Maximal degree in a Poisson-Delaunay graph

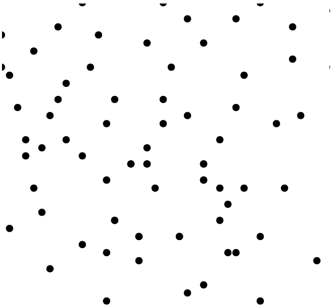
gilles.bonnet@rub.de

Joint work with Nicolas Chenavier (Calais, France)

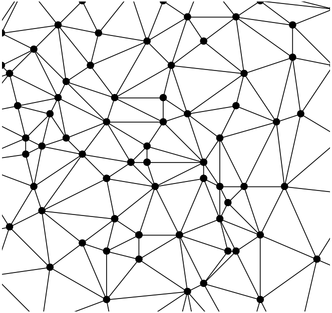
STAR Workshop on Random Graph
Nijmegen, 13th April 2018

RUHR
UNIVERSITÄT
BOCHUM

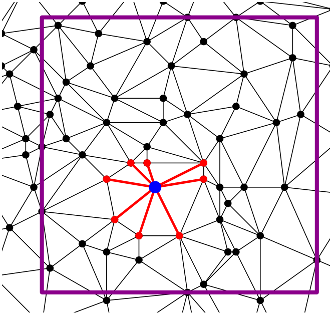
RUB



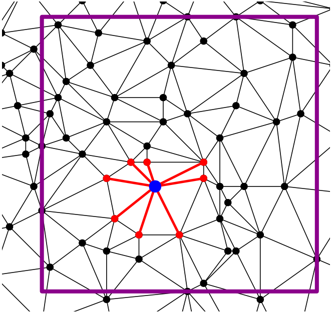
1. Poisson point process η of intensity 1



1. Poisson point process η of intensity 1
2. For any triple of points ($d = 2$):
   Draw triangle if the circumscribed circle is empty



1. Poisson point process η of intensity 1
2. For any triple of points ($d = 2$):
   Draw triangle if the circumscribed circle is empty
3. Let $\rho > 0$.
   $\Delta_\rho = \max_{x \in \eta \cap [0, \rho^{1/2}]^2} \deg_\eta(x)$.



1. Poisson point process η of intensity 1
2. For any triple of points ($d = 2$): Draw triangle if the circumscribed circle is empty
3. Let $\rho > 0$. $\Delta_\rho = \max_{x \in \eta \cap [0, \rho^{1/2}]^2} \deg_\eta(x)$.

Theorem [Bern, Eppstein, Yao '91]

$$\mathbb{E}\Delta_\rho = \Theta\left(\frac{\log \rho}{\log \log \rho}\right).$$

Theorem [Broutin, Devillers, Hemsley '14]

Assume $d = 2$. For any $\xi > 0$,

$$\mathbb{P}\left(\Delta_\rho \leq (\log \rho)^{2+\xi}\right) \to 1, \text{ as } \rho \to \infty.$$

Results for other random graphs

Random combinatorial graphs (non exhaustive list!)

- Erdös-Rényi graph $G_{n,p}$ with $p = o(\frac{\log n}{n})$ [Bollobás '85]
- Uniformly distributed among a class of graphs with n vertices
    - Labelled tree [Carr, Goh, Schmutz '94]
    - Planar [Drmota, Giménez, Panagiotou, Steger '14]
    - Triangulation of a n-gon [Gao, Wormald '00]
    - Excluded minor [Giménez, Mitsche, Noy '16]

Random geometric graphs

- Gilbert graph [Penrose '03]

$$\Delta_n := \text{maximal degree over} \begin{cases} \text{all vertices} & \text{(combinatorial graphs)} \\ \text{all vertices in } [0, n^{1/d}]^d & \text{(geometric graphs)} \end{cases}$$

$$\Delta_n \simeq \frac{\log n}{f(n)}$$

where $1 \leq f(n) \leq \log\log n$ is a correcting factor depending on the model.

Results for other random graphs

Random combinatorial graphs (non exhaustive list!)

- Erdös-Rényi graph $G_{n,p}$ with $p = o(\frac{\log n}{n})$ [Bollobás '85]
- Uniformly distributed among a class of graphs with n vertices
  - Labelled tree [Carr, Goh, Schmutz '94]
  - Planar [Drmota, Giménez, Panagiotou, Steger '14]
  - Triangulation of a n-gon [Gao, Wormald '00]
  - Excluded minor [Giménez, Mitsche, Noy '16]

Random geometric graphs

- Gilbert graph [Penrose '03]

$$\Delta_n := \text{maximal degree over} \begin{cases} \text{all vertices} & \text{(combinatorial graphs)} \\ \text{all vertices in } [0, n^{1/d}]^d & \text{(geometric graphs)} \end{cases}$$

$$\Delta_n \simeq \frac{\log n}{f(n)}$$

where $1 \leq f(n) \leq \log \log n$ is a correcting factor depending on the model.

For the *Erdös-Rényi* graphs, *labelled tree* model and *Gilbert* graphs, it is also know that

Δ_n concentrates on two consecutive values.

Our results (arXiv:1804.01416)

$\Delta_\rho =$ maximal degree over all vertices of a Poisson-Delaunay graph in $[0, \rho^{\frac{1}{d}}]^d$.

Theorem 1

Assume $d = 2$. There exists a map $\rho \mapsto I_\rho$ such that

1. $\mathbb{P}\left(\Delta_\rho \in \{I_\rho, I_\rho + 1\} \right) \to 1$;
2. $I_\rho \sim \frac{\log \rho}{2 \log \log \rho}$.

Our results (arXiv:1804.01416)

$\Delta_\rho =$ maximal degree over all vertices of a Poisson-Delaunay graph in $[0, \rho^{\frac{1}{d}}]^d$.

Theorem 1

Assume $d = 2$. There exists a map $\rho \mapsto I_\rho$ such that

1. $\mathbb{P}\left(\Delta_\rho \in \{I_\rho, I_\rho + 1\} \right) \to 1$;
2. $I_\rho \sim \frac{\log \rho}{2 \log \log \rho}$.

Theorem 2

For any $d \geq 2$. There exists a map $\rho \mapsto J_\rho$, such that

1. $\mathbb{P}\left(\Delta_\rho \in \{J_\rho, J_\rho + 1, \dots, J_\rho + \ell_d\} \right) \to 1$, where $\ell_d = \left\lfloor \frac{d+3}{2} \right\rfloor$;
2. $J_\rho \sim \frac{\log \rho}{\frac{2}{d-1} \log \log \rho}$.

Our results (arXiv:1804.01416)

$\Delta_\rho =$ maximal degree over all vertices of a Poisson-Delaunay graph in $[0, \rho^{\frac{1}{d}}]^d$.

Theorem 1

Assume $d = 2$. There exists a map $\rho \mapsto I_\rho$ such that

1. $\mathbb{P}\left(\Delta_\rho \in \{I_\rho, I_\rho + 1\} \right) \to 1$;
2. $I_\rho \sim \frac{\log \rho}{2 \log \log \rho}$.

Theorem 2

For any $d \geq 2$. There exists a map $\rho \mapsto J_\rho$, such that

1. $\mathbb{P}\left(\Delta_\rho \in \{J_\rho, J_\rho + 1, \dots, J_\rho + \ell_d\} \right) \to 1$, where $\ell_d = \left\lfloor \frac{d+3}{2} \right\rfloor$;
2. $J_\rho \sim \frac{\log \rho}{\frac{2}{d-1} \log \log \rho}$.

Corollaries

1. For $d = 2$, there exists a sequence ρ_i such that $\mathbb{P}\left(\Delta_{\rho_i} = I_{\rho_i} \right) \to 1$.
2. For any $d \geq 2$, $\mathbb{E}\, \Delta_\rho \sim \frac{\log \rho}{\frac{2}{d-1} \log \log \rho}$.

Typical degree

$\eta \ldots$ stationary Poisson point process
$\mathcal{D}^0 = \deg_{\eta \cup \{0\}}(0) =$ typical degree

Typical degree

$\eta \ldots$ stationary Poisson point process
$\mathcal{D}^0 = \deg_{\eta\cup\{0\}}(0) =$ typical degree

Theorem [B. '16]

There exist constants $c_1, c_2, c_3 > 0$, depending on d, such that,

$$c_1^k k^{\frac{-2}{d-1}k} \leq \mathbb{P}\left(\mathcal{D}^0 = k\right) \leq c_2 k^{\frac{-2}{d-1}} \mathbb{P}\left(\mathcal{D}^0 = k-1\right) \leq \ldots \leq c_3^k k^{\frac{-2}{d-1}k}$$

Typical degree

$\eta \ldots$ stationary Poisson point process
$\mathcal{D}^0 = \deg_{\eta \cup \{0\}}(0) =$ typical degree

Theorem [B. '16]

There exist constants $c_1, c_2, c_3 > 0$, depending on d, such that,

$$c_1^k k^{\frac{-2}{d-1}k} \leq \mathbb{P}\left(\mathcal{D}^0 = k\right) \leq c_2 k^{\frac{-2}{d-1}} \mathbb{P}\left(\mathcal{D}^0 = k-1\right) \leq \ldots \leq c_3^k k^{\frac{-2}{d-1}k}$$

Fast decay of $k \mapsto \mathbb{P}\left(\mathcal{D}^0 = k\right) \implies \mathbb{P}\left(\mathcal{D}^0 \geq k\right) \sim \mathbb{P}\left(\mathcal{D}^0 = k\right)$.

Typical degree

$\eta \dots$ stationary Poisson point process
$\mathcal{D}^0 = \deg_{\eta \cup \{0\}}(0) =$ typical degree

Theorem [B. '16]

There exist constants $c_1, c_2, c_3 > 0$, depending on d, such that,

$$c_1^k k^{\frac{-2}{d-1}k} \leq \mathbb{P}\left(\mathcal{D}^0 = k\right) \leq c_2 k^{\frac{-2}{d-1}} \mathbb{P}\left(\mathcal{D}^0 = k-1\right) \leq \ldots \leq c_3^k k^{\frac{-2}{d-1}k}$$

Fast decay of $k \mapsto \mathbb{P}\left(\mathcal{D}^0 = k\right) \implies \mathbb{P}\left(\mathcal{D}^0 \geq k\right) \sim \mathbb{P}\left(\mathcal{D}^0 = k\right)$.

Lemma

There exists a map $\rho \in \mathbf{R}_+ \mapsto I_\rho \in \mathbf{N}$ with the following properties:

1. $\rho \mathbb{P}\left(\mathcal{D}^0 \geq I_\rho + 2\right) \to 0$;
2. $\rho \mathbb{P}\left(\mathcal{D}^0 \geq I_\rho\right) \to \infty$;

Typical degree

$\eta \ldots$ stationary Poisson point process
$\mathcal{D}^0 = \deg_{\eta\cup\{0\}}(0) =$ typical degree

Theorem [B. '16]

There exist constants $c_1, c_2, c_3 > 0$, depending on d, such that,

$$c_1^k k^{\frac{-2}{d-1}k} \leq \mathbb{P}\left(\mathcal{D}^0 = k\right) \leq c_2 k^{\frac{-2}{d-1}} \mathbb{P}\left(\mathcal{D}^0 = k-1\right) \leq \ldots \leq c_3^k k^{\frac{-2}{d-1}k}$$

Fast decay of $k \mapsto \mathbb{P}\left(\mathcal{D}^0 = k\right) \implies \mathbb{P}\left(\mathcal{D}^0 \geq k\right) \sim \mathbb{P}\left(\mathcal{D}^0 = k\right)$.

Lemma

There exists a map $\rho \in \mathbf{R}_+ \mapsto I_\rho \in \mathbf{N}$ with the following properties:

1. $\rho\,\mathbb{P}\left(\mathcal{D}^0 \geq I_\rho + 2\right) \to 0$;
2. $\rho\,\mathbb{P}\left(\mathcal{D}^0 \geq I_\rho\right) \to \infty$;

2bis. $\left(\frac{\log\log\rho}{\log\rho}\right)^{\frac{2\ell}{d-1}} \rho\,\mathbb{P}\left(\mathcal{D}^0 \geq I_\rho - \ell\right) \to \infty$, for any $\ell \in \mathbf{N}$;

Typical degree

$\eta \dots$ stationary Poisson point process
$\mathcal{D}^0 = \deg_{\eta \cup \{0\}}(0) =$ typical degree

Theorem [B. '16]

There exist constants $c_1, c_2, c_3 > 0$, depending on d, such that,

$$c_1^k k^{\frac{-2}{d-1}k} \leq \mathbb{P}\left(\mathcal{D}^0 = k\right) \leq c_2 k^{\frac{-2}{d-1}} \mathbb{P}\left(\mathcal{D}^0 = k-1\right) \leq \ldots \leq c_3^k k^{\frac{-2}{d-1}k}$$

Fast decay of $k \mapsto \mathbb{P}\left(\mathcal{D}^0 = k\right) \implies \mathbb{P}\left(\mathcal{D}^0 \geq k\right) \sim \mathbb{P}\left(\mathcal{D}^0 = k\right)$.

Lemma

There exists a map $\rho \in \mathbf{R}_+ \mapsto I_\rho \in \mathbf{N}$ with the following properties:

1. $\rho \, \mathbb{P}\left(\mathcal{D}^0 \geq I_\rho + 2\right) \to 0$;
2. $\rho \, \mathbb{P}\left(\mathcal{D}^0 \geq I_\rho\right) \to \infty$;

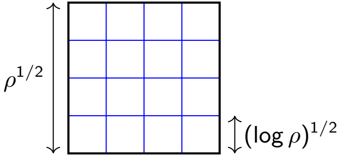
2bis. $\left(\frac{\log \log \rho}{\log \rho}\right)^{\frac{2\ell}{d-1}} \rho \, \mathbb{P}\left(\mathcal{D}^0 \geq I_\rho - \ell\right) \to \infty$, for any $\ell \in \mathbf{N}$;

3. $I_\rho \sim \frac{\log \rho}{\frac{2}{d-1} \log \log \rho}$.

Proof I (any $d \geq 2$): Easy part

$$\begin{aligned}
\mathbb{P}\left(\Delta_\rho \geq I_\rho + 2\right) &= \mathbb{P}\left(\bigcup_{x \in \eta \cap [0,\rho^{1/d}]^d} \{d_\eta(x) \geq I_\rho + 2\}\right) \\
&\leq \mathbb{E}\left[\sum_{x \in \eta \cap [0,\rho^{1/d}]^d} \mathbb{1}(d_\eta(x) \geq I_\rho + 2)\right] \\
&= \rho\, \mathbb{P}\left(\mathcal{D}^0 \geq I_\rho + 2\right) \\
&\to 0. \quad \square
\end{aligned}$$

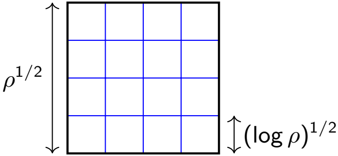
Proof II ($d = 2$): $\mathbb{P}\left(\Delta_\rho < I_\rho\right) \to 0$.



1. Divide $[0, \rho^{1/2}]^2$ into squares of area $\simeq \log \rho$.

$$\mathbb{P}\left(\Delta_\rho < I_\rho\right) \leq \mathbb{P}\left(\Delta_{\log \rho} < I_\rho\right)^{c\frac{\rho}{\log \rho}} \sim \exp\left(-c\frac{\rho}{\log \rho}\mathbb{P}\left(\Delta_{\log \rho} \geq I_\rho\right)\right).$$

Proof II ($d = 2$): $\mathbb{P}(\Delta_\rho < l_\rho) \to 0$.



1. Divide $[0, \rho^{1/2}]^2$ into squares of area $\simeq \log \rho$.

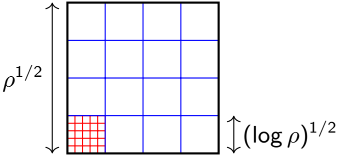
$$\mathbb{P}(\Delta_\rho < l_\rho) \leq \mathbb{P}(\Delta_{\log \rho} < l_\rho)^{c\frac{\rho}{\log \rho}} \sim \exp\left(-c\frac{\rho}{\log \rho}\mathbb{P}(\Delta_{\log \rho} \geq l_\rho)\right).$$

2. In each of these squares there is at most 4 vertices with degree $\geq l_\rho$.

Lemma

Let S be a set of 5 vertices in a planar graph. Then there exist 2 vertices in S which have at most 23 neighbors in common.

Proof II ($d = 2$): $\mathbb{P}\left(\Delta_\rho < l_\rho\right) \to 0$.



1. Divide $[0, \rho^{1/2}]^2$ into squares of area $\simeq \log \rho$.

$$\mathbb{P}\left(\Delta_\rho < l_\rho\right) \leq \mathbb{P}\left(\Delta_{\log \rho} < l_\rho\right)^{c\frac{\rho}{\log \rho}} \sim \exp\left(-c\frac{\rho}{\log \rho}\mathbb{P}\left(\Delta_{\log \rho} \geq l_\rho\right)\right).$$

2. In each of these squares there is at most 4 vertices with degree $\geq l_\rho$.

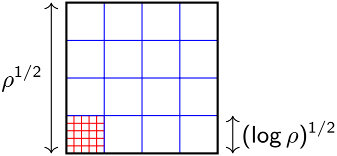
Lemma

Let S be a set of 5 vertices in a planar graph. Then there exist 2 vertices in S which have at most 23 neighbors in common.

3. Subdivide $[0, (\log \rho)^{1/2}]^2$ cubes into unit squares $S_1, \ldots, S_{\log \rho}$.

$$\mathbb{P}\left(\Delta_{\log \rho} \geq l_\rho\right) = \mathbb{P}\left(\bigcup_{i \leq \log \rho} \{\Delta_{S_i} \geq l_\rho\}\right) \geq \frac{1}{4}\sum_{i \leq \log \rho} \mathbb{P}\left(\Delta_{S_i} \geq l_\rho\right) \geq \frac{\log \rho}{20}\mathbb{P}\left(\mathcal{D}^0 \geq l_\rho\right).$$

Proof II ($d = 2$): $\mathbb{P}(\Delta_\rho < l_\rho) \to 0$.



1. Divide $[0, \rho^{1/2}]^2$ into squares of area $\simeq \log\rho$.

$$\mathbb{P}(\Delta_\rho < l_\rho) \le \mathbb{P}(\Delta_{\log\rho} < l_\rho)^{c\frac{\rho}{\log\rho}} \sim \exp\left(-c\frac{\rho}{\log\rho}\mathbb{P}(\Delta_{\log\rho} \ge l_\rho)\right).$$

2. In each of these squares there is at most 4 vertices with degree $\ge l_\rho$.

Lemma

Let S be a set of 5 vertices in a planar graph. Then there exist 2 vertices in S which have at most 23 neighbors in common.

3. Subdivide $[0, (\log\rho)^{1/2}]^2$ cubes into unit squares $S_1, \ldots, S_{\log\rho}$.

$$\mathbb{P}(\Delta_{\log\rho} \ge l_\rho) = \mathbb{P}\left(\bigcup_{i \le \log\rho}\{\Delta_{S_i} \ge l_\rho\}\right) \ge \frac{1}{4}\sum_{i \le \log\rho}\mathbb{P}(\Delta_{S_i} \ge l_\rho) \ge \frac{\log\rho}{20}\mathbb{P}\left(\mathcal{D}^0 \ge l_\rho\right).$$

4. Put everything together. ☐

Thank you!