Concentration and Cumulants for Stabilizing Functionals of Point Processes

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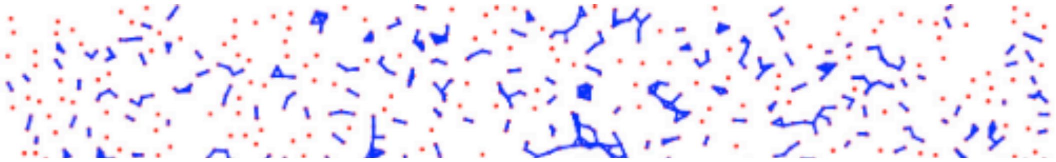
Stabilizing Functionals

Example 1: Random geometric graph statistics

$\mathscr{X}$: point process in $\mathbb{R}^d$.

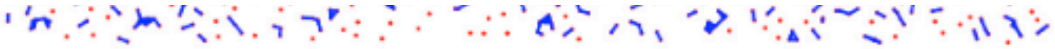
The **random geometric graph** of radius ρ>0 associated with $\mathscr{X}$ is formed by connecting by an edge any two distinct points $x, y \in \mathscr{X}$ if $\|x - y\| < \rho$.



Edge counting : $$\frac{1}{2} \sum_{x,y \in \mathscr{X}^2_{\neq}} \mathbf{1}(\|x-y\| < \rho) = \sum_{x \in \mathscr{X}} \frac{1}{2} \sum_{y \in \mathscr{X} \setminus \{x\}} \mathbf{1}(\|x-y\| < \rho)$$

Total edge length : $$\sum_{x \in \mathscr{X}} \frac{1}{2} \sum_{y \in \mathscr{X} \setminus \{x\}} \mathbf{1}(\|x-y\| < \rho) \|x-y\|$$

Length power : $$\sum_{x \in \mathscr{X}} \frac{1}{2} \sum_{y \in \mathscr{X} \setminus \{x\}} \mathbf{1}(\|x-y\| < \rho) \|x-y\|^{\alpha}, \quad \alpha \in \mathbb{R}$$



Picture from [Bachmann, Peccati, 2016]

Stabilizing Functionals

Example 2 : Voronoi set approximation

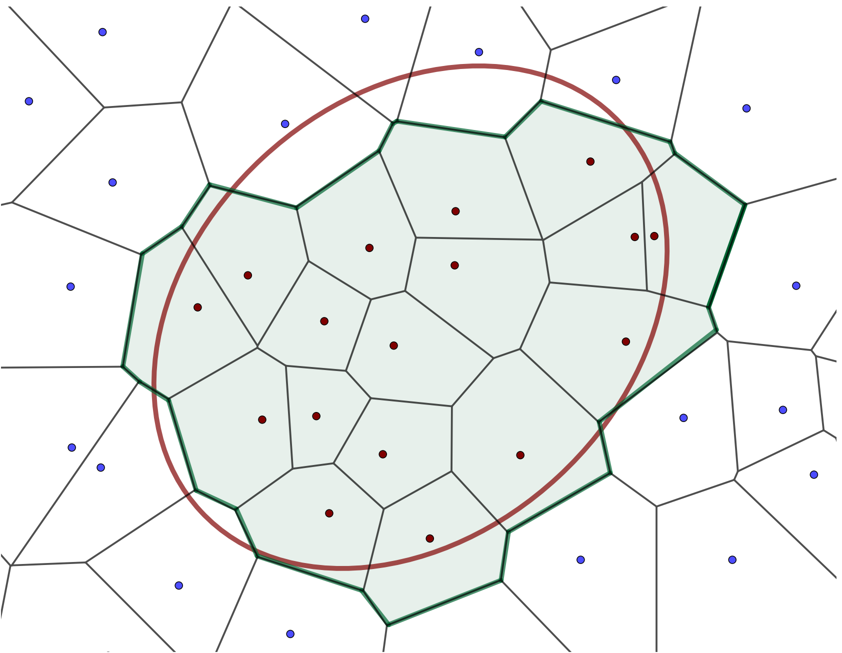
$K \subset \mathbb{R}^d$: set with nice properties (e.g. convex or smooth boundary)

$\mathcal{X}$: point process in $\mathbb{R}^d$.

The Voronoi cell with centre $x \in \mathcal{X}$ is the set
$C(x, \mathcal{X}) = \{z \in \mathbb{R}^d : \|z - x\| \leq \|z - y\|, \forall y \in \mathcal{X}\}$

The voronoi set approximation of K is $\cup_{x \in \mathcal{X}} C(x, \mathcal{X})$

Its volume is $$\mathrm{Vol}(K) + \sum_{x \in \mathcal{X}} \mathbf{1}(C(x, \mathcal{X}) \cap \partial K \neq \varnothing \ , \ x \in K)\mathrm{Vol}(C(x, \mathcal{X}) \backslash K) - \sum_{x \in \mathcal{X}} \mathbf{1}(C(x, \mathcal{X}) \cap \partial K \neq \varnothing \ , \ x \notin K)\mathrm{Vol}(C(x, \mathcal{X}) \cap K)$$



Stabilizing Functionals

Example 3 : Random polytopes

$X_1, \ldots, X_n \in \mathbb{R}^d$: i.i.d. random points

$P_n = \mathrm{conv}(X_1, \ldots, X_n)$: convex hull of the random points

Number of vertices of P_n

$$f_0(P_n) = \sum_{i=1}^{n} \mathbf{1}(X_i \text{ is a vertex of } P_n)$$

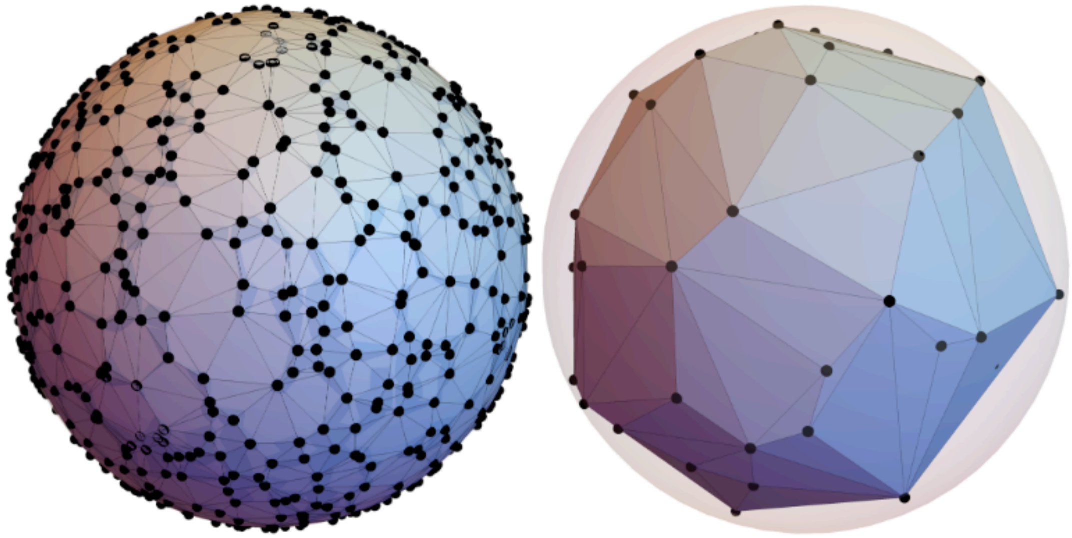


Figure 1.1: Convex hull of $n = 1000$ uniformly distributed points on the sphere (left figure, beta polytope with $\beta = -1$) and the ball (right figure, beta polytope with $\beta = 0$).

Illustration from [Kabluchko, Thäle, Zaporozhets '20]

Stabilizing Functionals

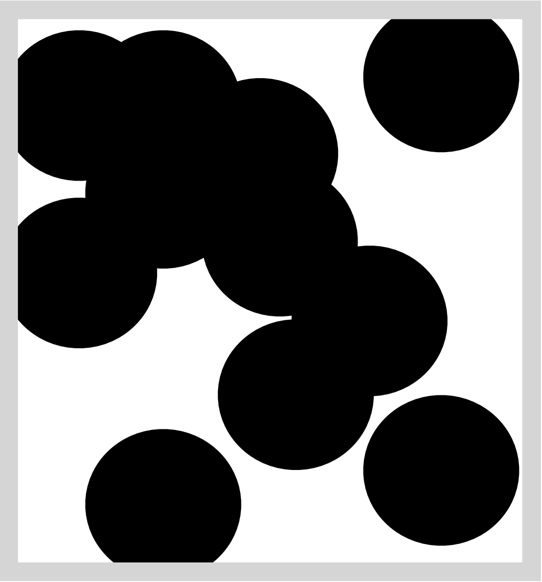
Example 4 : Boolean and cylinders process

$\mathscr{X}$: point process in $\mathbb{R}^d$.

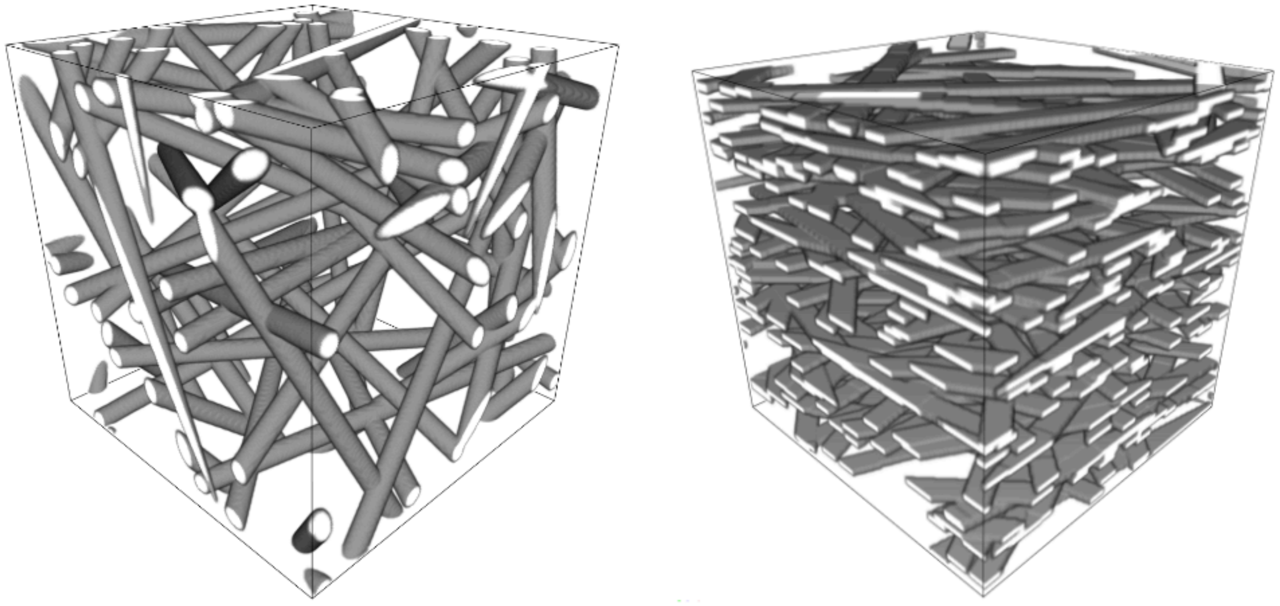
$\cup_{x\in\mathscr{X}} B(x,\rho)$: Boolean process, $\rho > 0$.

$\mathscr{Y}$: line process in $\mathbb{R}^d$.

$\cup_{y\in\mathscr{Y}} (y + B(0,\rho))$: Cylinder process, $\rho > 0$.



Boolean process

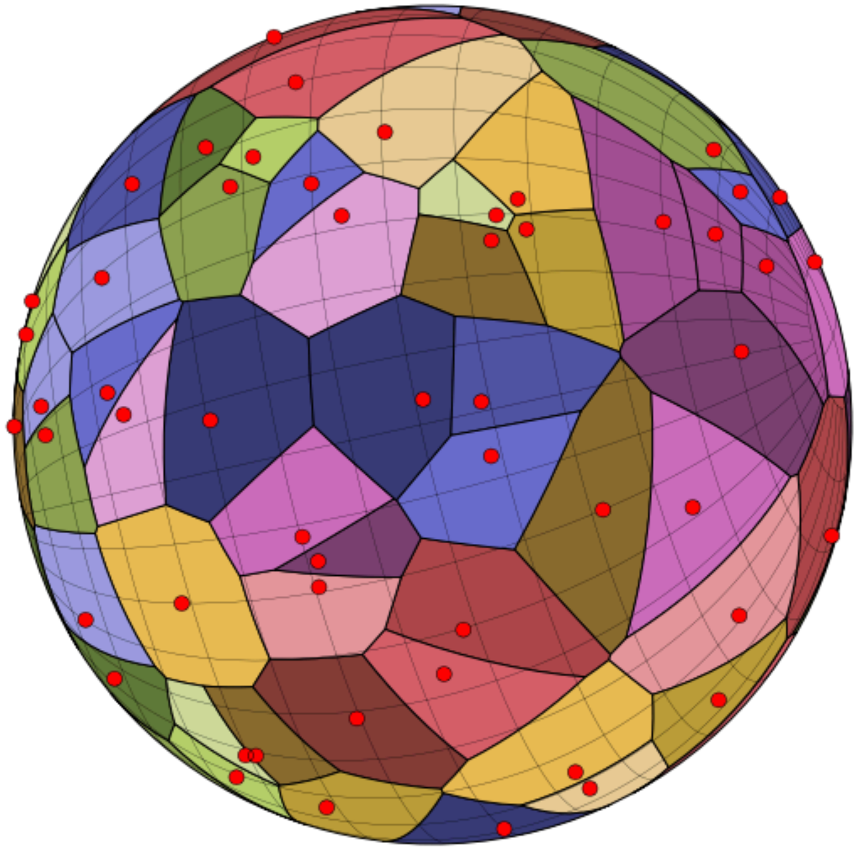


Simulations of Poisson cylinders processes
(credit: ClaudiaRedenbach, Kaiserslautern)

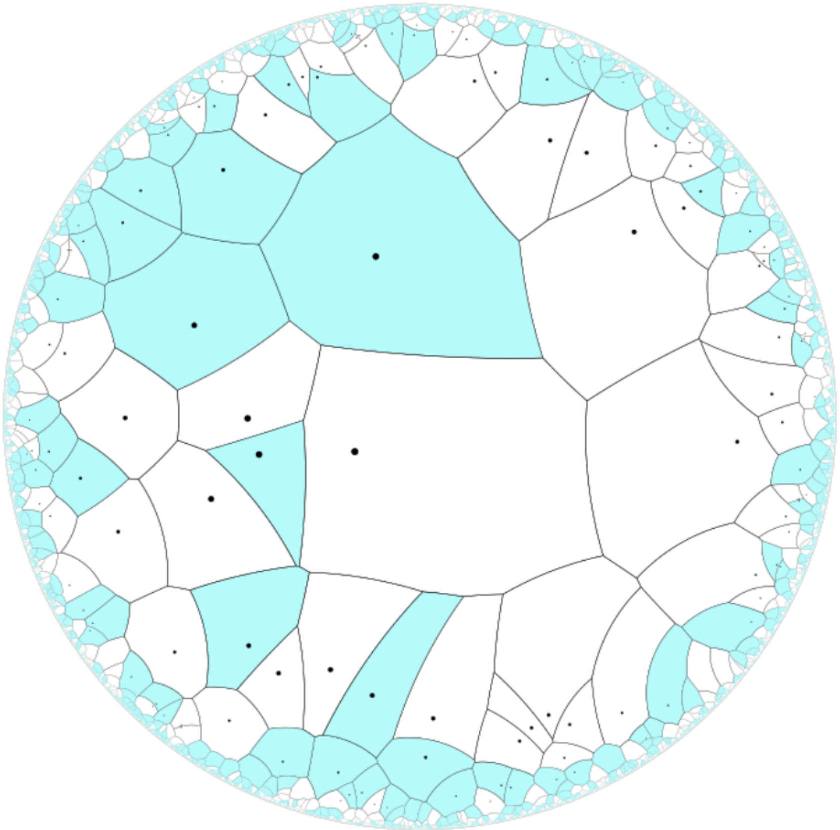
Stabilizing Functionals

Example 5 : In non euclidean geometry

Each of the previous models can also be considered in non euclidean setting.



Voronoi tessellation on the sphere
Created using the java applet
https://www.jasondavies.com/maps/voronoi/



Voronoi tessellation in the hyperbolic plane
Courtesy of Tobias Müller
http://www.math.rug.nl/~tobias/

Stabilizing Functional

General set up

$(\mathbb{X}, \mathscr{F})$: measurable space equipped with a metric d.

$\mathbb{Q}$: σ-finite measure on $\mathbb{X}$.

$\mathscr{X}_s$: point process of intensity measure $\mathbb{E}\mathscr{X}_s(\,\cdot\,) = s\mathbb{Q}(\,\cdot\,)$, $s > 0$.

$\xi(x, \mathscr{X}_s)$: a score function which depends only *locally* on the point configuration $\mathscr{X}_s$.

R : (random) radius of stabilisation.

- It must satisfy $\xi(x, \mathscr{X}_s) = \xi(x, \mathscr{X}_s \cap B_R(x))$ for all $x \in \mathscr{X}_s$.
- Typical assumptions : R is constant or has sub-exponential tails.

$h(\mathscr{X}_s) := \sum_{x \in \mathscr{X}_s} \xi(x, \mathscr{X}_s)$: Stabilizing functional.

Sometime the model is refined:

$\xi(x, \mathscr{X}_s, M)$: the score function depends also on the distance between x and a fixed submanifold $M \subset \mathbb{R}^d$.

$h(\mathscr{X}_s) := \sum_{x \in \mathscr{X}_s} \xi(x, \mathscr{X}_s, M)$: Surface-order stabilizing functional.

Concentration bounds

One of the goals of our project is to establish *concentration bounds for some stabilizing functional*, i.e. get non asymptotic bounds for the probabilities

- $\mathbb{P}(h(\mathscr{X}) - \mathbb{E}h(\mathscr{X}) \geq r)$, $r > 0,$
- $\mathbb{P}(h(\mathscr{X}) - \mathbb{E}h(\mathscr{X}) \leq -r)$, $r > 0,$
- $\mathbb{P}(h(\mathscr{X}) - \mathbb{M}h(\mathscr{X}) \geq r)$, $r > 0,$
- $\mathbb{P}(h(\mathscr{X}) - \mathbb{M}h(\mathscr{X}) \leq -r)$, $r > 0.$

Poisson analysis method 1

Entropy method : Herbst argument

Entropy of a real random variable $X \geq 0$: $\mathrm{Ent}(X) := \mathbb{E}\Psi(X) - \Psi(\mathbb{E}X)$ where $\Psi(x) = x \log x$.

$$G(u) := \log \mathbb{E}e^{u(X-\mathbb{E}X)}$$

$$\mathbb{P}(X \geq \mathbb{E}X + r) \leq \frac{\mathbb{E}e^{u(X-\mathbb{E}X)}}{e^{ur}} = \exp\left(-u\left(r - \frac{G(u)}{u}\right)\right)$$

$$\lim_{u \to 0^+} \frac{G(u)}{u} = 0$$

$$\frac{\partial}{\partial u}\left(\frac{G(u)}{u}\right) = \cdots = \frac{1}{u^2}\mathrm{Ent}\left(\frac{e^{uX}}{\mathbb{E}e^{uX}}\right),$$

$$\frac{G(u)}{u} = \int_0^u \frac{1}{v^2}\mathrm{Ent}\left(\frac{e^{vX}}{\mathbb{E}e^{vX}}\right)\mathrm{d}v$$

$$\mathbb{P}(X \geq \mathbb{E}X + r) \leq \exp\left(-u\left(r - \int_0^u \frac{1}{v^2}\mathrm{Ent}\left(\frac{e^{vX}}{\mathbb{E}e^{vX}}\right)\mathrm{d}v\right)\right)$$

Poisson analysis method 1

Entropy method

η : Poisson point process of intensity measure Λ

$F = F(\eta)$: Poisson functional

Entropy of F : $\mathrm{Ent}(F) := \mathbb{E}\Psi(F) - \Psi(\mathbb{E}F)$ where $\Psi(x) = x \log x$.

Difference operator : $D_x F := F(\eta + \delta_x) - F(\eta)$, $x \in \mathbb{X}$.

Modified log-Sobolev inequality for Poisson functionals [Wu ’00]:

$$\mathrm{Ent}(F) \leq \mathbb{E} \int [D_x\Psi(F) - \Psi'(F)D_xF] \; \Lambda(\mathrm{d}x)$$

Bounds on the entropy lead to concentration bound via the Herbst argument

This method have been exploited in [Bachmann, Peccati ’16] and [Bachmann, Reitzner ’18] in the context of stochastic geometry, especially random geometric graphs.

Poisson analysis method 1 bis

Φ-Entropy method

η : Poisson point process of intensity measure Λ

$F = F(\eta)$: Poisson functional

$\Phi : \mathbb{R}_+ \to \mathbb{R}$: convex + continuous + twice differentiable + (affine OR $\Phi'' > 0$ and $1/\Phi''$ concave)

Φ-Entropy of F : $\mathrm{Ent}_\Phi(F) := \mathbb{E}\Phi(F) - \Phi(\mathbb{E}F)$.

Difference operator : $D_xF := F(\eta + \delta_x) - F(\eta)$, $x \in \mathbb{X}$.

Modified Φ-Sobolev inequality for Poisson functionals [Chafaï '04]:

$$\mathrm{Ent}_\Phi(F) \leq \mathbb{E}\int [D_x\Phi(F) - \Phi'(F)D_xF]\ \Lambda(\mathrm{d}x)$$

Bounds on the Φ-entropy lead to concentration bound via the Herbst argument

Applied to Poisson cylinders and Poisson random polytopes in [Gusakova, Sambale, Thäle '20].

Poisson analysis method 2

[Gierieger, Last ´18]

η : Poisson process of distribution Π_Λ

η_t : t-thinning of η

$F = F(\eta)$: Poisson functional

$$s_F := \sup\{s \geq 0 : e^{sF} \in L^2(\mathbb{P}), D_{(\cdot)}e^{sF} \in L^2(\mathbb{P} \otimes \Lambda)\} \in [0,\infty],$$

$$V_F(s) := \int (e^{sD_xF} - 1) \int_0^1 \int D_x f(\eta_t + \mu)\, \Pi_{(1-t)\Lambda}(\mathrm{d}\mu)\, \mathrm{d}t\, \Lambda(\mathrm{d}x), \quad s \in [0,s_F)$$

If $F \in L^2(\mathbb{P})$ and $D_{(\cdot)}F \in L^2(\mathbb{P} \otimes \Lambda)$, and if almost surely $V_F(s) \leq v(s)$ for some measurable function $v : [0,s_F) \to \mathbb{R}$, one has that $\mathbb{P}(F - \mathbb{E}F \geq r) \leq \exp\Big(\inf_{s\in[0,s_F)} \Big(\int_0^s v(u)\, \mathrm{d}u - rs \Big)\Big)$, $r \geq 0$.

Applied to functionals of a Boolean model in [Gierieger, Last ´18]

Applied to functionals of a Poisson cylinder process in [Baci, Betken, Gusakova, Thäle, ´19]

Cumulant bounds

[Saulis, Statulevicius ´91]

$X, X_1, X_2, \dots$: real random variables

If $\mathbb{E}|X|^k < \infty$, its k-th cumulant is defined as $c^k[X] = (-\mathfrak{i})^k \frac{\mathrm{d}^k}{\mathrm{d}t^k} \log \mathbb{E}[\exp(\mathfrak{i}tX)]\Big|_{t=0}$ where $\mathfrak{i}$ is the imaginary unit.

If

- $\mathbb{E}[X_n] = 0$ and $\mathrm{Var}[X_n] = 1$ for all $n \in \mathbb{N}$,
- There exists a constant $\gamma \in [0,\infty)$, such that for all $k \in \{3,4,\dots\}$ and sufficiently large n, one has $|c^k[X_n]| \leq \frac{(k!)^{1+\gamma}}{(\Delta_n)^{k-2}}$,

Then

$$\mathbb{P}(|X_n| \geq y) \leq 2\exp\left(-\frac{1}{4}\min\left\{\frac{y^2}{2^{1+\gamma}}, (y\,\Delta_n)^{1/(1+\gamma)}\right\}\right).$$

Thank you !