

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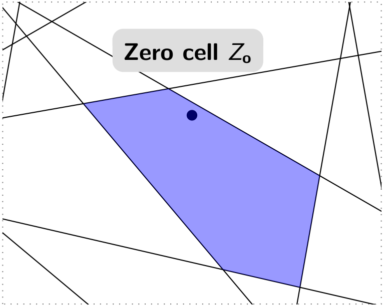
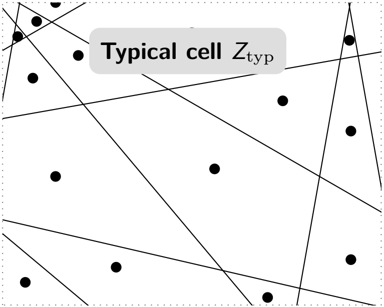

**Zero cell $Z_o$**

Always defined


**Typical cell $Z_{\mathrm{typ}}$**

Require stationarity:
$r = 1$ and $\varphi$ even.

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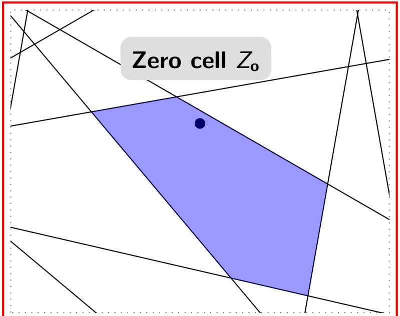

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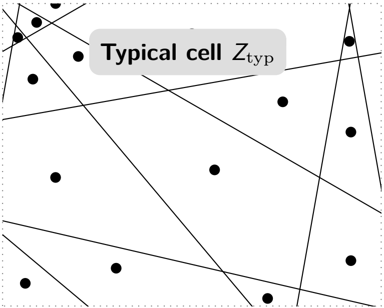

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**Theorem**

Let $n \geq d + 1$.

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   Conditionally on $f(Z_{\mathbf{o}}) = n$
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**⇒ The distribution of the number of facets is essential!**

# Distribution of $f_{d-1}(Z_{\mathbf{o}})$

**Theorem (Upper bound)**

$$\mathbb{P}(f(Z_{\mathbf{o}}) = n) \leq c_1^n n^{-\frac{2n}{d-1}}$$

$+$ there exists $n_\varphi$ such that $\mathbb{P}(f(Z_{\mathbf{o}}) = n)$ is either vanishing or decreasing for $n > n_\varphi$.

If $\varphi$ is the rotation invariant (or *well spread*) we also have:

**Theorem (Lower bound)**

$$\mathbb{P}(f(Z_{\mathbf{o}}) = n) \geq c_2^n n^{-\frac{2n}{d-1}}$$

# Polytopal approximation

$d_H(\cdot,\cdot)\dots$ Hausdorff distance
$\mathcal{P}_n = \{$polytopes with $n$ facets $\}$
$d_H(K,\mathcal{P}_n) = \min_{P\in\mathcal{P}_n} d_H(K,P)$

**Classical result**

Let $K \subset B^d$ be a convex body.

$$d_H(K,\mathcal{P}_n) < cn^{-\frac{2}{d-1}}$$

… where $c$ is independent from $K$ and $n$.

# Idea of proof: Upper bound

$\Downarrow$ **Complementary theorem**

$$\mathbb{P}\left(f(Z_{\mathbf{o}}) = n\right) = \int\limits_{(\mathbb{S}^{d-1}\times\mathbb{R})^n} \mathbb{1}\left(P_{[n]} \in \mathcal{P}_n\right) \mathbb{1}\left(\Phi\left(P_{[n]}\right) < 1\right) \mathrm{d}t_1 \varphi(\mathrm{d}u_1) \cdots \mathrm{d}t_n \varphi(\mathrm{d}u_n)$$

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$$< c\, n^{-\frac{2}{d-1}} \int\limits_{(\mathbb{S}^{d-1}\times\mathbb{R})^{n-1}} \mathbb{1}\left(P_{[n-1]} \in \mathcal{P}_{n-1}\right) \mathbb{1}\left(\Phi\left(P_{[n-1]}\right) < 1\right) \mathrm{d}t_1\varphi(\mathrm{d}u_1)\cdots\mathrm{d}t_{n-1}\varphi(\mathrm{d}u_{n-1})$$

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$\Downarrow$ **Iterations**

$$\mathbb{P}\left(f(Z_{\mathbf{o}}) = n\right) \le c^2 \left(n(n-1)\right)^{-\frac{2}{d-1}}\, \mathbb{P}\left(f(Z_{\mathbf{o}}) = n-2\right)$$

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# Idea of proof: Lower bound (isotropic case)

$$\mathbb{P}\left(f(Z_o) = n\right) = \int_{(\mathbb{S}^{d-1}\times\mathbb{R})^n} \mathbb{1}\left(P_{[n]} \in \mathcal{P}_n\right) \mathbb{1}\left(\Phi\left(P_{[n]}\right) < 1\right) \mathrm{d}t_1\varphi(\mathrm{d}u_1)\cdots\mathrm{d}t_n\varphi(\mathrm{d}u_n)$$

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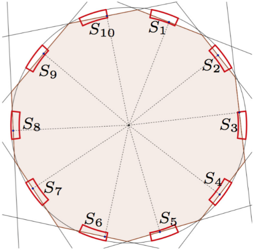

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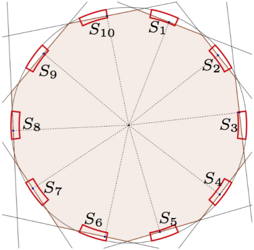
$$\begin{aligned}\mathbb{P}(f(Z_o)=n) &= \int_{(\mathbb{S}^{d-1}\times\mathbb{R})^n} \mathbb{1}\left(P_{[n]}\in\mathcal{P}_n\right)\mathbb{1}\left(\Phi\left(P_{[n]}\right)<1\right)\mathrm{d}t_1\varphi(\mathrm{d}u_1)\cdots\mathrm{d}t_n\varphi(\mathrm{d}u_n)\\ &\geq n!\int_{(\mathbb{S}^{d-1}\times\mathbb{R})^n}\mathbb{1}\left((u_1,t_1)\in S_1\right)\ldots\mathbb{1}\left((u_n,t_n)\in S_n\right)\mathrm{d}t_1\varphi(\mathrm{d}u_1)\cdots\mathrm{d}t_n\varphi(\mathrm{d}u_n)\\ &= n!\left(\int_{\mathbb{S}^{d-1}\times\mathbb{R})}\mathbb{1}\left((u_1,t_1)\in S_1\right)\mathrm{d}t_1\varphi(\mathrm{d}u_1)\right)^n\\ &\geq n!\left(cn^{-\frac{d+1}{d-1}}\right)^n \geq c^n n^{-\frac{2n}{d-1}}\end{aligned}$$



# D.G. Kendall's problem: Shape of Big cells

**Conjecture (D.G. Kendall, 1987)**

(planar, stationary and isotropic case)

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**Theorem (Hug, Reitzner, Schneider '04), (Hug, Schneider '07)**

There exists $c$, $c' > 0$ such that, for any $\varepsilon > 0$ and $a > 0$,

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$$\Leftrightarrow \limsup_{a\to\infty} -a^{-\frac{1}{d}} \ln\left(\mathbb{P}\left(\frac{\Phi(Z_{\mathbf{o}})}{V_d(Z_{\mathbf{o}})^{\frac{1}{d}}} > \tau + \varepsilon \mid V_d(Z_{\mathbf{o}}) > a\right)\right) \leq c'\varepsilon.$$

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$$\Leftrightarrow \limsup_{a\to\infty} -a^{-\frac{1}{d}} \ln\left(\mathbb{P}\left(\frac{\Phi(Z_{\mathbf{o}})}{V_d(Z_{\mathbf{o}})^{\frac{1}{d}}} > \tau + \varepsilon \mid V_d(Z_{\mathbf{o}}) > a\right)\right) \leq c'\varepsilon.$$

**Theorem**

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# Shape of cells with many facets

Assume that $\varphi$ is rotation invariant.

**Conjecture**

Conditionally on $f(Z_{\mathbf{o}}) = n \to \infty$,

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$d_H(K, \mathcal{K}_i) = \min_{K_i \in \mathcal{K}_i} d_H(K, K_i)$ where $\mathcal{K}_i$... convex bodies of dimension $i$

**Theorem (Polytopal approximation of elongated convex bodies)**

Let $1 < i < \frac{d-1}{2}$. Let $K \subset B^d$ be a convex body. For any $\varepsilon > 0$ there exists $\delta > 0$ such that, for $n > N(\varepsilon)$,

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**Theorem (Big cells are not elongated)**

Let $1 < i < \frac{d-1}{2}$. There exists $\delta > 0$ such that

$$\lim_{n \to \infty} \mathbb{P}\left(d_H(Z_{\mathbf{o}}, \mathcal{K}_i) < \delta \mid f(Z_{\mathbf{o}}) = n\right) = 0.$$

# Size distribution

**Theorem (Distribution of $\Phi$)**

For $a > 0$

$$\mathbb{P}(\Phi(Z) > a) < \exp\left(-a + c_1 a^{\frac{d+1}{d-1}}\right).$$

If $\varphi$ is rotation invariant (or well spread), then for $a > c_3$

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Similar with other size measurements, e.g. the volume.

# Small typical cells

$\Sigma : \mathcal{K} \to \mathbb{R}_+ \ldots$ size measurement, e.g. $\Phi$, $V_d$, ...

**Theorem**

Assume that $\varphi$ is absolutely continuous.

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$$\lim_{a\to 0} \mathbb{P}(f(Z_{\mathrm{typ}}) = d+1\,,\, \mathfrak{s}(Z_{\mathrm{typ}}) \in S | \Sigma(Z_{\mathrm{typ}}) < a) = \frac{c_\varphi(S)}{c_\varphi(\mathcal{P}_{d+1,\mathfrak{c},\Phi})}.$$

where $\frac{c_\varphi(\cdot)}{c_\varphi(\mathcal{P}_{d+1,\mathfrak{c},\Phi})}$ is a probability measure on
$\mathcal{P}_{d+1,\mathfrak{c},\Phi} = \{\text{simplices } P \ : \ \Phi(P) = 1, \mathfrak{c}(P) = \mathbf{o}\}$.

# THANK YOU!

- *Poisson hyperplane tessellation: Asymptotic probabilities of the zero and typical cells.*
  Ph.D. thesis (2016)
- *Polytopal approximation of elongated convex bodies.*
  Advances in Geometry (accepted)
- *Cells with many facets in a Poisson hyperplane tessellation.*
  joint work with P. Calka and M. Reitzner
  arXiv:1608.07979
- *Small cells in a Poisson hyperplane tessellation.*
  arXiv:1702.01964