Random convex hulls in high dimension

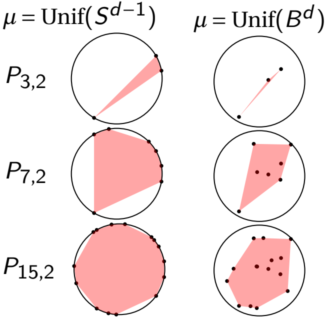
Gilles Bonnet

Ruhr University Bochum

12 March 2020

Model

- $X_1, X_2, \ldots \in \mathbb{R}^d$ i.i.d. random vectors distributed according to a probability measure $\mu = \mu_d$
- $P_{n,d} = \mathrm{conv}(X_1, \ldots, X_n)$



What does a **high dimensional** random polytope look like ?

Distribution	Regime		Characteristic
Uniform on the sphere,	$n \simeq \alpha d$	linear,	Number of facets,
Uniform in the ball,	$n \simeq d^\alpha$	polynomial,	Height of the facets,
Gaussian,	$n \simeq \alpha^d$	exponential,	Volume,
Beta distributions,	$n \simeq d^{\alpha d}$	super exponential,	$\mathbf{1}(0 \in P_{n,d})$,
...	...		...

Distribution

- Uniform on the sphere,

Regime

- $n \simeq \alpha d$ linear.

Characteristic

- $\mathbf{1}(0 \in P_{n,d})$,

Theorem (Wendel, 1962)

$$\mathbb{P}(0 \notin P_{n,d}) = \frac{1}{2^{n-1}} \sum_{k=0}^{d-1} \binom{n-1}{k} = \mathbb{P}(B_{n-1} < d).$$

where B_{n-1} is a Binomial random variable with parameters $n-1$ and $\frac{1}{2}$.
Since $[2B_{n-1} - (n-1)]/\sqrt{n-1} \xrightarrow{d} Z \sim \mathcal{N}(0,1)$ we have

Corollary (Threshold phenomena)

$$\lim_{d\to\infty} \mathbb{P}(0 \in P_{n,d}) = \begin{cases} 0 & \text{if } n \le (2-\epsilon)d, \\ 1 & \text{if } n \ge (2+\epsilon)d. \end{cases}$$

Corollary (Phase transition)

If $n = 2d + x\sqrt{d} + o(\sqrt{d})$ *then* $\lim_{d\to\infty} \mathbb{P}(0 \in P_{n,d}) = \Phi(x).$

where $\Phi(x) = \mathbb{P}(Z \le x)$ is the CDF of a standard normal random variable.

Height of a facet

Let F be a facet of $P_{n,d}$.
Set $H = \mathrm{affineHull}(F)$ its supporting hyperplane
Let $u \in S^{d-1}$ and $h \in \mathbb{R}$ such that $H = \{x \in \mathbb{R}^d : \langle x, u\rangle = h\}$ and $P_{n,d} \subset \{x \in \mathbb{R}^d : \langle x, u\rangle \leq h\}$. We call h the height of the facet F.

(a) A facet with positive height.

(b) A facet with negative height.

The **typical height** H_{typ} is a random variable defined by

$$\mathbb{P}(H_{\mathrm{typ}} \in \cdot) = \mathbb{P}([X_1, \ldots, X_d] \text{ has height } \in \cdot \mid [X_1, \ldots, X_d] \text{ is a facet of } P_{n,d}).$$

Typical height

Distribution assumption: **uniform on the sphere**.
Let $[h_1, h_2] \subset [-1, 1]$.

$$\mathbb{P}(H_{\text{typ}} \in [h_1, h_2]) = \frac{\mathbb{P}([X_1, \dots, X_d] \text{ is facet \& its height is } \in [h_1, h_2])}{\mathbb{P}([X_1, \dots, X_d] \text{ is a facet of } P_{n,d})} = \frac{I_{[h_1,h_2]}}{I_{[-1,1]}},$$

where

$$I_{[h_1,h_2]} = \mathbb{P}([X_1, \dots, X_d] \text{ is facet \& its height is } \in [h_1, h_2])$$

$\vdots$ Toolbox from integral geometry

$$= \int_{h_1}^{h_2} c_1 (1-h)^{\frac{d^2-2d-1}{2}} \left(c_2 \int_{-1}^{h} (1-s^2)^{\frac{d-3}{2}} \, \mathrm{d}s \right)^{n-d} \mathrm{d}h.$$

Typical height

Theorem (B., O'Reilly, 2019+)

Distribution assumption: ***Uniform on the sphere.***

- *(sub-exponential)* If $(\ln n)/d \to 0$, then $H_{\text{typ}} \xrightarrow{\mathbb{P}} 0$.
  - If $n-d = O(\sqrt{d})$, then $dH_{\text{typ}} - \frac{n-d}{\sqrt{d}} r_1 \xrightarrow{d} Z \sim \mathcal{N}(0,1)$.
  - If $\sqrt{d} \ll n-d \ll d$, then $\frac{d^{3/2}}{n-d} H_{\text{typ}} \xrightarrow{\mathbb{P}} r_2$.
  - If $n \sim \rho d$, then $\sqrt{d}\, H_{\text{typ}} \xrightarrow{\mathbb{P}} r_3(\rho)$.
  - If $\ln n \ll d \ll n$, then $\sqrt{\frac{d}{\ln(n/d)}} H_{\text{typ}} \xrightarrow{\mathbb{P}} r_4$.
- *(exponential)* If $(\ln n)/d \to \alpha$, then $H_{\text{typ}} \xrightarrow{\mathbb{P}} \sqrt{1-e^{-\alpha}}$.
- *(super-exponential)* If $(\ln n)/d \to \infty$, then $H_{\text{typ}} \xrightarrow{\mathbb{P}} 1$.
  - If $\ln n \gg d$, then $-\frac{d-1}{\ln n} \ln\left(1 - H_{\text{typ}}^2\right) \xrightarrow{\mathbb{P}} r_5$.
  - If $\ln n \gg d \ln d$, then $r_6 \frac{n}{d} (1 - H_{\text{typ}}^2)^{\frac{d}{2}} - \sqrt{d} \xrightarrow{\text{tv}} Z \sim \mathcal{N}(0,1)$.
  - If "d is fixed", then $nc_d (1 - H_{\text{typ}}^2)^{\frac{d-1}{2}} \xrightarrow{\text{tv}} \Gamma_{d-1}$.

where $r_1 = \sqrt{\frac{2}{\pi}}$, $r_2 = \frac{2}{\pi}$, $r_3 = \operatorname{argmax}(r \mapsto (\rho-1)\ln\Phi(r) - r^2/2)$, $r_4 = \sqrt{2}$, $r_5 = 2$, $r_6 = \frac{1}{2\sqrt{\pi}}$.

Interval containing the heights of all the facets

$$H_{\min} := \min\{\mathrm{height}(F) \mid F \in \mathrm{Facets}(P_{n,d})\}$$
$$H_{\max} := \max\{\mathrm{height}(F) \mid F \in \mathrm{Facets}(P_{n,d})\}$$

These random variables have a similar asymptotic as the typical height:

Theorem (B., O'Reilly, 2019+)

Distribution assumption: ***Uniform on the sphere.***

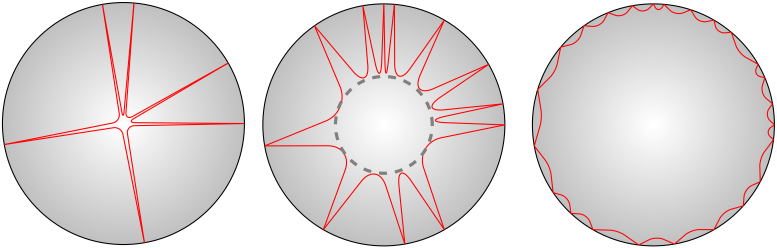
For $H \in \{H_{\min}, H_{\mathrm{typ}}, H_{\max}\}$, we have

- *(sub-exponential)* *If* $(\ln n)/d \to 0$, *then* $H \xrightarrow{\mathbb{P}} 0$.
  - *If* $n - d \ll d$, *then* $\sqrt{d}H \xrightarrow{\mathbb{P}} 0$.
  - *If* $n \sim \rho d$, *then* $\mathbb{P}(r_7(\rho) \le \sqrt{d}H \le r_8(\rho)) \to 1$.
  - *If* $\ln n \ll d \ll n$, *then* $\sqrt{\frac{d}{\ln(n/d)}}H \xrightarrow{\mathbb{P}} r_4$.
- *(exponential)* *If* $(\ln n)/d \to \alpha$, *then* $H \xrightarrow{\mathbb{P}} \sqrt{1 - e^{-\alpha}}$.
- *(super-exponential)* *If* $(\ln n)/d \to \infty$, *then* $H \xrightarrow{\mathbb{P}} 1$.
  - *If* $\ln n \gg d$, *then* $-\frac{d-1}{\ln n}\ln\left(1 - H^2\right) \xrightarrow{\mathbb{P}} r_5$.

Three main regimes

Intuitive representations

Distribution	**Characteristic**
• Uniform on the sphere,	• Height of the facets,



(a) **sub-exponential regime**
$(\ln n)/d \to 0$.
Facets' heights $\simeq O(1/\sqrt{d})$.

(b) **exponential regime**
$(\ln n)/d \to \alpha$.
Facets' heights $\simeq \sqrt{1-e^{-\alpha}}$.

(c) **super-exponential regime**
$(\ln n)/d \to \infty$.
Facets' heights $\simeq 1$.

Distance between $P_{n,d}$ and B^d

Hausdorff distance Vs Volume ratio.

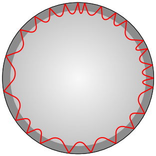
Distribution assumption: **Uniform on the sphere**

Hausdorff distance:

$$\mathrm{d_H}(P_{n,d}, B^d) = 1 - H_{\min}$$

Corollary

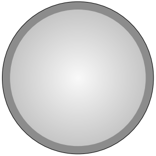
$\mathrm{d_H}(P_{n,d}, B^d) \to 0 \Leftrightarrow (\ln n)/d \to \infty.$



Volume ratio:

$$\frac{\mathbb{E}\, V_d(P_{n,d})}{V_d(B^d)}$$

⚠ The volume of the ball B^d is concentrated in a thin shell of width of order $\frac{1}{d}$.



Threshold

Theorem: Volume Threshold **2007**
Pivovarov

Distribution assumption: **Uniform on the sphere/ball.**
Fix $\varepsilon \in (0,1)$.
Then
$$\lim_{d\to\infty} \frac{\mathbb{E}\, V_d(P_{n,d})}{V_d(B^d)} = \begin{cases} 0 & \text{if } n \le d^{(1-\varepsilon)(\frac{d}{2})}, \\ 1 & \text{if } n \ge d^{(1+\varepsilon)(\frac{d}{2})}. \end{cases}$$

Theorem: Volume Threshold (Extension of Pivovarov's result) **2018**
B., Chasapis, Grote, Temesvari, Turchi

Distribution assumption: **Beta distribution**: density $\propto \mathbb{1}(x \in B^d)(1-\|x\|^2)^{\beta}$.
Fix $\varepsilon \in (0,1)$ and let $\beta = \beta(d) > -1$.
Then
$$\lim_{d\to\infty} \frac{\mathbb{E}\, V_d(P_{n,d})}{V_d(B^d)} = \begin{cases} 0 & \text{if } n \le d^{(1-\varepsilon)(\beta+\frac{d}{2})}, \\ 1 & \text{if } n \ge d^{(1+\varepsilon)(\beta+\frac{d}{2})}. \end{cases}$$

In the same paper we show similar results for **intrinsic volumes** and for **Gaussian distribution**.

Phase Transition

Theorem: Phase transition for random polytopes in the unit sphere 2019+ B., Kabluchko, Turchi

Distribution assumption: **Beta distribution**: density $\propto \mathbb{1}(x \in B^d)(1 - \|x\|^2)^\beta$.

Let $\beta = \beta(d) > -1$. If $x \in \mathbb{R}$ is fixed and $n = \left(\frac{d}{2x + o(1)}\right)^{\frac{d}{2}+\beta}$, then

$$\lim_{d\to\infty} \frac{\mathbb{E}\, V_d(P_{n,d})}{V_d(B^d)} = e^{-x}$$

In the same paper we show similar results for:

- **intrinsic volumes**.
- asymptotic for the **number of vertices** for the uniform distribution on the ball.

RUHR UNIVERSITÄT BOCHUM RUB

Threshold phenomena for high-dimensional random polytopes
G.B., G. Chasapis, J. Grote, D. Temesvari and N. Turchi
Communications in Comtemporary Mathematics, vol 21, no 5 (2019)

Phase transition for the volume of high-dimensional random polytopes
G.B., Z. Kabluchko and N. Turchi
arXiv:1911.12696

Facets of spherical random polytopes
G.B. and E. O'Reilly
arXiv:1908.04033

Thank you!