High Dimensional Random Polytopes

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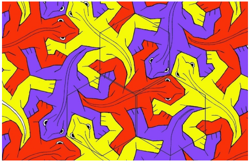
Interview TT position Statistics (220081) / Stochastics (220079)

Outline

1. Research Overview: Stochastic Geometry (~15 min)
2. Recent Research Topic: High Dimensional Random Polytopes (~15 min)
3. Questions (~15 min)

Research Overview : Stochastic **geometry**

Polytopes
Convex hull of a finite number of points.

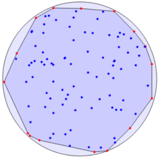


Tessellations
Covering of the space by non intersecting sets.

Networks
Collection of vertices and edges.

Research Overview : **Stochastic** geometry

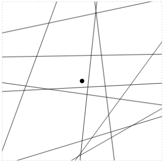
Three examples and applications



Polytopes

Random convex hull.

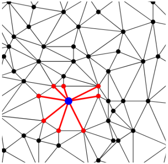
→Approximation of convex set



Tessellations

Random hyperplane tessellation.

→Compress sensing



Networks

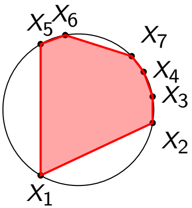
Random Delaunay graph.

→Delaunay tessellation field estimator [Rien van de Weijgaert]

Research Overview: Stochastic Geometry

Toy example in dimension 2

Let $X_1, \ldots, X_n \sim \mathrm{Unif}(S^1)$ i.i.d.
Let $P_n = \mathrm{conv}(X_1, \ldots, X_n)$.
Let $V_1(P_n)$ be its perimeter.



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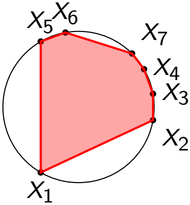

$$\mathbb{E} V_1(P_n) = \mathbb{E} \sum_{1 \le i < j \le n} \mathbf{1}([X_i, X_j] \text{ is an edge of } P_n) V_1([X_i, X_j])$$

$$= \binom{n}{2} \mathbb{E}[\mathbf{1}([X_1, X_2] \text{ is an edge of } P_n) V_1([X_1, X_2])]$$

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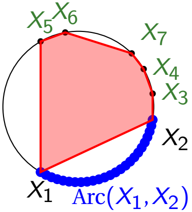

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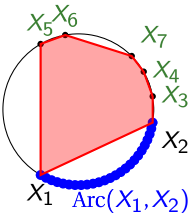

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$$= \binom{n}{2} 2 \int_0^{2\pi} \mathbb{P}(X_3 \notin \mathrm{Arc}(X_1, X_2) \mid \mathrm{Length}(\mathrm{Arc}(X_1, X_2)) = u)^{n-2} 2 \sin\left(\frac{u}{2}\right) \frac{\mathrm{d}u}{2\pi}$$

$$= 2n(n-1) \int_0^1 v^{n-2} \sin(\pi v) \, \mathrm{d}v$$

Research Overview: Stochastic Geometry

Toy example in dimension 2

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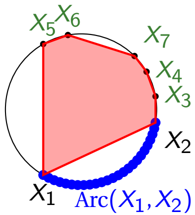

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\begin{aligned}
\mathbb{E}V_1(P_n) &= \mathbb{E}\sum_{1\le i<j\le n} \mathbf{1}([X_i,X_j] \text{ is an edge of } P_n)V_1([X_i,X_j]) \\
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&= 2n(n-1)\int_0^1 v^{n-2}\sin(\pi v)\,\mathrm{d}v \\
&= 2n(n-1)\left(\frac{\pi}{n(n-1)} - O\left(\frac{1}{n^4}\right)\right) = 2\pi - O\left(\frac{1}{n^2}\right)
\end{aligned}
$$

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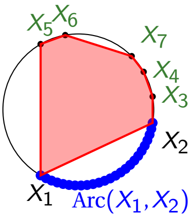

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$$= 2n(n-1)\left(\frac{\pi}{n(n-1)} - O\left(\frac{1}{n^4}\right)\right) = 2\pi - O\left(\frac{1}{n^2}\right) \overset{!!!}{=} V_1(\mathrm{Optimal}_n).$$

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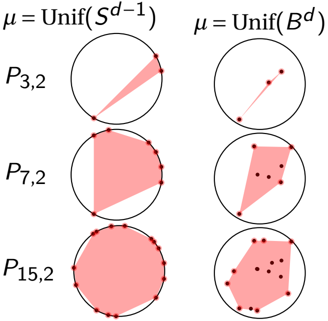
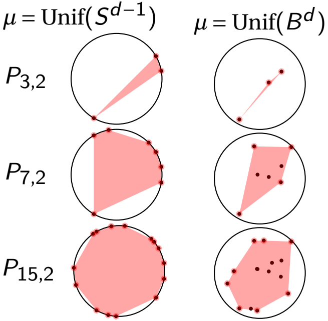
Model

- $X_1, X_2, \ldots \in \mathbb{R}^d$ i.i.d. random vectors distributed according to a probability measure $\mu = \mu_d$
- $P_{n,d} = \mathrm{conv}(X_1, \ldots, X_n)$

What does a **high dimensional** random polytope look like ?



Model



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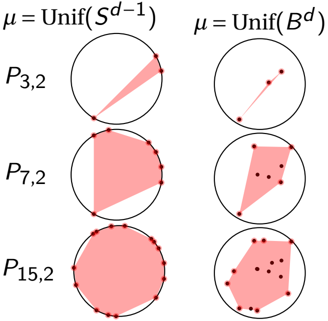
What does a **high dimensional** random polytope look like ?

Distribution

- Uniform on the sphere,
- Uniform in the ball,
- Gaussian,
- Beta distributions,
- ...

Model



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What does a **high dimensional** random polytope look like ?

Distribution	**Regime**	
• Uniform on the sphere,	• $n \simeq \alpha d$	linear,
• Uniform in the ball,	• $n \simeq d^\alpha$	polynomial,
• Gaussian,	• $n \simeq \alpha^d$	exponential,
• Beta distributions,	• $n \simeq d^{\alpha d}$	super exponential,
• ...	• ...	

Model

- $X_1, X_2, \ldots \in \mathbb{R}^d$ i.i.d. random vectors distributed according to a probability measure $\mu = \mu_d$
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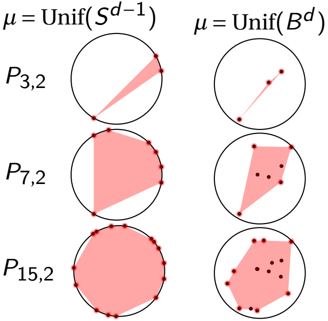
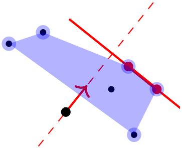


What does a **high dimensional** random polytope look like ?

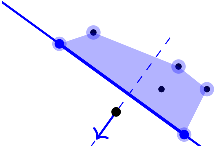
Distribution	**Regime**		**Characteristic**
Uniform on the sphere,	$n \simeq \alpha d$	linear,	Number of facets,
Uniform in the ball,	$n \simeq d^{\alpha}$	polynomial,	Height of the facets,
Gaussian,	$n \simeq \alpha^{d}$	exponential,	Volume,
Beta distributions,	$n \simeq d^{\alpha d}$	super exponential,	$\mathbf{1}(0 \in P_{n,d})$,
...	...		...

Height of a facet

Let F be a facet of $P_{n,d}$.
Set $H = \mathrm{affineHull}(F)$ its supporting hyperplane
Let $u \in S^{d-1}$ and $h \in \mathbb{R}$ such that $H = \{x \in \mathbb{R}^d : \langle x, u \rangle = h\}$ and $P_{n,d} \subset \{x \in \mathbb{R}^d : \langle x, u \rangle \leq h\}$. We call h the height of the facet F.



(a) A facet with positive height.



(b) A facet with negative height.

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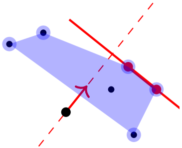

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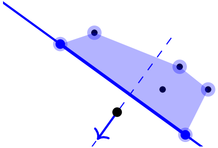

(b) A facet with negative height.

The **typical height** H_{typ} is a random variable defined by

$$\mathbb{P}(H_{\text{typ}} \in \cdot) = \mathbb{P}([X_1, \dots, X_d] \text{ has height } \in \cdot \mid [X_1, \dots, X_d] \text{ is a facet of } P_{n,d}).$$

Typical height

Distribution assumption: **uniform on the sphere**.
Let $[h_1, h_2] \subset [-1, 1]$.

$$\mathbb{P}(H_{\text{typ}} \in [h_1, h_2]) = \frac{\mathbb{P}([X_1, \ldots, X_d] \text{ is facet \& its height is } \in [h_1, h_2])}{\mathbb{P}([X_1, \ldots, X_d] \text{ is a facet of } P_{n,d})}$$

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$$= \frac{I_{[h_1,h_2]}}{I_{[-1,1]}},$$

where

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$\vdots$ Toolbox from integral geometry

$$= \int_{h_1}^{h_2} c_1 (1-h)^{\frac{d^2-2d-1}{2}} \left(c_2 \int_{-1}^{h} (1-s^2)^{\frac{d-3}{2}} \, \mathrm{d}s \right)^{n-d} \mathrm{d}h.$$

Typical height

Theorem (B., O'Reilly, 2019+)

Distribution assumption: ***Uniform on the sphere.***

- *(sub-exponential)* If $(\ln n)/d \to 0$, then $H_{\text{typ}} \xrightarrow{\mathbb{P}} 0$.
- *(exponential)* If $(\ln n)/d \to \alpha$, then $H_{\text{typ}} \xrightarrow{\mathbb{P}} \sqrt{1 - e^{-\alpha}}$.
- *(super-exponential)* If $(\ln n)/d \to \infty$, then $H_{\text{typ}} \xrightarrow{\mathbb{P}} 1$.

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- *(sub-exponential)* If $(\ln n)/d \to 0$, then $H_{\mathrm{typ}} \xrightarrow{\mathbb{P}} 0$.
    - If $n-d = O(\sqrt{d})$, then $dH_{\mathrm{typ}} - \frac{n-d}{\sqrt{d}} r_1 \xrightarrow{d} Z \sim \mathcal{N}(0,1)$.
    - If $\sqrt{d} \ll n-d \ll d$, then $\frac{d^{3/2}}{n-d} H_{\mathrm{typ}} \xrightarrow{\mathbb{P}} r_2$.
    - If $n \sim \rho d$, then $\sqrt{d} H_{\mathrm{typ}} \xrightarrow{\mathbb{P}} r_3(\rho)$.
    - If $\ln n \ll d \ll n$, then $\sqrt{\frac{d}{\ln(n/d)}} H_{\mathrm{typ}} \xrightarrow{\mathbb{P}} r_4$.
- *(exponential)* If $(\ln n)/d \to \alpha$, then $H_{\mathrm{typ}} \xrightarrow{\mathbb{P}} \sqrt{1-e^{-\alpha}}$.
- *(super-exponential)* If $(\ln n)/d \to \infty$, then $H_{\mathrm{typ}} \xrightarrow{\mathbb{P}} 1$.
    - If $\ln n \gg d$, then $-\frac{d-1}{\ln n} \ln\left(1 - H_{\mathrm{typ}}^2\right) \xrightarrow{\mathbb{P}} r_5$.
    - If $\ln n \gg d \ln d$, then $r_6 \frac{n}{d} (1 - H_{\mathrm{typ}}^2)^{\frac{d}{2}} - \sqrt{d} \xrightarrow{\mathrm{tv}} Z \sim \mathcal{N}(0,1)$.
    - If "d is fixed", then $nc_d (1 - H_{\mathrm{typ}}^2)^{\frac{d-1}{2}} \xrightarrow{\mathrm{tv}} \Gamma_{d-1}$.

where $r_1 = \sqrt{\frac{2}{\pi}}$, $r_2 = \frac{2}{\pi}$, $r_3 = \mathrm{argmax}(r \mapsto (\rho-1)\ln\Phi(r) - r^2/2)$, $r_4 = \sqrt{2}$, $r_5 = 2$, $r_6 = \frac{1}{2\sqrt{\pi}}$.

Interval containing the heights of all the facets

$$H_{\min} := \min\{\mathrm{height}(F) \mid F \in \mathrm{Facets}(P_{n,d})\}$$
$$H_{\max} := \max\{\mathrm{height}(F) \mid F \in \mathrm{Facets}(P_{n,d})\}$$

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These random variables have a similar asymptotic as the typical height:

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For $H \in \{H_{\min}, H_{\text{typ}}, H_{\max}\}$, we have

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  - *If $n - d \ll d$,* *then $\sqrt{d}H \xrightarrow{\mathbb{P}} 0$.*
  - *If $n \sim \rho d$,* *then $\mathbb{P}(r_7(\rho) \le \sqrt{d}H \le r_8(\rho)) \to 1$.*
  - *If $\ln n \ll d \ll n$,* *then $\sqrt{\frac{d}{\ln(n/d)}}H \xrightarrow{\mathbb{P}} r_4$.*
- *(exponential)* *If $(\ln n)/d \to \alpha$, then $H \xrightarrow{\mathbb{P}} \sqrt{1-e^{-\alpha}}$.*
- *(super-exponential)* *If $(\ln n)/d \to \infty$, then $H \xrightarrow{\mathbb{P}} 1$.*
  - *If $\ln n \gg d$,* *then $-\frac{d-1}{\ln n}\ln\left(1-H^2\right) \xrightarrow{\mathbb{P}} r_5$.*

Three main regimes

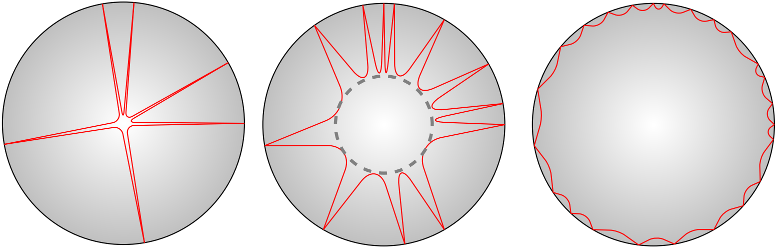
Intuitive representations

Distribution

- Uniform on the sphere,

Characteristic

- Height of the facets,



(a) **sub-exponential regime**
$(\ln n)/d \to 0$.
Facets' heights $\simeq O(1/\sqrt{d})$.

(b) **exponential regime**
$(\ln n)/d \to \alpha$.
Facets' heights $\simeq \sqrt{1-e^{-\alpha}}$.

(c) **super-exponential regime**
$(\ln n)/d \to \infty$.
Facets' heights $\simeq 1$.

Distance between $P_{n,d}$ and B^d

Hausdorff distance Vs Volume ratio.

Distribution assumption: **Uniform on the sphere**

Hausdorff distance:

$$\mathrm{d_H}(P_{n,d}, B^d) = 1 - H_{\min}$$

Volume ratio:

$$\frac{\mathbb{E}\, V_d(P_{n,d})}{V_d(B^d)}$$

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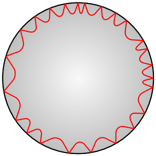
Corollary

$\mathrm{d_H}(P_{n,d}, B^d) \to 0 \Leftrightarrow (\ln n)/d \to \infty.$



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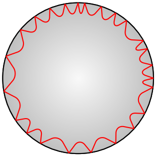
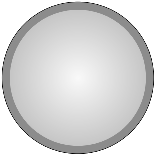


Volume ratio:

$$\frac{\mathbb{E}\, V_d(P_{n,d})}{V_d(B^d)}$$

⚠ The volume of the ball B^d is concentrated in a thin shell of width of order $\frac{1}{d}$.



Threshold

Theorem: Volume Threshold 2007
Pivovarov

Distribution assumption: **Uniform on the sphere/ball.**
Fix $\varepsilon \in (0,1)$.

Then
$$\lim_{d\to\infty} \frac{\mathbb{E}\, V_d(P_{n,d})}{V_d(B^d)} = \begin{cases} 0 & \text{if } n \le d^{(1-\varepsilon)(\frac{d}{2})}, \\ 1 & \text{if } n \ge d^{(1+\varepsilon)(\frac{d}{2})}. \end{cases}$$

Threshold

Theorem: Volume Threshold **2007**
Pivovarov

Distribution assumption: **Uniform on the sphere/ball.**
Fix $\varepsilon \in (0,1)$.
Then

$$\lim_{d\to\infty} \frac{\mathbb{E}\, V_d(P_{n,d})}{V_d(B^d)} = \begin{cases} 0 & \text{if } n \leq d^{(1-\varepsilon)(\frac{d}{2})}, \\ 1 & \text{if } n \geq d^{(1+\varepsilon)(\frac{d}{2})}. \end{cases}$$

Beta distribution ...
$$\begin{cases} \text{density } \propto \mathbb{1}(x \in B^d)(1-\|x\|^2)^\beta & \text{if } \beta > -1; \\ \text{uniform distribution on the sphere} & \text{if } \beta = -1. \end{cases}$$

Theorem: Volume Threshold (Extension of Pivovarov's result) **2018**
B., Chasapis, Grote, Temesvari, Turchi

Distribution assumption: **Beta distribution**. Fix $\varepsilon \in (0,1)$ and let $\beta = \beta(d) \geq -1$.
Then

$$\lim_{d\to\infty} \frac{\mathbb{E}\, V_d(P_{n,d})}{V_d(B^d)} = \begin{cases} 0 & \text{if } n \leq d^{(1-\varepsilon)(\beta+\frac{d}{2})}, \\ 1 & \text{if } n \geq d^{(1+\varepsilon)(\beta+\frac{d}{2})}. \end{cases}$$

⊕ <u>similar results</u> for **intrinsic volumes** and for **Gaussian distribution**.

Phase Transition

Beta distribution ... $\begin{cases} \text{density } \propto \mathbb{1}(x \in B^d)(1-\|x\|^2)^\beta & \text{if } \beta > -1; \\ \text{uniform distribution on the sphere} & \text{if } \beta = -1. \end{cases}$

Theorem: Phase transition for random polytopes in the unit sphere 2019+ B., Kabluchko, Turchi

Distribution assumption: **Beta distribution**. Let $\beta = \beta(d) \geq -1$.

If $x \in \mathbb{R}$ is fixed and $n = \left(\frac{d}{2x+o(1)}\right)^{\frac{d}{2}+\beta}$, then

$$\lim_{d\to\infty} \frac{\mathbb{E}\, V_d(P_{n,d})}{V_d(B^d)} = e^{-x}$$

In the same paper we show similar results for:

- **intrinsic volumes**.
- asymptotic for the **number of vertices** for the uniform distribution on the ball.

Threshold phenomena for high-dimensional random polytopes
G.B., G. Chasapis, J. Grote, D. Temesvari and N. Turchi
Communications in Comtemporary Mathematics, vol 21, no 5 (2019)

Phase transition for the volume of high-dimensional random polytopes
G.B., Z. Kabluchko and N. Turchi
arXiv:1911.12696

Facets of spherical random polytopes
G.B. and E. O'Reilly
arXiv:1908.04033

Thank you!