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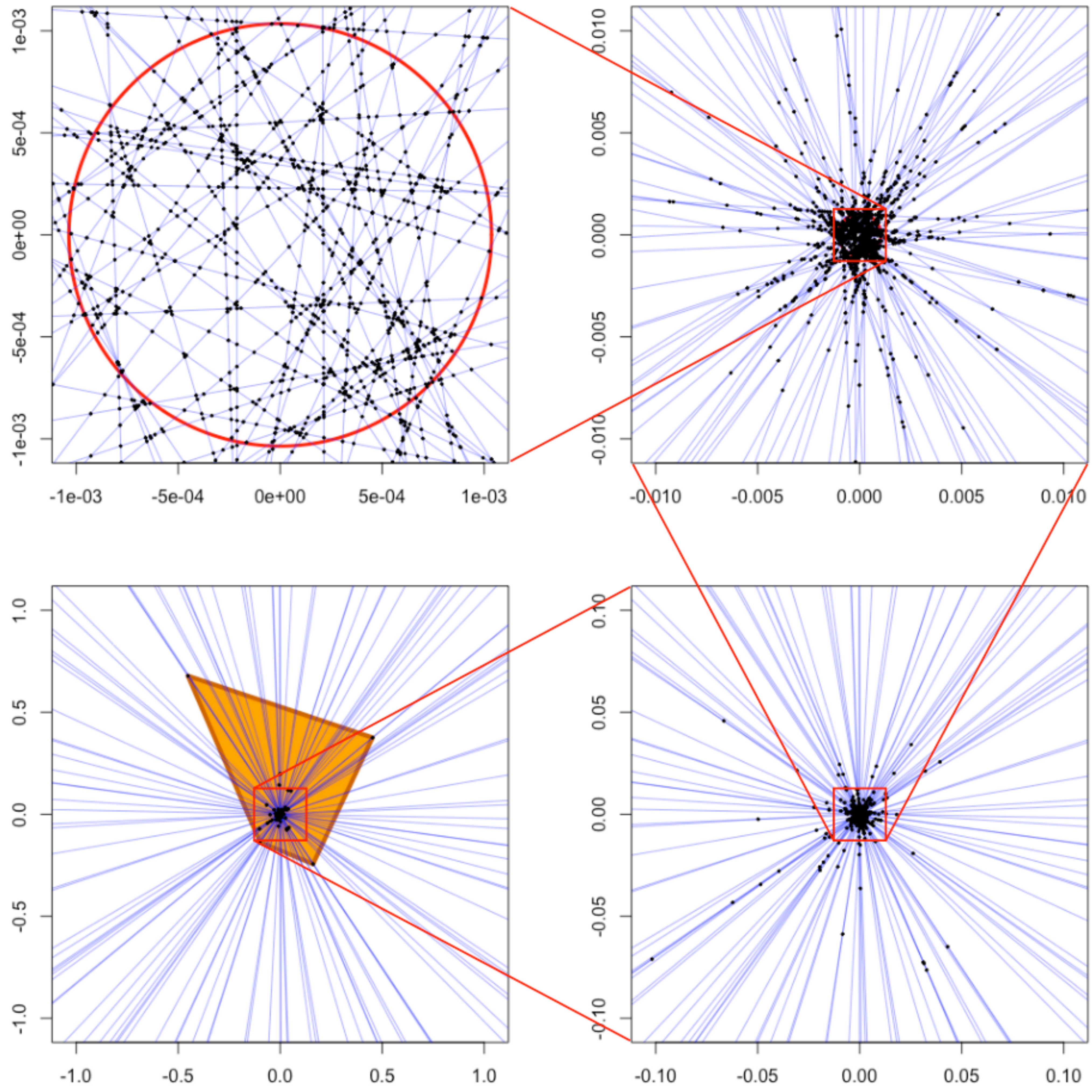
# WEAK CONVERGENCE OF THE INTERSECTION POINT PROCESS OF POISSON HYPERPLANES

Joint work with Anastas Baci and Christoph Thäle (arXiv:2007.06398)
Accepté pour publication dans les Annales de l'Institut Henri Poincaré (B) Probabilités et Statistique

$t = 30\,000$
$R = t^{-2/3} \simeq 0.001$



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   - $U_1, \ldots, U_N \in S^{d-1}$ i.i.d. unit vectors uniformly distributed on the unit sphere,
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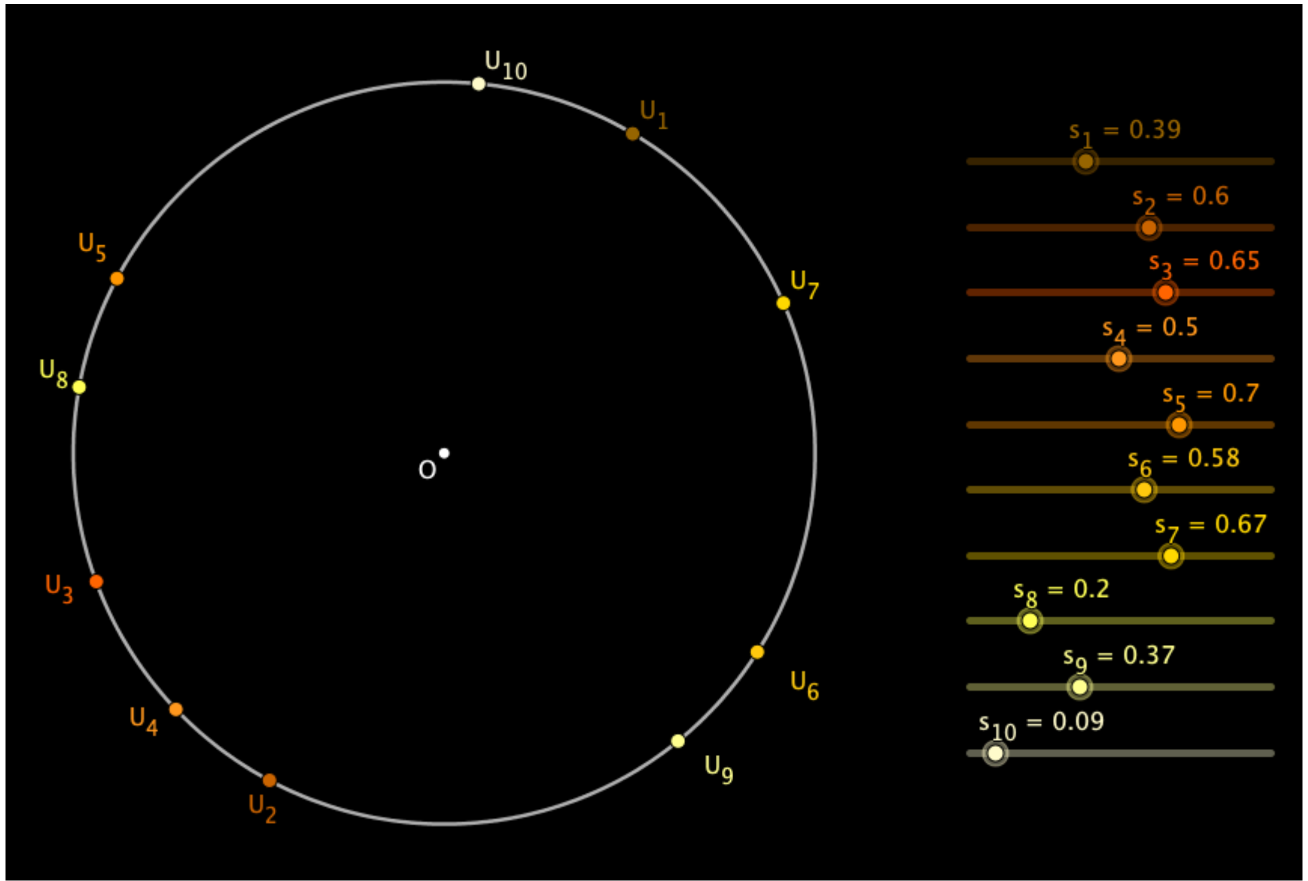

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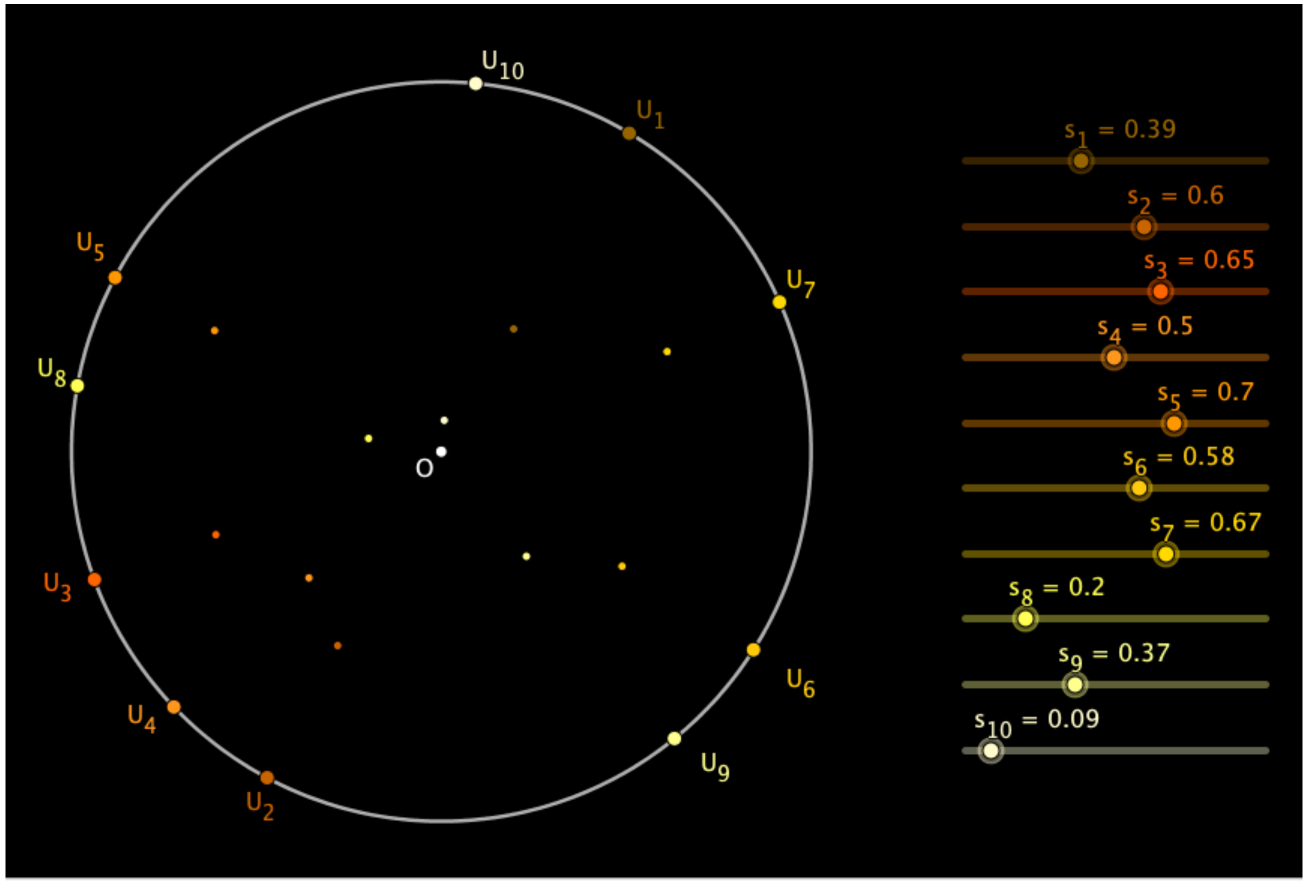

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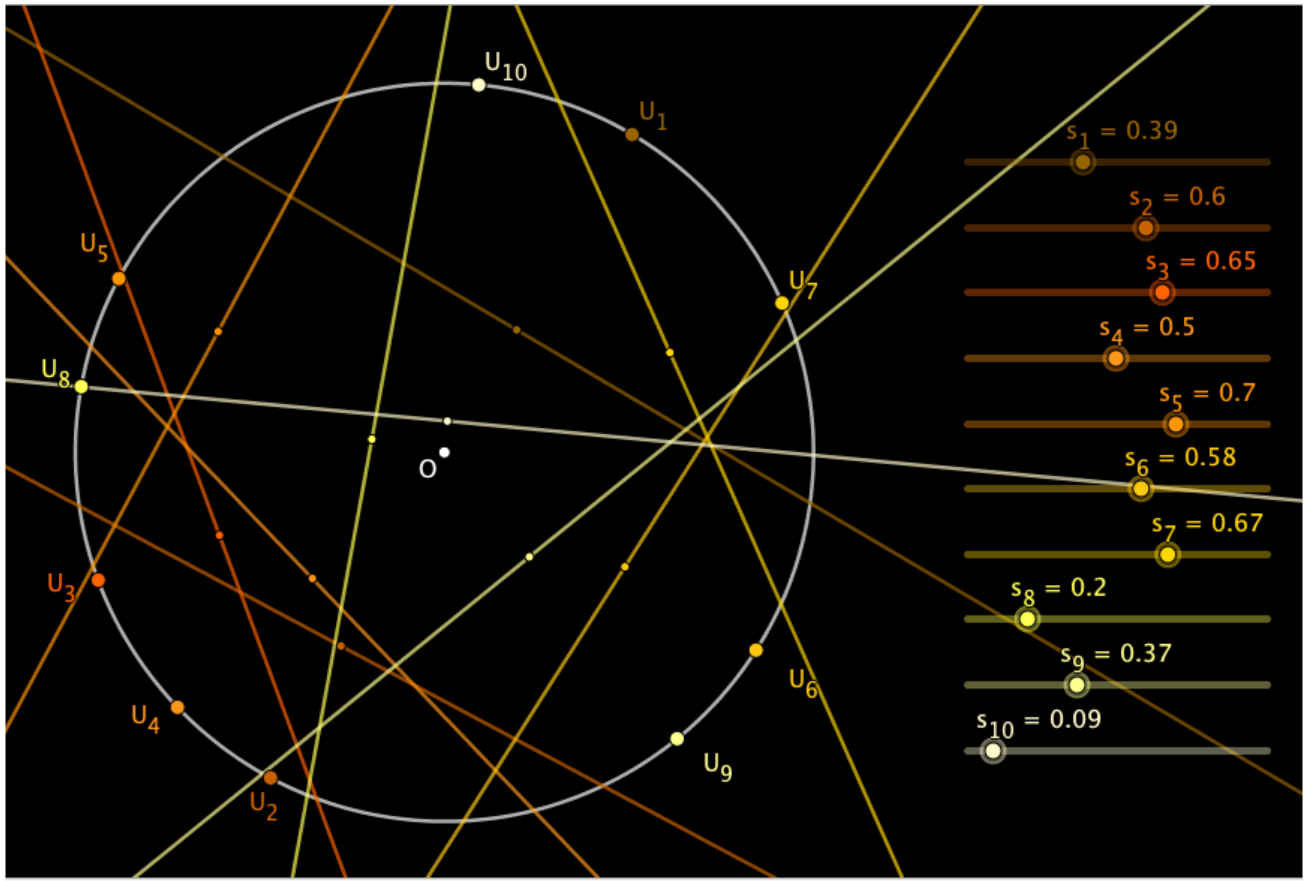

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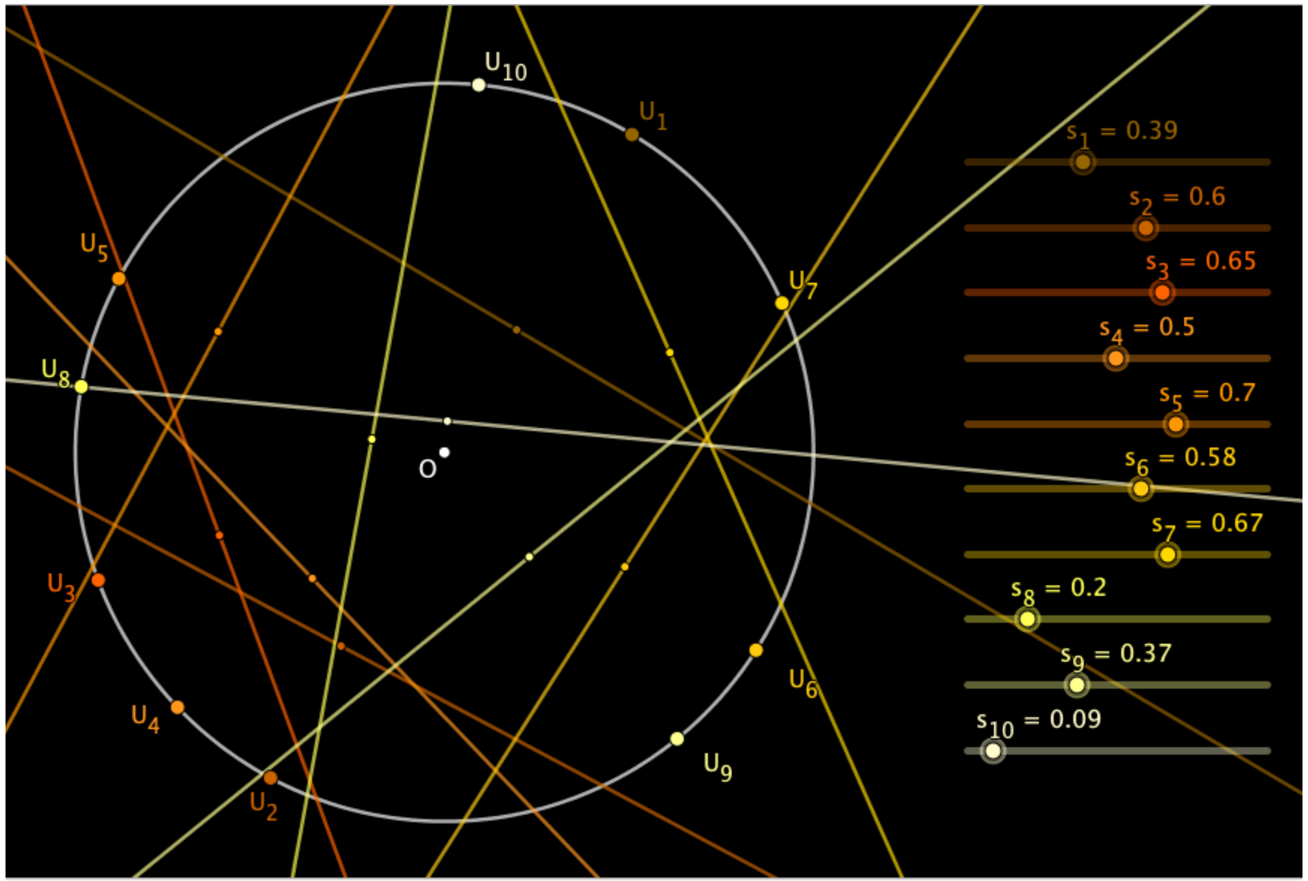

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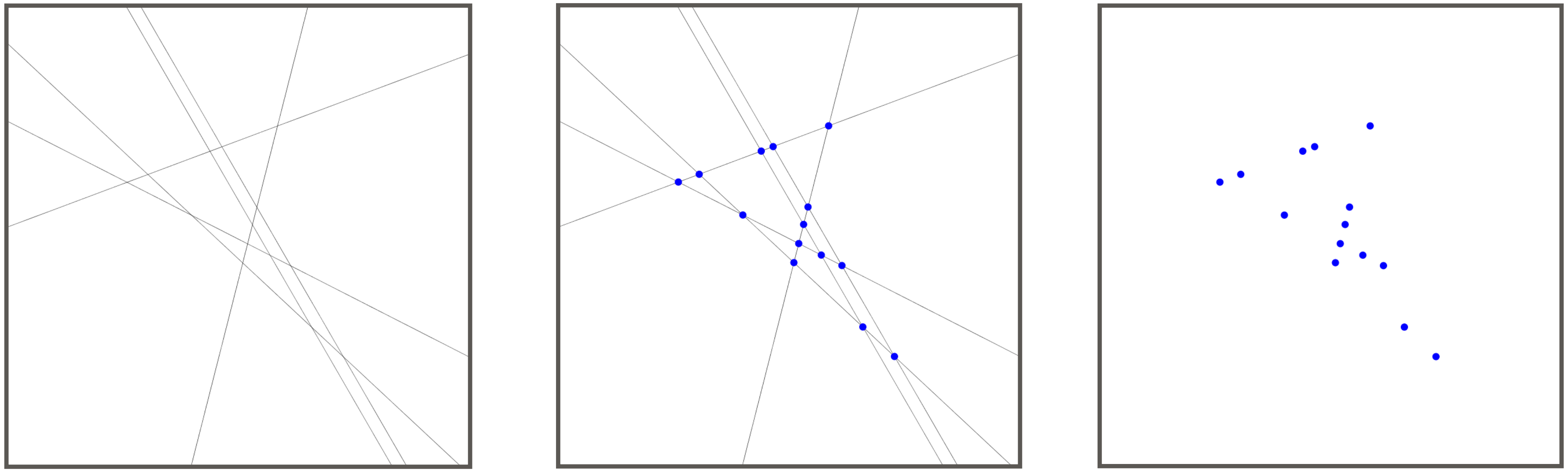

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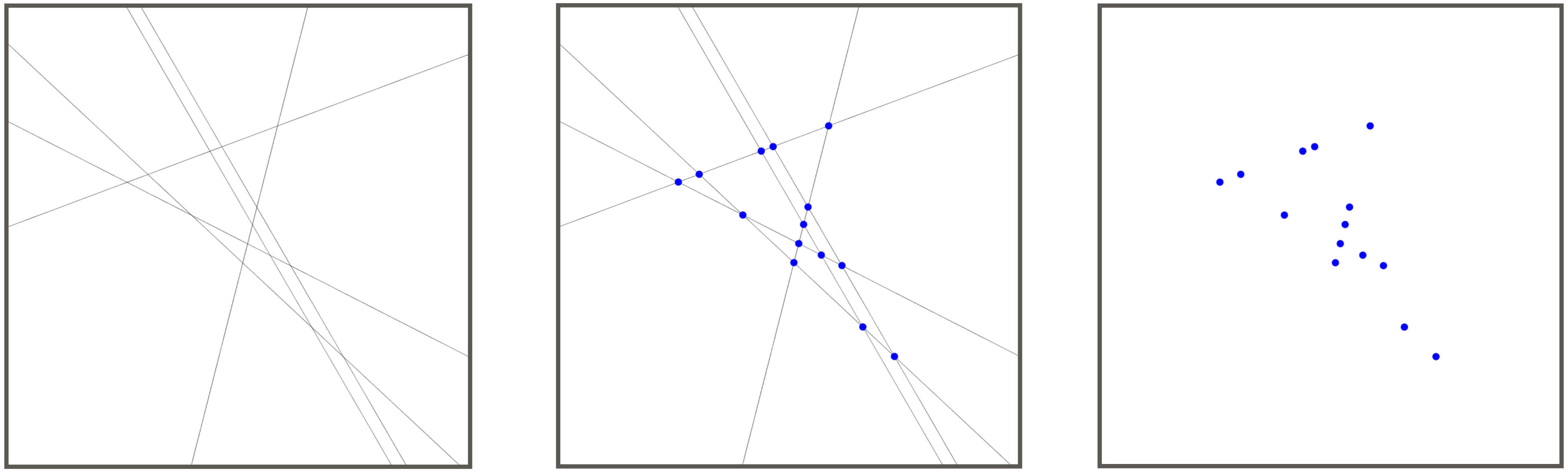

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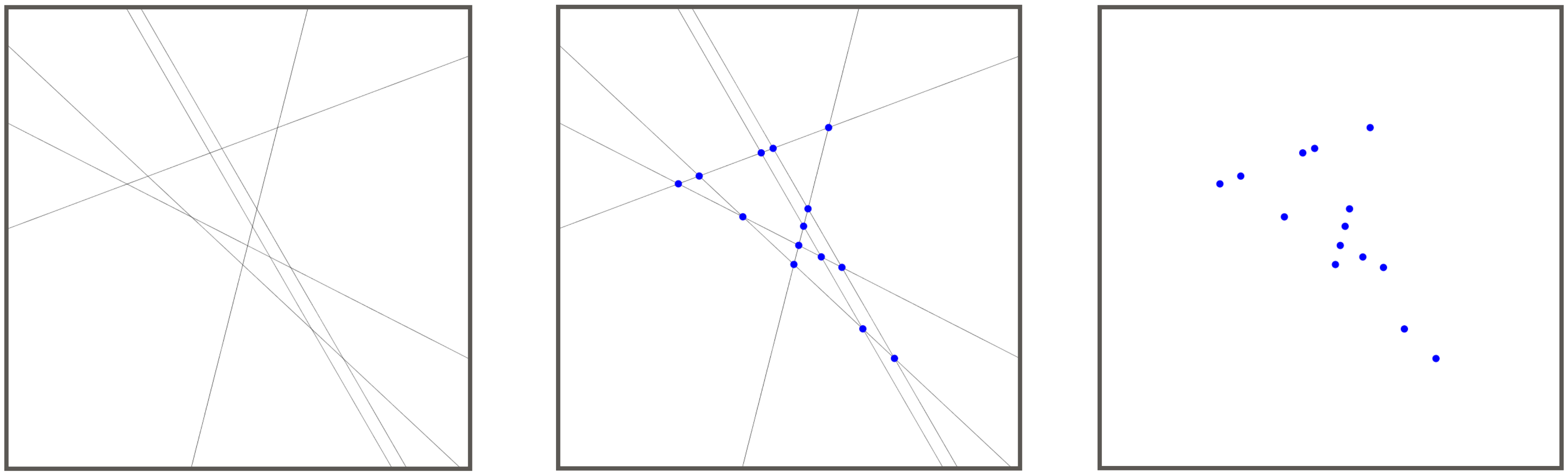

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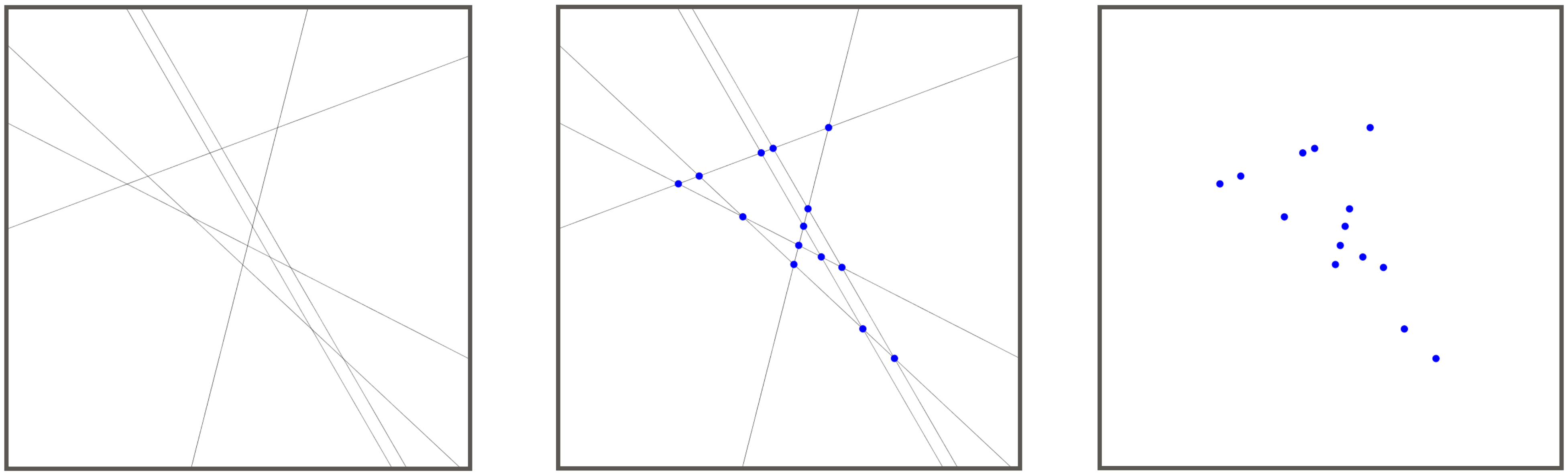

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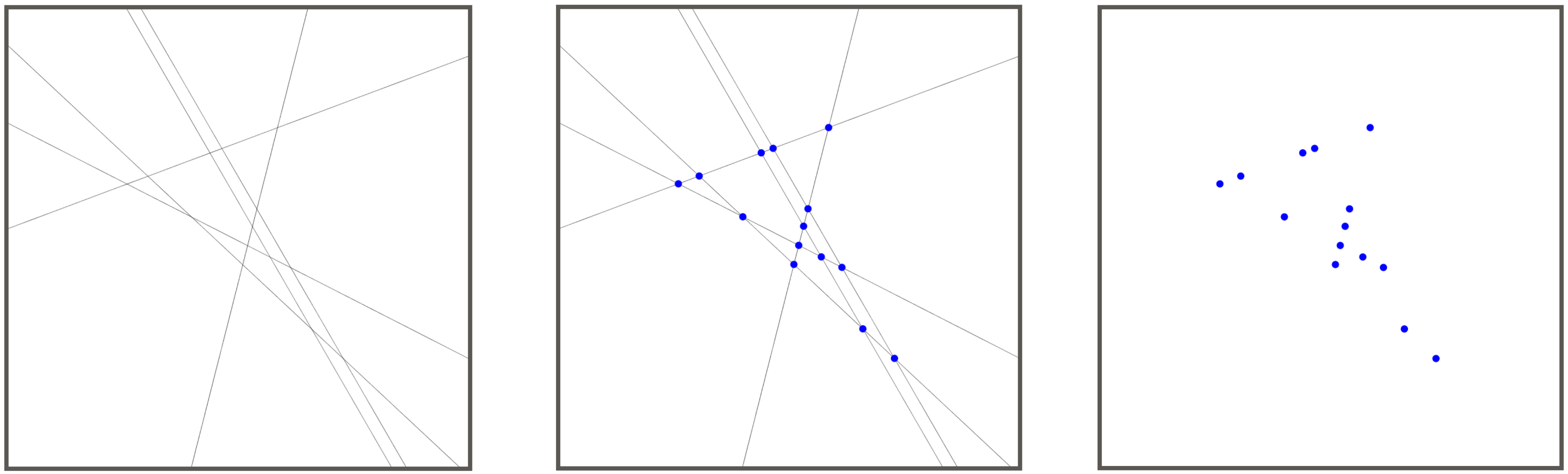

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2nd option (potentially) :

- $t \to \infty$,
- $R = R(t) \to 0$.

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The **intensity measure** $L_{t,R}$ of the intersection process $\Xi_{t,R}$ is the measure on $\mathbb{R}^d$ defined by

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for any $S \in \mathscr{B}(\mathbb{R}^d)$. Let $f_{t,R}$ be **its density**.

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$$f_{t,R}(x) = \mathbf{1}\{\|x\| < R\}c_1 t^d + \mathbf{1}\{\|x\| \geq R\}R^{d+1}t^d\|x\|^{-(d+1)}\left[c_2 + O\left(\frac{R^2}{\|x\|^2}\right)\right]$$

In particular, if $R = t^{-\frac{d}{d+1}}$ and $x \in \mathbb{R}^d \backslash \{0\}$ is fixed, $\lim_{t\to\infty} f_{t,R}(x) = c_2 \|x\|^{-(d+1)}$.

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The proof relies on the Integral Geometry toolbox.

$$L_{t,R}(S) = \frac{t^d}{d!} \int_{A(d,d-1)^d} \mathbf{1}(H_1 \cap \ldots \cap H_d \in S) \prod_{i=1}^{d} \mathbf{1}(H_i \cap B_R \neq \emptyset)\ \mu_{d-1}^{\otimes d}(\mathrm{d}(H_1, \ldots, H_d)), \qquad \text{(multivariate Mecke)}$$

$$= \int_S \frac{t^d}{d!} \int_{(A(x,d-1) \cap [B_R^d])^d} [H_1, \ldots, H_d] \mu_{d-1}^{x}(\mathrm{d}(H_1, \ldots, H_d)) \mathrm{d}x, \qquad \text{(Blaschke Petkanschin)}$$

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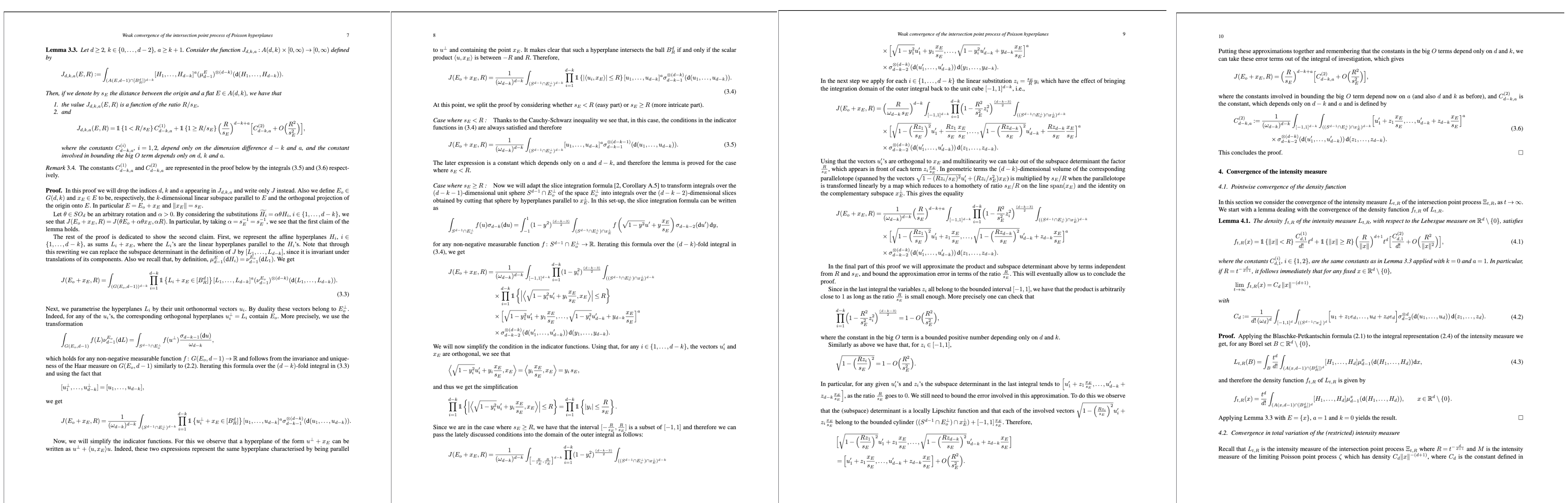

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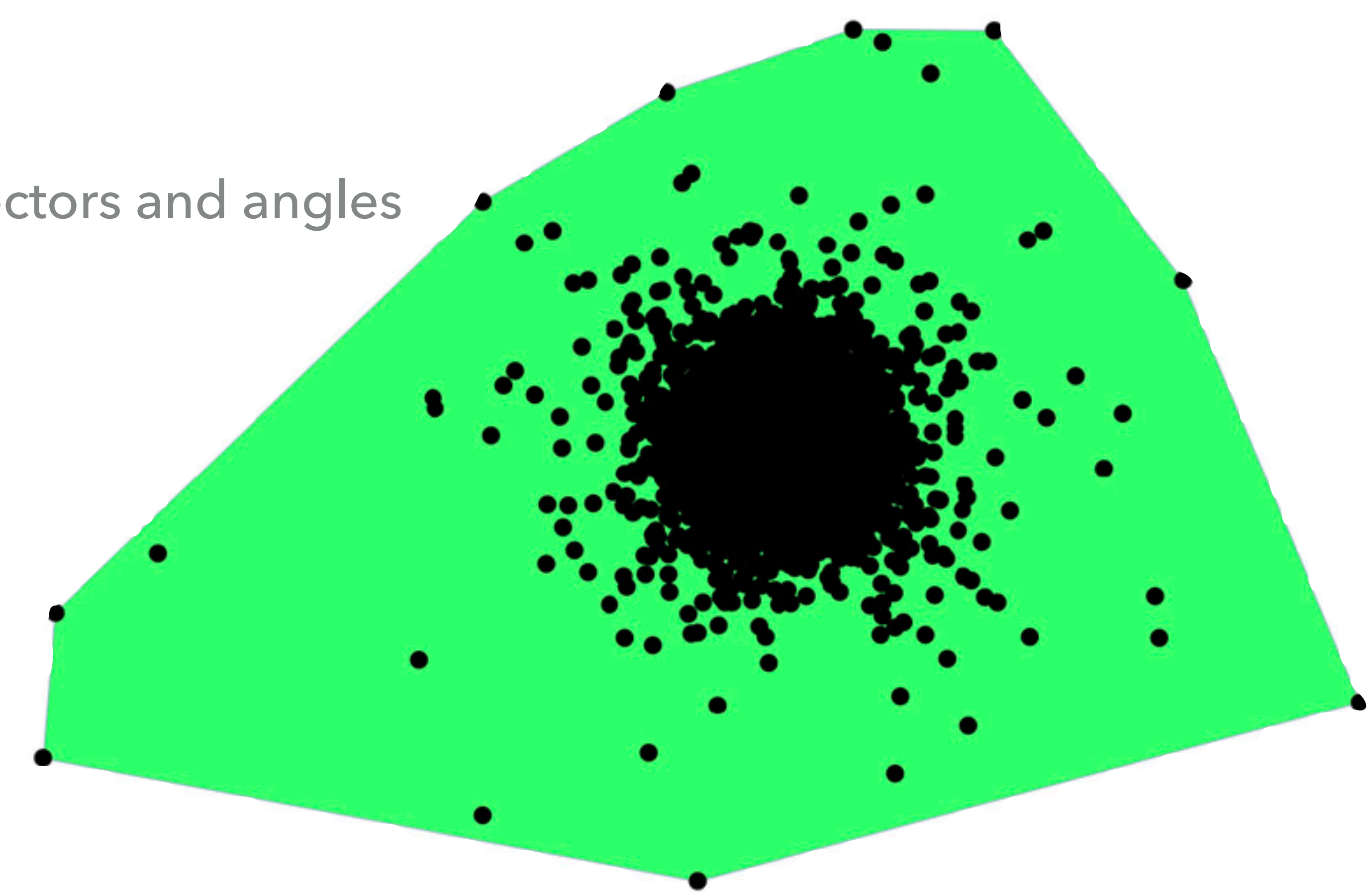
Illustration:
Beta polytopes and **Poisson polyhedra**: *f*-vectors and angles
Kabluchko, Thäle, Zaporozhets
Advances in Mathematics (2020)



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Simple point process = random discrete subset of $\mathbb{R}^d$.

Total variation distance between discrete sets  subsets $S_1, S_2 \subset \mathbb{R}^d$ :

$$d_{\mathrm{TV}}(S_1, S_2) = \max(\#(S_1 \backslash S_2), \#(S_2 \backslash S_1)) \ .$$

Remark : This is the total variation distance between the counting measures of $S_1$ and $S_2$.

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**THEOREM** (Bound on the K-R distance)

Assume that $R = t^{-\frac{d}{d+1}}$ and $0 < R < r < 1$. Then, we have that

$$d_{\mathrm{KR}}(\Xi_{t,R_{|(B_r)^c}}, \zeta_{|(B_r)^c}) \leq c t^{-\frac{1}{d+1}} \ln(t) r^{-3},$$

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As a corollary, we get :

**THEOREM** (Convergence in distribution of the point process)

Assume that $R = t^{-\frac{d}{d+1}}$. Then

$$\Xi_{t,R} \xrightarrow{d} \zeta, \quad t \to \infty ,$$

on $\mathbb{R}^d \backslash \{0\}$ .

The convergence in distribution means here that $\Xi_{t,R}(B) \xrightarrow{d} \zeta(B)$ for all Borel sets $B \subset \mathbb{R}^d \backslash \{0\}$, relatively compact (in the space $\mathbb{R}^d \backslash \{0\}$) and with boundary of zero Lebesgue measure.

By Theorem 3.1 of [Decreusefond, Schulte, Thäle, Ann. Probab. 2016], we bound the K-R distance

$$d_{\mathrm{KR}}(\Xi_{t,R_{|(B_r)^c}}, \zeta_{|(B_r)^c}) \leq d_{\mathrm{TV}}(L_{t,R_{|(B_r)^c}}, M_{|(B_r)^c}) + \frac{2^{d+1}}{d!}\rho_{t,R}(r),$$

where $\rho_{t,R}(r) = \max_{1 \leq \ell \leq d-1} I_{\ell,t,R}(r)$, with

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and where $[B_r] = \{H \in A(d, d-1) : H \cap B_r \neq 0\}$.

# IDEA OF THE PROOF

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We have already seen that $d_{\mathrm{TV}}(L_{t,R_{|(B_r)^c}}, M_{|(B_r)^c}) \leq C t^{-\frac{2d}{d+1}} r^{-3}$.

By Theorem 3.1 of [Decreusefond, Schulte, Thäle, Ann. Probab. 2016], we bound the K-R distance

$$d_{\mathrm{KR}}(\Xi_{t,R_{|(B_r)^c}}, \zeta_{|(B_r)^c}) \leq d_{\mathrm{TV}}(L_{t,R_{|(B_r)^c}}, M_{|(B_r)^c}) + \frac{2^{d+1}}{d!}\rho_{t,R}(r),$$

where $\rho_{t,R}(r) = \max_{1 \leq \ell \leq d-1} I_{\ell,t,R}(r)$ , with

$$I_{\ell,t,R}(r) = \int_{[B_r]^\ell} t^\ell \left( t^{d-\ell} \int_{[B_r]^{d-\ell}} \mathbf{1}\big(\|H_1 \cap \ldots \cap H_d\| \geq r\big)\ \mu_{d-1}^{\otimes(d-\ell)}(\mathrm{d}(H_{\ell+1}, \ldots, H_d)) \right)^2 \mu_{d-1}^{\otimes \ell}(\mathrm{d}(H_1, \ldots, H_\ell)),$$

and where $[B_r] = \{H \in A(d, d-1) : H \cap B_r \neq 0\}$.

We have already seen that $d_{\mathrm{TV}}(L_{t,R_{|(B_r)^c}}, M_{|(B_r)^c}) \leq Ct^{-\frac{2d}{d+1}} r^{-3}$.

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$I_{\ell,t,R}(r) = \ldots = \ldots = \ldots = \ldots = \ldots = \ldots \leq \ldots = \ldots \leq \ldots = \ldots \leq Ct^{-\frac{1}{d+1}} \ln(t) r^{-2}$. ■

# IDEA OF THE PROOF

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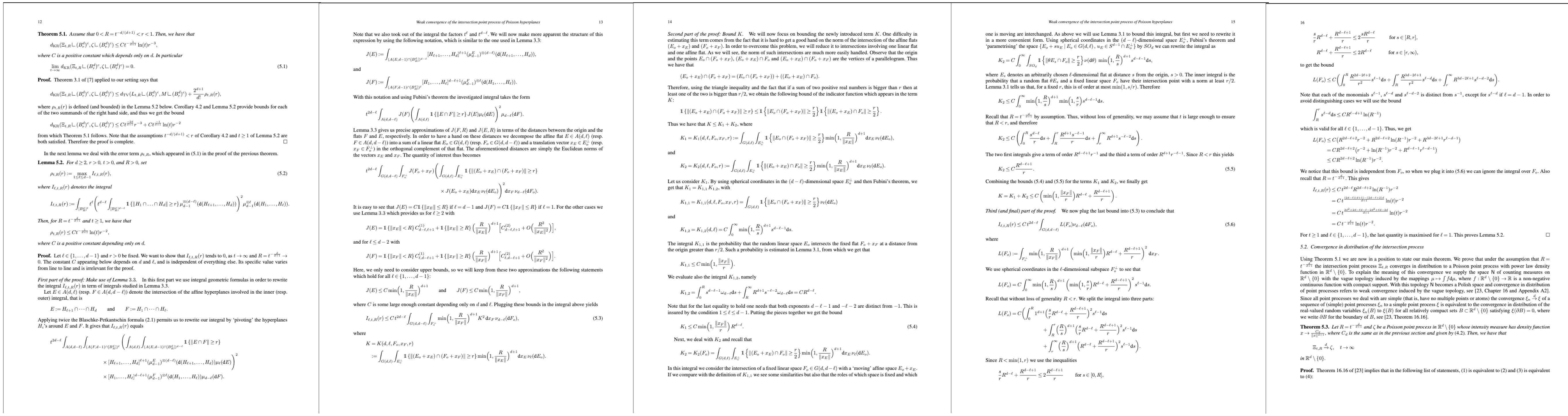
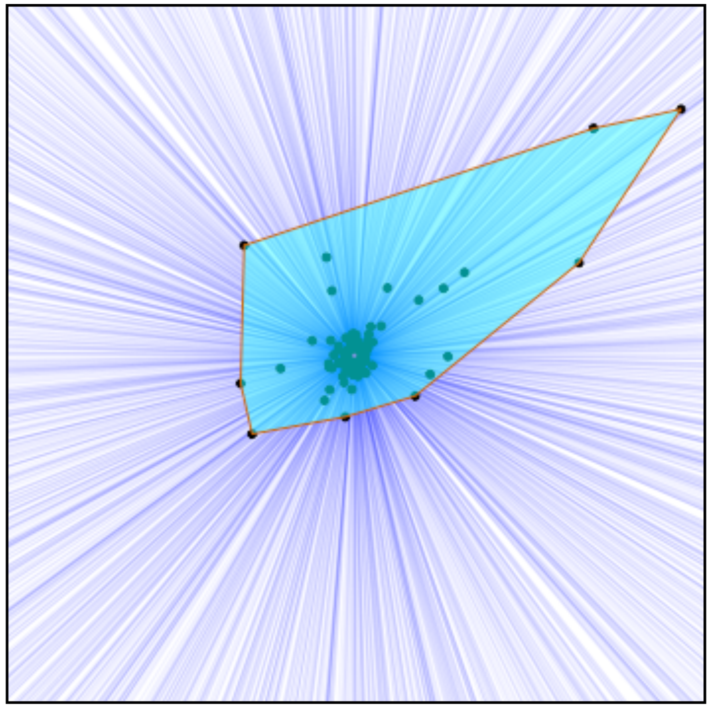


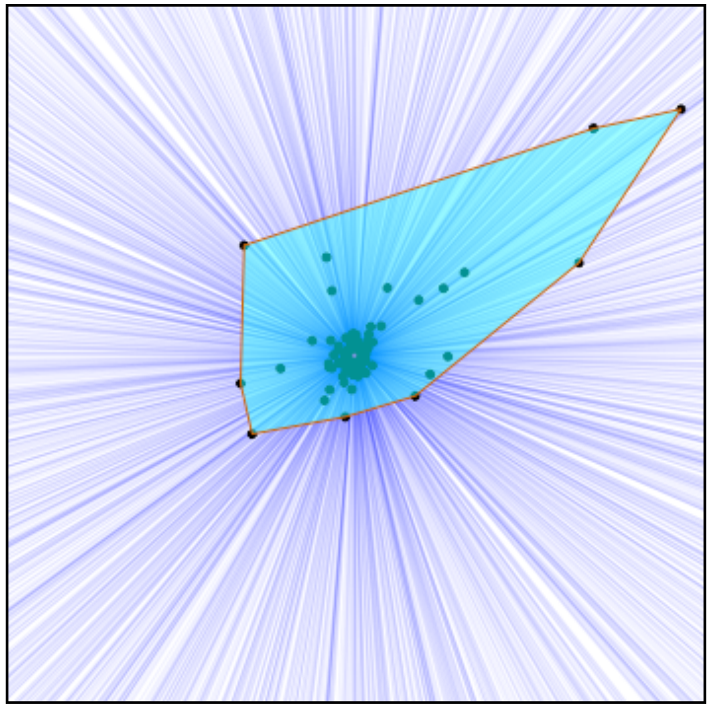


For a set $X \subset \mathbb{R}^d$, we denote :

- $\text{conv}\, X$ its convex hull

For a polytope $P \subset \mathbb{R}^d$ we denote :

- $f_k(P)$ the number of its $k$-dimensional faces, $k \in \{0,\ldots,d\}$.



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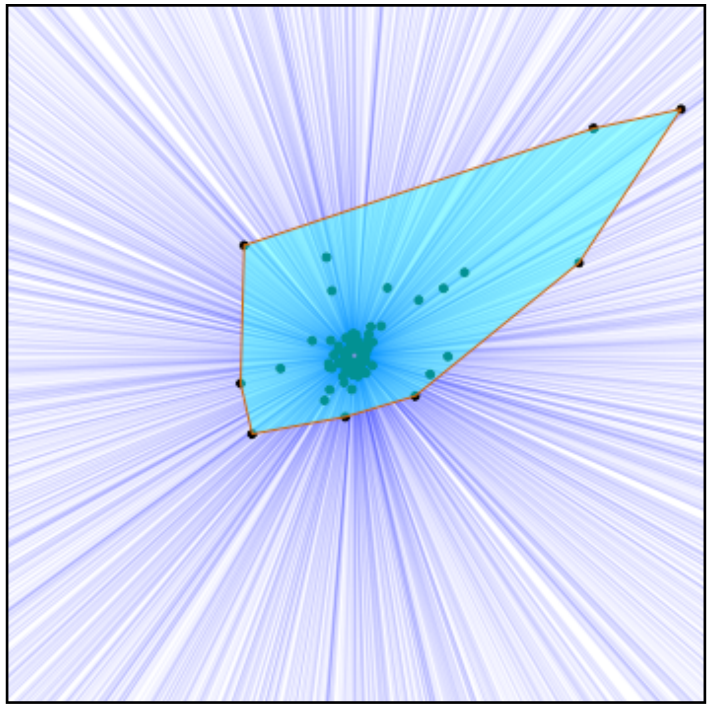
**COROLARRY** (Convergence of the convex hull)

Assume that $R = t^{-\frac{d}{d+1}}$. Then

$$\mathrm{conv}\, \Xi_{t,R} \xrightarrow{d} \mathrm{conv}\, \zeta, \quad t \to \infty.$$

and, for any $k \in \{0,\ldots,d\}$,

$$f_k(\mathrm{conv}\, \Xi_{t,R}) \xrightarrow{d} f_k(\mathrm{conv}\, \zeta), \quad t \to \infty.$$



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With Fatou's lemma and a result from [Kabluchko, Marynych, Temesvari and Thäle, PTRF 2019] we get

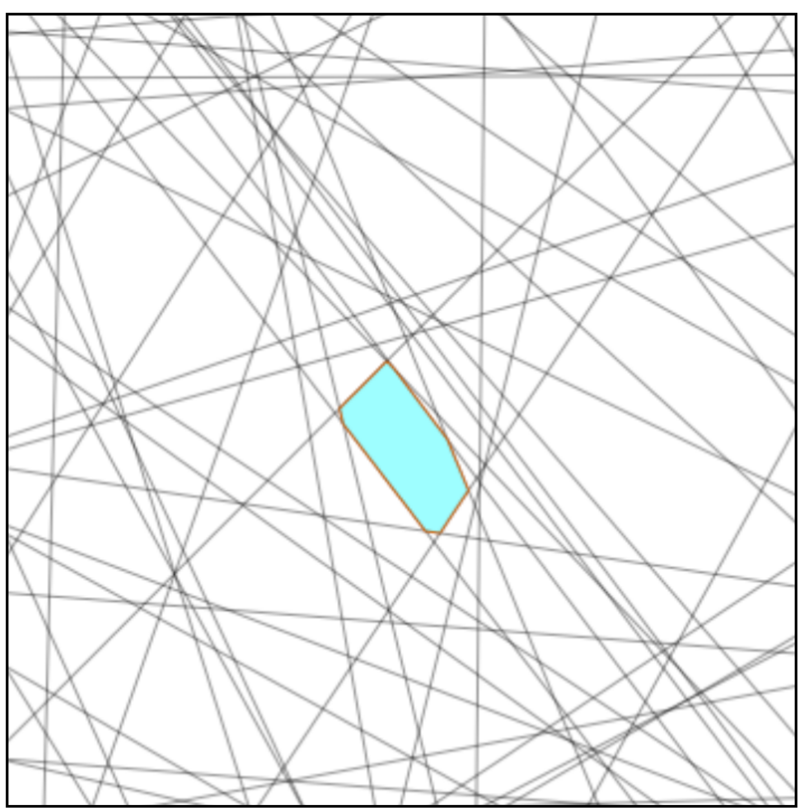
**COROLLARY** (Disproval of a conjecture from [Devroye,Toussaint, J. Algorithms 1993] )
Assume that $d = 2$ and $R = t^{-\frac{d}{d+1}}$. Then

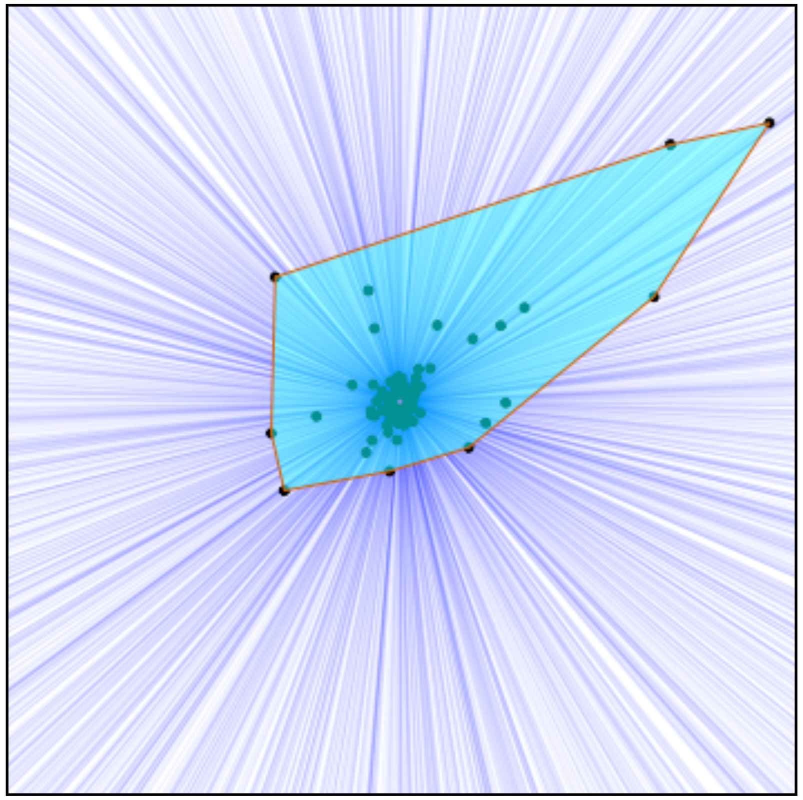
$$\liminf_{t\to\infty} \mathbb{E} f_0(\mathrm{conv}\ \Xi_{t,R}) \geq \mathbb{E} f_0(\mathrm{conv}\ \zeta) = \frac{\pi^2}{2} > 4.$$

# ZERO CELL OF A POISSON HYPERPLANE PROCESS

- $Z_\gamma$ : Zero cell of a stationary and isotropic Poisson hyperplane process of intensity,
- $Z_\gamma^\circ = \{x \in \mathbb{R}^d : \langle x, y \rangle \leq 1 \text{ for all } y \in Z_\gamma\}$ its dual.

$$C_d := \frac{1}{d!\,(\omega_d)^d} \int_{[-1,1]^d} \int_{((S^{d-1}\cap e_d^\perp)^d} \left[u_1 + z_1 e_d, \dots, u_d + z_d e_d\right] \sigma_{d-2}^{\otimes d}(\mathrm{d}(u_1, \dots, u_d))\, \mathrm{d}(z_1, \dots, z_d)\,.$$





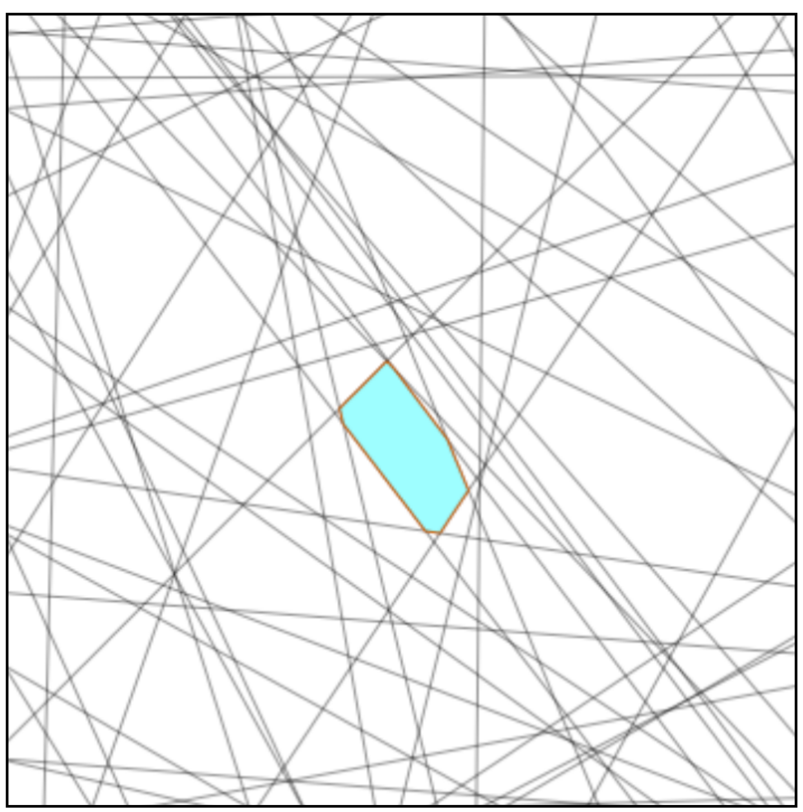
- $Z_\gamma$ : Zero cell of a stationary and isotropic Poisson hyperplane process of intensity,
- $Z_\gamma^\circ = \{x \in \mathbb{R}^d : \langle x, y\rangle \leq 1 \text{ for all } y \in Z_\gamma\}$ its dual.

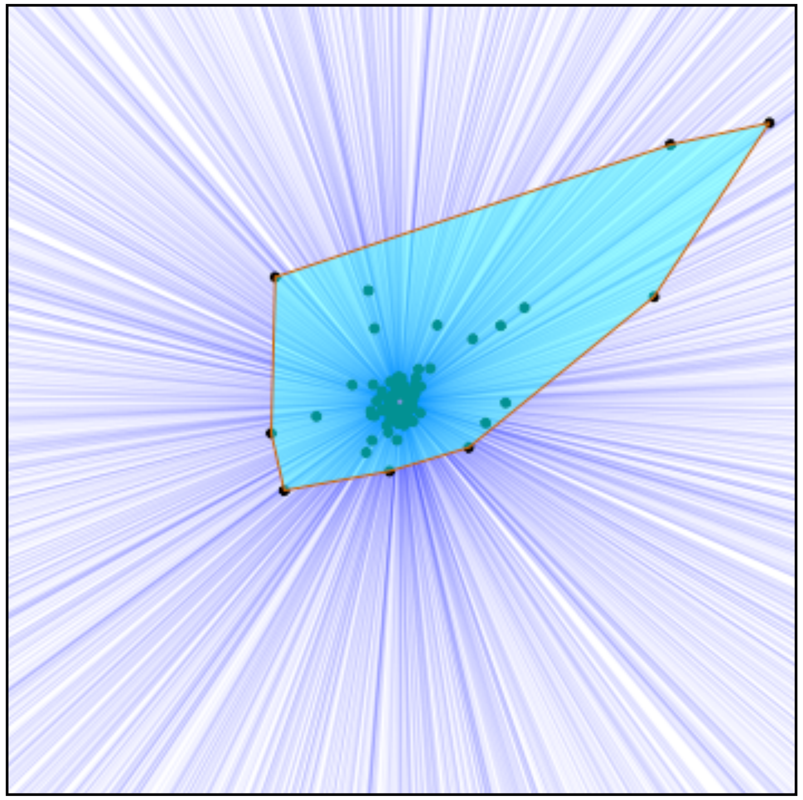
$$C_d := \frac{1}{d!\,(\omega_d)^d} \int_{[-1,1]^d} \int_{((S^{d-1}\cap e_d^\perp)^d} \left[u_1 + z_1 e_d, \ldots, u_d + z_d e_d\right] \sigma_{d-2}^{\otimes d}(\mathrm{d}(u_1, \ldots, u_d))\, \mathrm{d}(z_1, \ldots, z_d)\,.$$

**COROLARRY** (Convergence of the convex hull to the polar of the zero cell)

Let $R = t^{-\frac{d}{d+1}}$ and put $\gamma_d := \frac{1}{2} C_d \omega_d$ . Then $\Xi_{t,R} \xrightarrow{d} Z_\gamma^\circ$, as $t \to \infty$.

Moreover, for all $k \in \{0,1,\ldots,d-1\}$, $f_k(\mathrm{conv}\ \Xi_{t,R}) \xrightarrow{d} f_{d-k-1}(Z_{\gamma_d})$, as $t \to \infty$.





# MERCI POUR VOTRE ATTENTION !

## REFERENCE

Weak convergence of the intersection point process of Poisson hyperplanes,

Annales de l'Institut Henri Poincaré (B) Probabilités et Statistique (accepté)

A. Baci, G. B. and C. Thäle

# ADVERTISEMENTS

## WORKSHOP : RANDOMNESS UNLEASHED GEOMETRY, TOPOLOGY AND DATA

University of Groningen (in person and online). 22/09/2021 to 24/09/2021

https://sites.google.com/rug.nl/rug-td/

Alessandra Cipriani (TU Delft),
Markus Heydenreich (LMU Munich),
Daniel Hug (KIT),
Benedikt Jahnel (WIAS Berlin),
Sabine Jansen (LMU Munich),
Wolfgang König (WIAS Berlin),
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Tobias Müller (University of Groningen),
Takashi Owada (Purdue University),
Florian Pausinger (Queen's University Belfast),
Claudia Redenbach (TU Kaiserslautern),
Daniel Valesin (University of Groningen),
Rasmus Waagepetersen (Aalborg University)

Co-organized with: Magnus B. Botnan (VU Amsterdam) and Christian Hirsch (University of Groningen)

## JOB ADVERTISEMENT

Please contact me if you know a (good) potential candidate for a PhD in the area of stochastic geometry.
Start: anytime between September 1st 2021 and September 1st 2022
Location: Groningen, Netherlands
Language: English
Contact: g.f.y.bonnet@rug.nl