Monotonicity of facet numbers of random convex hulls

gilles.bonnet@rub.de

Joint work with Julian Grote, Daniel Temesvari, Christoph Thäle, Nicola Turchi & Florian Wespi

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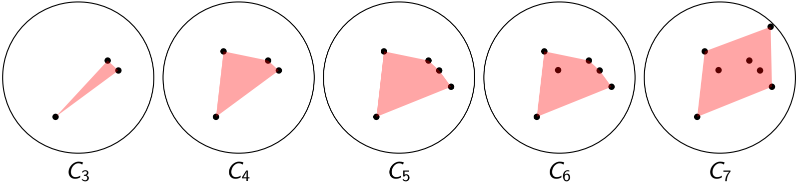
Setting

$X_1, X_2, \ldots \in \mathbb{R}^d$ i.i.d. random points

$C_n := \mathrm{conv}(X_1, \ldots, X_n)$



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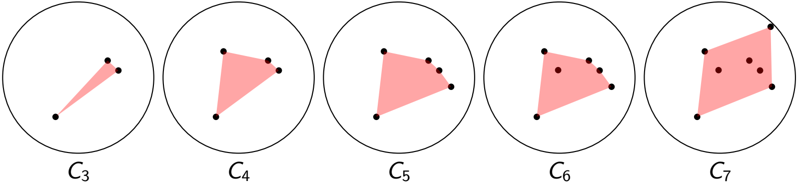

Trivial observations

$C_{d+1} \subset C_{d+2} \subset \ldots$
$V_d(C_{d+1}) \subset V_d(C_{d+2}) \subset \ldots$

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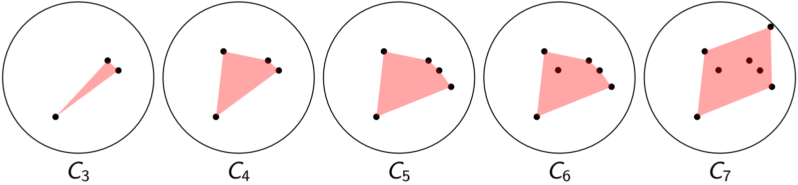

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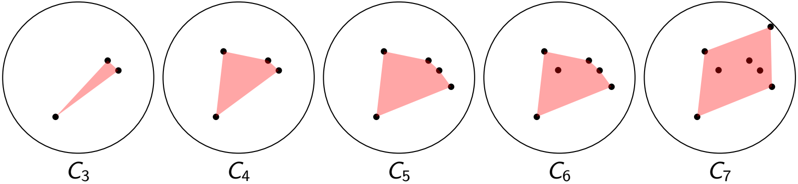

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Is it true that $\mathbb{E} f_i(C_n) \leq \mathbb{E} f_i(C_{n+1})$?
For which distributions?

$X_1, X_2, \ldots \in \mathbb{R}^d$ i.i.d. random points, $d \geq 2$. $C_n := \mathrm{conv}(X_1, \ldots, X_n)$, $n \geq d+1$

Theorem [Devillers, Glisse, Goaoc, Moroz, Reitzner, '13]

Assume that the points are uniformly distributed in a convex body K

- If $d = 2$, then $\mathbb{E} f_i(C_n) < \mathbb{E} f_i(C_{n+1})$, $i = 0, 1$.
- If $d \geq 3$ and K has a smooth boundary, then there exists n_K such that

$$\mathbb{E} f_i(C_n) < \mathbb{E} f_i(C_{n+1}), \ i = d-1, d-2, \ n \geq n_K.$$

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Theorem [Beerman, '14]

If the distribution is the standard Gaussian distribution, or the uniform distribution in the ball B^d, then

$$\mathbb{E} f_{d-1}(C_n) \leq \mathbb{E} f_{d-1}(C_{n+1}).$$

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Theorem [Thäle, Kabluchko, '18+]

- If the distribution is the standard Gaussian distribution, then

$$\mathbb{E} f_i(C_n) < \mathbb{E} f_i(C_{n+1}), \; i = 0, \ldots, d-1.$$

- Similar results for the symmetric Gaussian polytope and the Gaussian zonotope.

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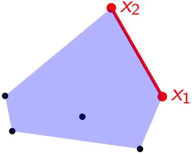

Density $p(\|x\|)$

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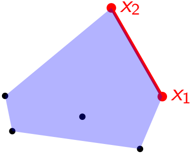

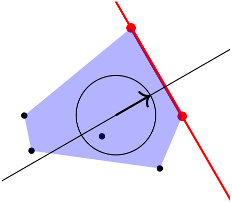

Density $p(\|x\|)$ Blaschke-Petkanschin

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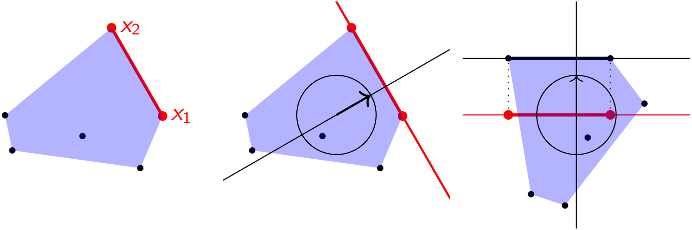

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Blaschke-Petkanschin

rotation invariance
+ orthogonality
+ independence

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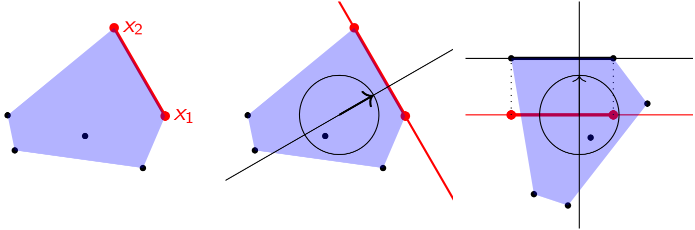

Density $p(\|x\|)$ | Blaschke-Petkanschin | rotation invariance + orthogonality + independence

$$\mathbb{E}f_{d-1}(C_n) = c\binom{n}{d}\int_{-\infty}^{\infty}\mathbb{P}((X_1)_d \le t)^{n-d}\int_{(\mathbb{R}^{d-1})^d}\prod_{i=1}^{d} p\left(\sqrt{\|y_i\|^2+t^2}\right) \times \Delta_{d-1}(y_1,\dots,y_d)\,\mathrm{d}(y_1,\dots,y_d)\,\mathrm{d}t$$

3 (very) nice classes of distributions with density in $\mathbb{R}^d$:

G	centred Gaussian	$p_{\mathbf{G},\sigma}(\|x\|) = c\exp\left(-\frac{\|x\|^2}{2\sigma^2}\right)$	$\sigma > 0$
H	heavy-tailed	$p_{\mathbf{H},\sigma,\beta}(\|x\|) = c\left(1+\frac{\|x\|^2}{\sigma^2}\right)^{-\beta}$	$\beta > \frac{d}{2},\ \sigma > 0$
B	beta type	$p_{\mathbf{B},\sigma,\beta}(\|x\|) = c\left(1-\frac{\|x\|^2}{\sigma^2}\right)^{\beta}$	$\beta > -1,\ \sigma > 0,\ x \in \mathbb{B}^d_\sigma$

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Lemma

$p_{\mathbf{G},\sigma}(\sqrt{r^2+t^2}) = \varphi(t)\, p_{\mathbf{G},\sigma}\left(\frac{r}{\psi(t)}\right)$ with $\psi(t) = 1$ and $\varphi(t) = \exp\left(\frac{-t^2}{2\sigma^2}\right)$.

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By [Miles '71] & [Ruben, Miles '80] (under certain assumption) these are the only distributions with this property.

We know that

$$\mathbb{E}f_{d-1}(C_n) = c\binom{n}{d}\int_{-\infty}^{\infty} \mathbb{P}((X_1)_d \leq t)^{n-d} \int_{(\mathbb{R}^{d-1})^d} \prod_{i=1}^{d} p(\sqrt{\|y_i\|^2 + t^2}) \times \Delta_{d-1}(y_1, \dots, y_d)\, \mathrm{d}(y_1, \dots, y_d)\, \mathrm{d}t$$

and $p(\sqrt{\|y_i\|^2 + t^2}) = \varphi(t)p\left(\frac{\|y_i\|}{\psi(t)}\right)$.

Thus

$$\mathbb{E}f_{d-1}(C_n) = c\binom{n}{d}\int_{-\infty}^{\infty} \mathbb{P}((X_1)_d \leq t)^{n-d} \varphi(t)^d \psi(t)^{d^2-1}\, \mathrm{d}t.$$

Thus

$$\mathbb{E}f_{d-1}(C_n) - \mathbb{E}f_{d-1}(P_{n-1})$$
$$= c\int_{-\infty}^{\infty}\left[\binom{n}{d}\mathbb{P}((X_1)_d \le t) - \binom{n-1}{d}\right]\mathbb{P}((X_1)_d \le t)^{n-1-d}\varphi(t)^d\psi(t)^{d^2-1}\,\mathrm{d}t$$

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$$= c\int_0^1 \underbrace{\left[\binom{n}{d}(1-s) - \binom{n-1}{d}\right]}_{\text{affine, negative slope, root } s^* \in [0,1]} \underbrace{(1-s)^{n-1-d}}_{>0} \quad \underbrace{L(s)}_{\ge 0,\text{strictly concave}}{}^{d-1}\,\mathrm{d}s$$

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$$> c\int_0^1 \left[\binom{n}{d}(1-s) - \binom{n-1}{d}\right](1-s)^{n-1-d}\left(\frac{L(s^*)}{s^*}s\right)^{d-1}\mathrm{d}s = \cdots > 0. \quad \square$$

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U uniform on centred spheres $\mathbb{S}^{d-1}_\sigma$.

Theorem

Let $X_1, \dots, X_n$ be i.i.d. according to a distribution belonging to one of the classes **G**, **H**, **B** or **U**. Then

$$\mathbb{E}f_{d-1}(C_n) > \mathbb{E}f_{d-1}(P_{n-1}).$$

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G. Bonnet, J. Grote, D. Temesvari, C. Thäle, N. Turchi & F. Wespi
Journal of Mathematical Analysis and Applications (2017)

The monotonicity of f-vectors of random polytopes
O. Devillers, M. Glisse, X. Goaoc, G. Moroz, M. Reitzner
Electronic Communications in Probability (2013)

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A canonical decomposition of the probability measure of sets of isotropic random points in $\mathbb{R}^n$
H. Ruben, R.E. Miles, Journal of Multivariate Analysis (1980)

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Z. Kabluchko, D. Temesvari, C. Thäle

Thank you!