Poisson hyperplane tessellation
Asymptotic probabilities of the zero and typical cells

Gilles Bonnet

Ruhr-Universität Bochum

gilles.bonnet@rub.de

DFG
GK 1916

DFG Research Training Group Combinatorial Structures in Geometry

Osnabrück, 11th October 2017

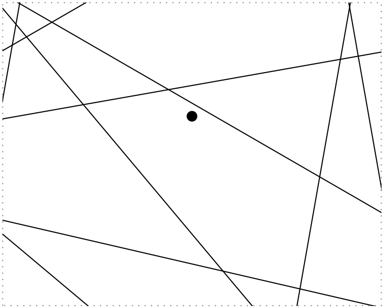
UNIVERSITÄT OSNABRÜCK

(homegeneous) Poisson hyperplane process (in $\mathbb{R}^d$)

Intensity measure

$$\gamma\mu(\cdot) = \gamma \int_{\mathbb{S}^{d-1}} \int_0^\infty \mathbb{1}\left(H(u,t) \in \cdot\right) t^{r-1} \mathrm{d}t \, \mathrm{d}\varphi(u),$$

- $\gamma > 0 \ldots$ intensity
- $\varphi \ldots$ directional distribution
- $r \geq 1 \ldots$ distance exponent



(homegeneous) Poisson hyperplane process (in $\mathbb{R}^d$) $\Rightarrow$ Random polytopes

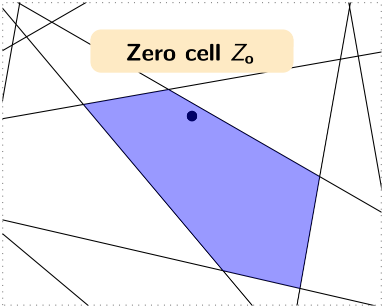
Intensity measure

$$\gamma\mu(\cdot) = \gamma \int_{\mathbb{S}^{d-1}} \int_0^\infty \mathbb{1}\left(H(u,t) \in \cdot\right) t^{r-1} \mathrm{d}t \, \mathrm{d}\varphi(u),$$

$\gamma > 0 \ldots$ intensity

$\varphi \ldots$ directional distribution

$r \geq 1 \ldots$ distance exponent



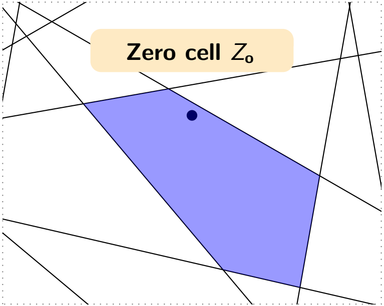
Always defined

(homegeneous) Poisson hyperplane process (in $\mathbb{R}^d$) $\Rightarrow$ Random polytopes

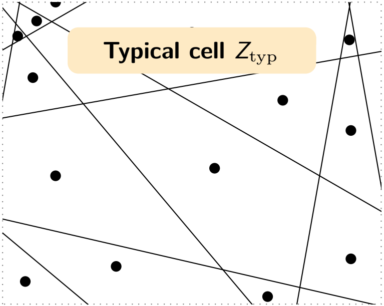
Intensity measure

$$\gamma\mu(\cdot) = \gamma \int_{\mathbb{S}^{d-1}} \int_0^\infty \mathbb{1}\left(H(u,t) \in \cdot\right) t^{r-1} \mathrm{d}t \, \mathrm{d}\varphi(u),$$

- $\gamma > 0 \ldots$ intensity
- $\varphi \ldots$ directional distribution
- $r \geq 1 \ldots$ distance exponent



Always defined



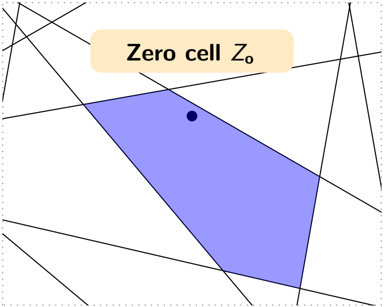
Require stationarity:
$r = 1$ and φ even.

(homegeneous) Poisson hyperplane process (in $\mathbb{R}^d$) $\Rightarrow$ Random polytopes

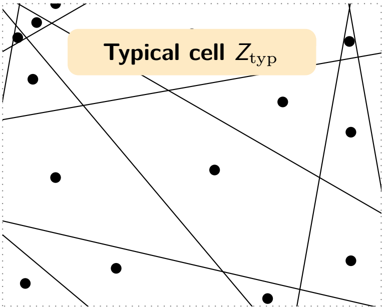
Intensity measure

$$\gamma\mu(\cdot) = \gamma \int_{\mathbb{S}^{d-1}} \int_0^\infty \mathbb{1}\left(H(u,t) \in \cdot\right) t^{r-1} \mathrm{d}t \, \mathrm{d}\varphi(u),$$

- $\gamma = 1 \ldots$ intensity
- $\varphi \ldots$ directional distribution
- $r = 1 \ldots$ distance exponent

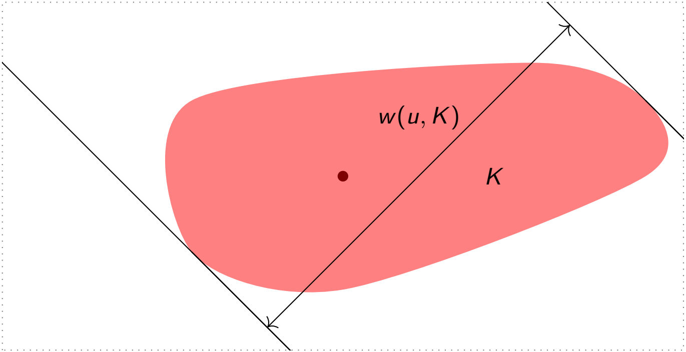


Always defined



Require stationarity:
$r = 1$ and φ even.

Φ-content



$$\Phi(K) = \frac{1}{2} \int_{\mathbb{S}^{d-1}} w(u, K) \, \mathrm{d}\varphi(u)$$

$$\mathbb{P}(K \subset Z_{\mathbf{o}}) = \exp(-\Phi(K))$$

Complementary Theorem (Miles, Møller, Zuyev, Cowan, Baumstark, Last,. . .)

$$f(Z_o) = \text{number of facets of } Z_o$$

$$\Phi(Z_o) = \frac{1}{2}\int_{\mathbb{S}^{d-1}} w(u, Z_o)\,\mathrm{d}\varphi(u)$$

$$\mathfrak{s}(Z_o) = \Phi(Z_o)^{-1} Z_o$$

Complementary Theorem (Miles, Møller, Zuyev, Cowan, Baumstark, Last,. . .)

$$f(Z_o) = \text{number of facets of } Z_o$$

$$\Phi(Z_o) = \frac{1}{2}\int_{\mathbb{S}^{d-1}} w(u, Z_o)\,\mathrm{d}\varphi(u)$$

$$\mathfrak{s}(Z_o) = \Phi(Z_o)^{-1} Z_o$$

Theorem

1. **(Complementary Theorem)**
   Conditionally on $f(Z_o) = n$
   1. $\Phi(Z_o)$ and $\mathfrak{s}(Z_o)$ are independent random variables,
   2. $\Phi(Z_o)$ is Γ_n distributed.

Complementary Theorem (Miles, Møller, Zuyev, Cowan, Baumstark, Last,...)

$$\Phi(Z_o) = \frac{1}{2}\int_{\mathbb{S}^{d-1}} w(u, Z_o)\,\mathrm{d}\varphi(u)$$

$$f(Z_o) = \text{number of facets of } Z_o$$

$$\mathfrak{s}(Z_o) = \Phi(Z_o)^{-1} Z_o$$

Theorem

1. **(Complementary Theorem)**
   Conditionally on $f(Z_o) = n$
   1. $\Phi(Z_o)$ and $\mathfrak{s}(Z_o)$ are independent random variables,
   2. $\Phi(Z_o)$ is Γ_n distributed.
2. $\mathbb{P}\left(f(Z_o) = n\,,\ \Phi(Z_o) \in A\,,\ \mathfrak{s}(Z_o) \in S\right)$

$$= \frac{1}{n!}\int_A t^{n-1}e^{-t}\mathrm{d}t \int_{(\mathbb{S}^{d-1}\times\mathbb{R})^n} \mathbb{1}\left(P_{[n]} \in \mathcal{P}_n\right)\mathbb{1}\left(\Phi\left(P_{[n]}\right) < 1\right)\mathbb{1}\left(\mathfrak{s}(Z_o) \in S\right)$$

$$\times\, \mathrm{d}t_1\varphi(\mathrm{d}u_1)\cdots\mathrm{d}t_n\varphi(\mathrm{d}u_n).$$

$$P_{[n]} = \cap_{i=1}^{n} H(u_i, t_i)^-$$

Complementary Theorem (Miles, Møller, Zuyev, Cowan, Baumstark, Last,...)

$f(Z_o) = \text{number of facets of } Z_o$

$$\Phi(Z_o) = \frac{1}{2}\int_{\mathbb{S}^{d-1}} w(u, Z_o)\,\mathrm{d}\varphi(u)$$

$$\mathfrak{s}(Z_o) = \Phi(Z_o)^{-1} Z_o$$

Theorem

1. **(Complementary Theorem)**
   Conditionally on $f(Z_o) = n$
   1. $\Phi(Z_o)$ and $\mathfrak{s}(Z_o)$ are independent random variables,
   2. $\Phi(Z_o)$ is Γ_n distributed.
2. $\mathbb{P}\left(f(Z_o) = n\,,\ \Phi(Z_o) \in A\,,\ \mathfrak{s}(Z_o) \in S\right)$

$$= \frac{1}{n!}\int_A t^{n-1} e^{-t}\mathrm{d}t \int_{(\mathbb{S}^{d-1}\times\mathbb{R})^n} \mathbb{1}\left(P_{[n]} \in \mathcal{P}_n\right)\mathbb{1}\left(\Phi\left(P_{[n]}\right) < 1\right)\mathbb{1}\left(\mathfrak{s}(Z_o) \in S\right)$$
$$\times\, \mathrm{d}t_1\varphi(\mathrm{d}u_1)\cdots\mathrm{d}t_n\varphi(\mathrm{d}u_n).$$

$P_{[n]} = \cap_{i=1}^n H(u_i, t_i)^-$

$\Rightarrow$ **The distribution of the number of facets is essential!**

Distribution of $f(Z_{\mathbf{o}})$

Theorem (Upper bound)

$$\mathbb{P}(f(Z_{\mathbf{o}}) = n) \leq c_1^n n^{-\frac{2n}{d-1}}$$

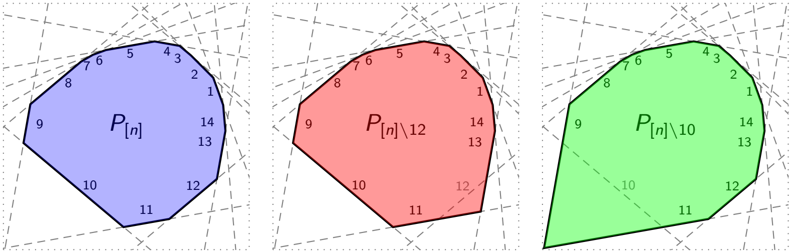
$+$ there exists n_φ such that $\mathbb{P}(f(Z_{\mathbf{o}}) = n)$ is either vanishing or decreasing for $n > n_\varphi$.

If φ is the rotation invariant (or *well spread*) we also have:

Theorem (Lower bound)

$$\mathbb{P}(f(Z_{\mathbf{o}}) = n) \geq c_2^n n^{-\frac{2n}{d-1}}$$

Polytopal approximation



$$P_I = \cap_{i \in I} H_i^-$$

Lemma (based on ideas from Reisner, Schütt and Werner '01)

There exists $I \subset [n]$ with $|I| \geq \frac{n}{4}$ such that, for any $i \in I$,

$$d_H(P_{[n]}, P_{[n]\setminus i}) < c\Phi(P_{[n]}) n^{-\frac{2}{d-1}},$$

and

$$\Phi(P_{[n]\setminus i}) < \exp\left(cn^{-\frac{d+1}{d-1}}\right) \Phi(P_{[n]}).$$

... where c is independent from n and $P_{[n]}$.

D.G. Kendall's problem (1987): Shape of Big cells

$$\tau := \min_K \frac{\Phi(K)}{V_d(K)^{\frac{1}{d}}} \overset{\text{(isotropy)}}{=} \frac{\Phi(B^d)}{V_d(B^d)^{\frac{1}{d}}}$$

Theorem (Hug, Reitzner, Schneider '04), (Hug, Schneider '07)

There exists c, $c' > 0$ such that, for any $\varepsilon > 0$ and $a > 0$,

$$\mathbb{P}\left(\frac{\Phi(Z_{\mathbf{o}})}{V_d(Z_{\mathbf{o}})^{\frac{1}{d}}} > \tau + \varepsilon \mid V_d(Z_{\mathbf{o}}) > a\right) \leq c \exp\left(-c' \varepsilon a^{\frac{1}{d}}\right).$$

$$\Leftrightarrow \limsup_{a \to \infty} -a^{-\frac{1}{d}} \ln\left(\mathbb{P}\left(\frac{\Phi(Z_{\mathbf{o}})}{V_d(Z_{\mathbf{o}})^{\frac{1}{d}}} > \tau + \varepsilon \mid V_d(Z_{\mathbf{o}}) > a\right)\right) \leq c' \varepsilon.$$

D.G. Kendall's problem (1987): Shape of Big cells

$$\tau := \min_K \frac{\Phi(K)}{V_d(K)^{\frac{1}{d}}} \overset{\text{(isotropy)}}{=} \frac{\Phi(B^d)}{V_d(B^d)^{\frac{1}{d}}}$$

Theorem (Hug, Reitzner, Schneider '04), (Hug, Schneider '07)

There exists c, $c' > 0$ such that, for any $\varepsilon > 0$ and $a > 0$,

$$\mathbb{P}\left(\frac{\Phi(Z_o)}{V_d(Z_o)^{\frac{1}{d}}} > \tau + \varepsilon \mid V_d(Z_o) > a\right) \leq c \exp\left(-c'\varepsilon a^{\frac{1}{d}}\right).$$

$$\Leftrightarrow \limsup_{a\to\infty} -a^{-\frac{1}{d}} \ln\left(\mathbb{P}\left(\frac{\Phi(Z_o)}{V_d(Z_o)^{\frac{1}{d}}} > \tau + \varepsilon \mid V_d(Z_o) > a\right)\right) \leq c'\varepsilon.$$

Theorem

For any $\varepsilon > 0$,

$$\lim_{a\to\infty} -a^{-\frac{1}{d}} \ln\left(\mathbb{P}\left(\frac{\Phi(Z_o)}{V_d(Z_o)^{\frac{1}{d}}} > \tau + \varepsilon \mid V_d(Z_o) > a\right)\right) = \varepsilon.$$

Shape of cells with many facets

Assume that φ is rotation invariant.

Conjecture

Conditionally on $f(Z_{\mathbf{o}}) = n \to \infty$,

$$\mathfrak{s}(Z_{\mathbf{o}}) \simeq B^d.$$

Theorem (Big cells are not elongated)

Assume that $d \geq 4$. There exists $\delta > 0$ such that

$$\lim_{n\to\infty} \mathbb{P}(\text{there exists } [a,b] \subset Z_{\mathbf{o}} \text{ such that } d_H([a,b], Z_{\mathbf{o}}) < \delta \mid f(Z_{\mathbf{o}}) = n) = 0.$$

Size distribution

Theorem (Distribution of Φ)

For $a > 0$

$$\mathbb{P}(\Phi(Z) > a) < \exp\left(-a + c_1 a^{\frac{d-1}{d+1}}\right).$$

If φ is rotation invariant (or well spread), then for $a > c_3$

$$\mathbb{P}(\Phi(Z) > a) > \exp\left(-a + c_2 a^{\frac{d-1}{d+1}}\right).$$

Similar with other size measurements, e.g. the volume.

Small typical cells

Theorem

Assume that φ is absolutely continuous.

$$\lim_{a\to 0} \mathbb{P}(f(Z_{\mathrm{typ}}) = d+1 | V_d(Z_{\mathrm{typ}}) < a) = 1.$$

More results:

- Case where φ has atoms.
- General size functionals instead of V_d.
- Shape distribution of small cells.

THANK YOU!

- *Poisson hyperplane tessellation: Asymptotic probabilities of the zero and typical cells.*
  Ph.D. thesis. (2016)
- *Polytopal approximation of elongated convex bodies.*
  Advances in Geometry (accepted)
- *Cells with many facets in a Poisson hyperplane tessellation.*
  joint work with P. Calka and M. Reitzner
  arXiv:1608.07979
- *Small cells in a Poisson hyperplane tessellation.*
  arXiv:1702.01964