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GILLES BONNET

RUHR UNIVERSITY BOCHUM, GERMANY

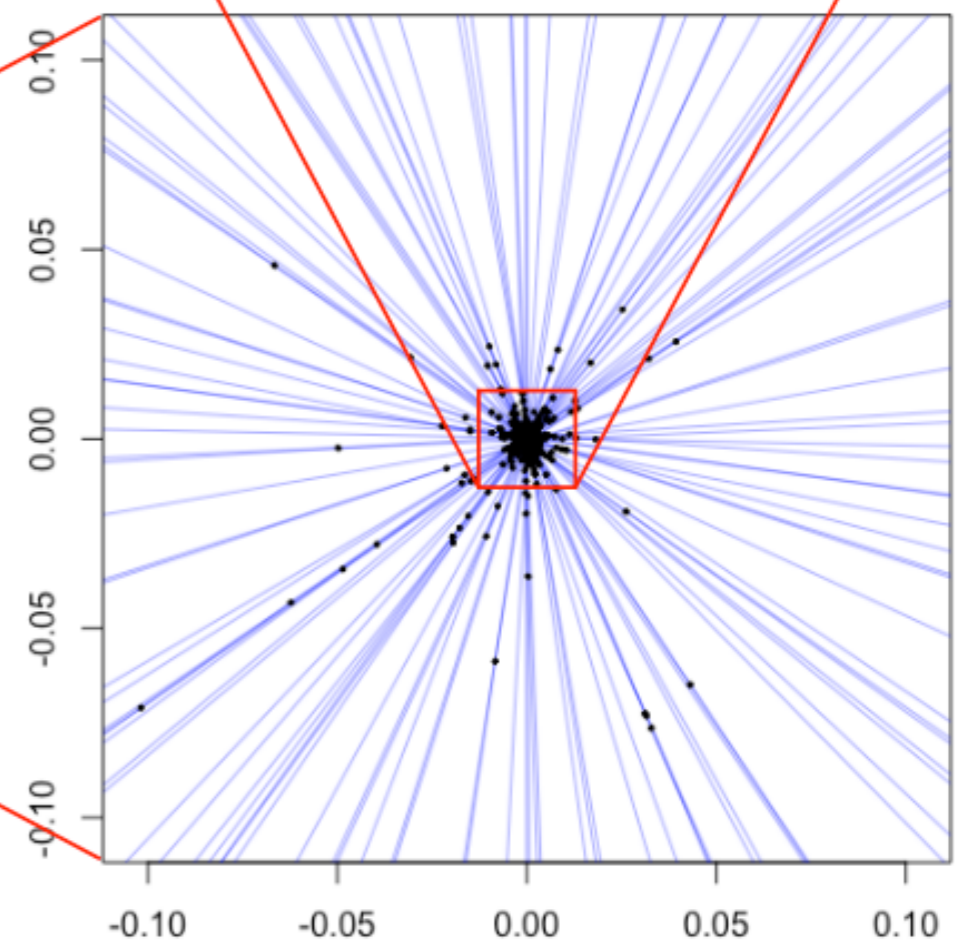
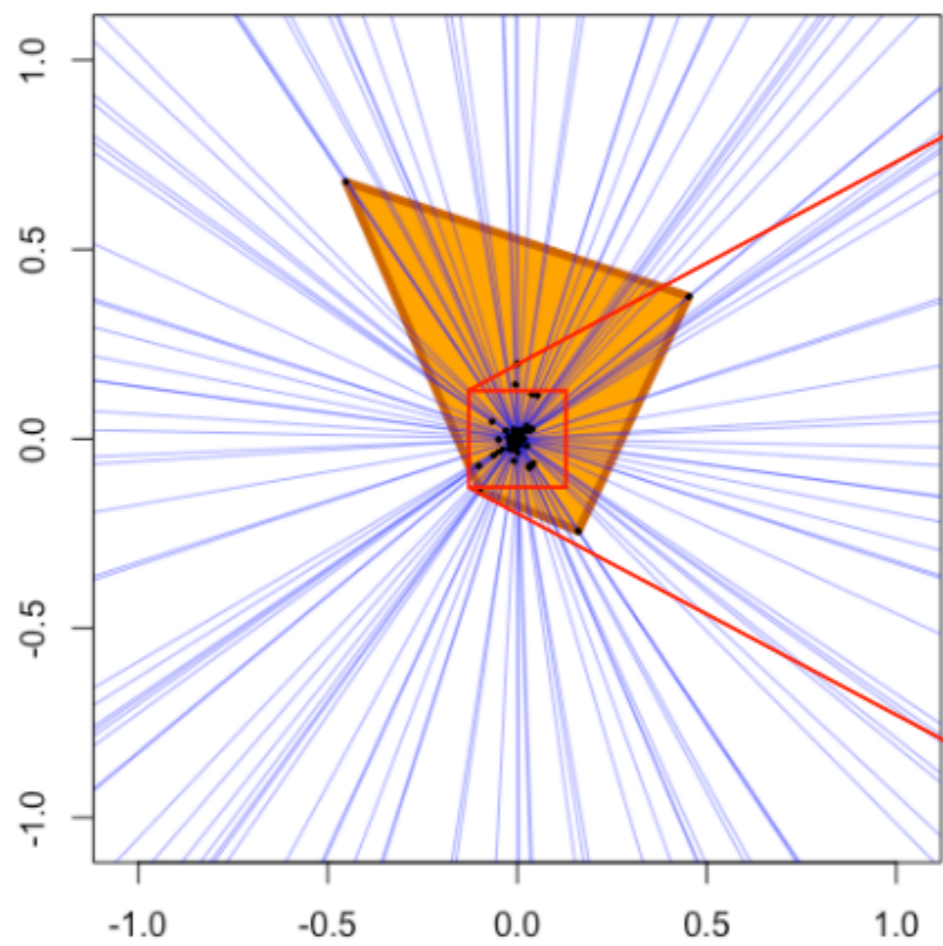
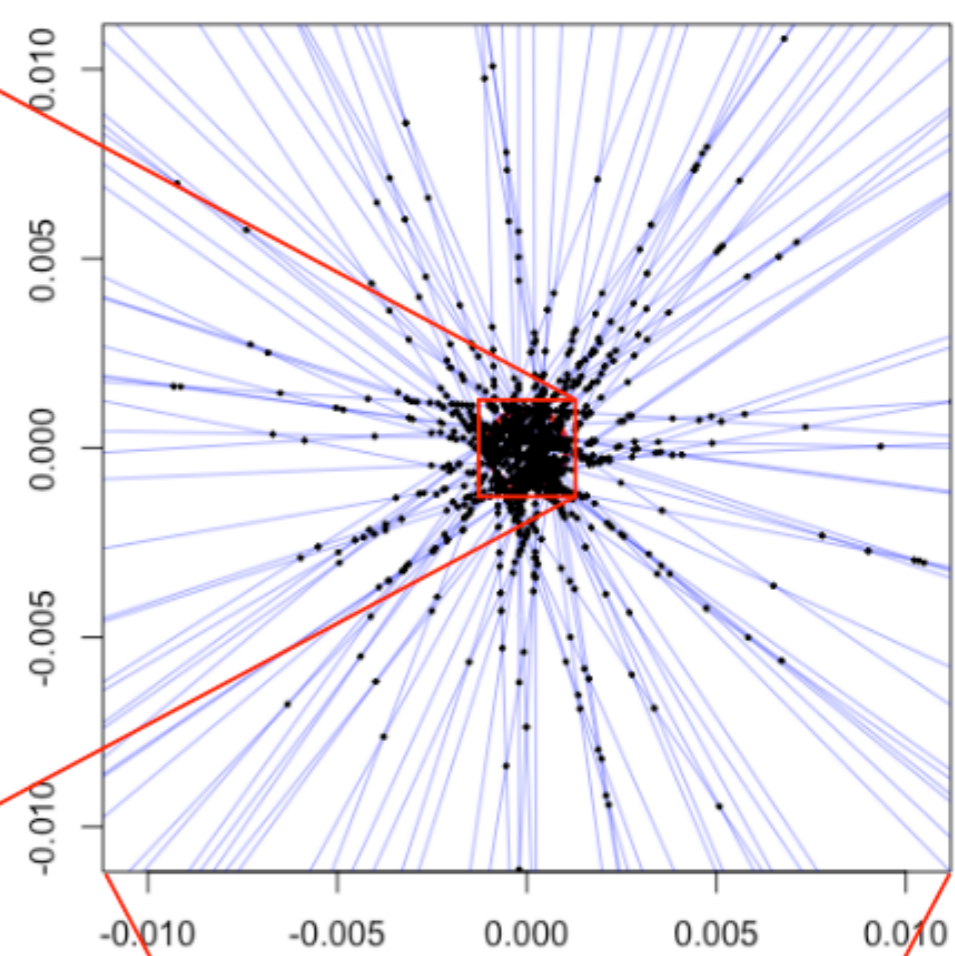
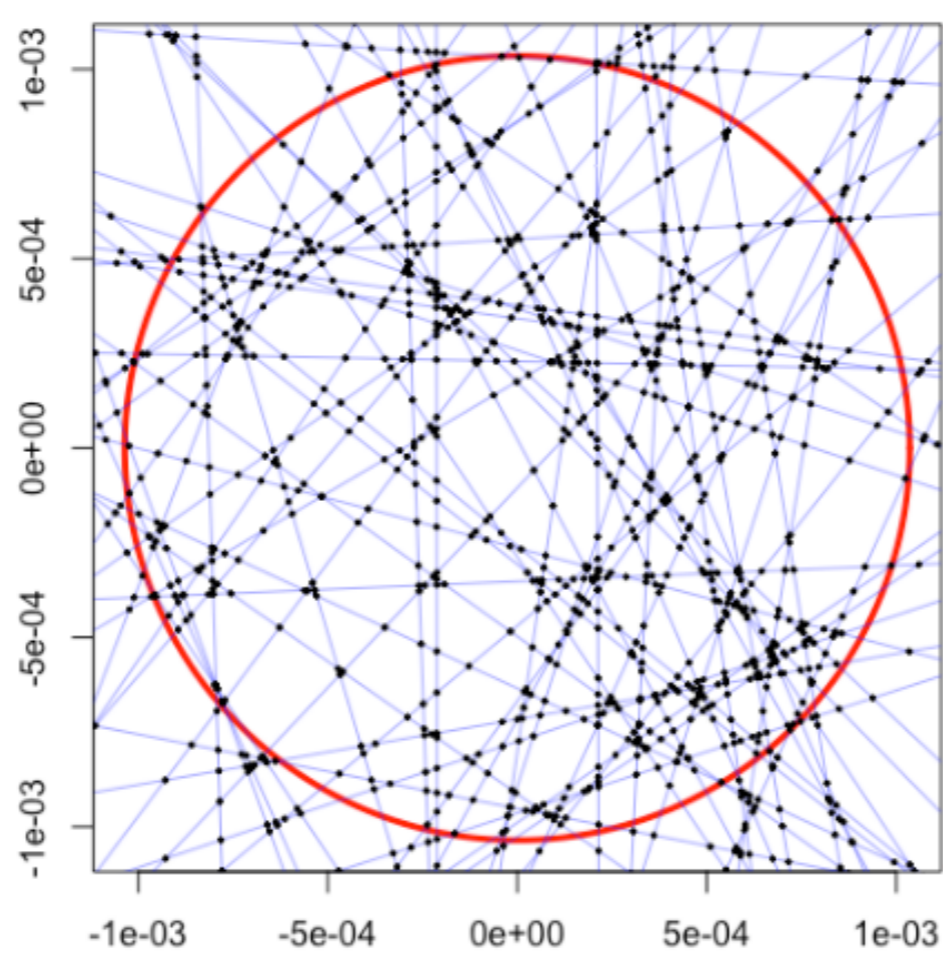
UNIVERSITY OF GRONINGEN, NETHERLANDS

WEAK CONVERGENCE OF THE INTERSECTION POINT PROCESS OF POISSON HYPERPLANES

Joint work with Anastas Baci and Christoph Thäle (arXiv:2007.06398)

Accepté pour publication dans les Annales de l'Institut Henri Poincaré (B) Probabilités et Statistique

$t = 30\,000$
 $R = t^{-2/3} \simeq 0.001$



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1. Two parameters:

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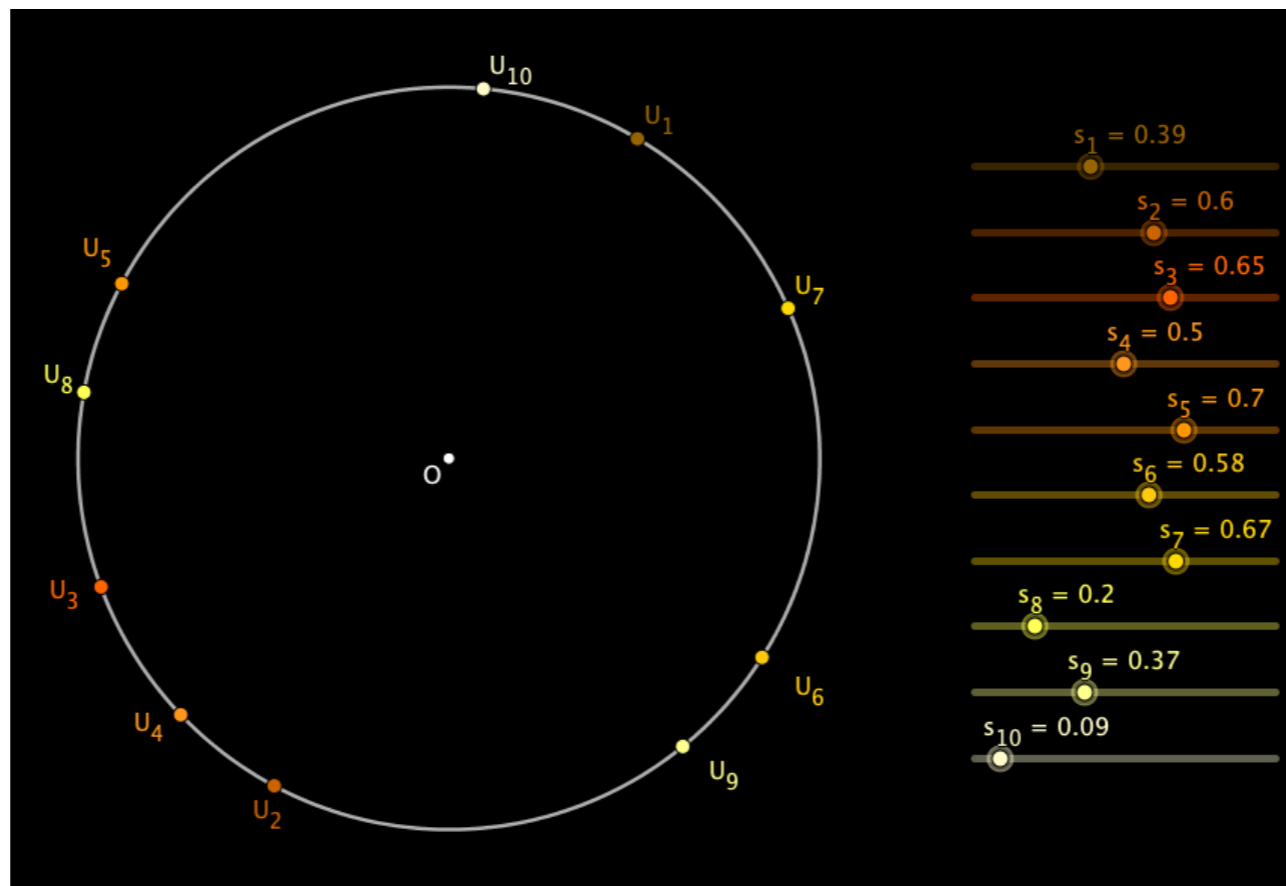
3. Construct N i.i.d. hyperplanes intersecting the ball B_R :

- $U_1, \dots, U_N \in S^{d-1}$ i.i.d. unit vectors uniformly distributed on the unit sphere,
- $s_1, \dots, s_N \in [0, R]$ i.i.d. random numbers uniformly distributed between 0 and R ,
- $H_i = \{x \in \mathbb{R}^d : \langle x, U_i \rangle = s_i\}$.

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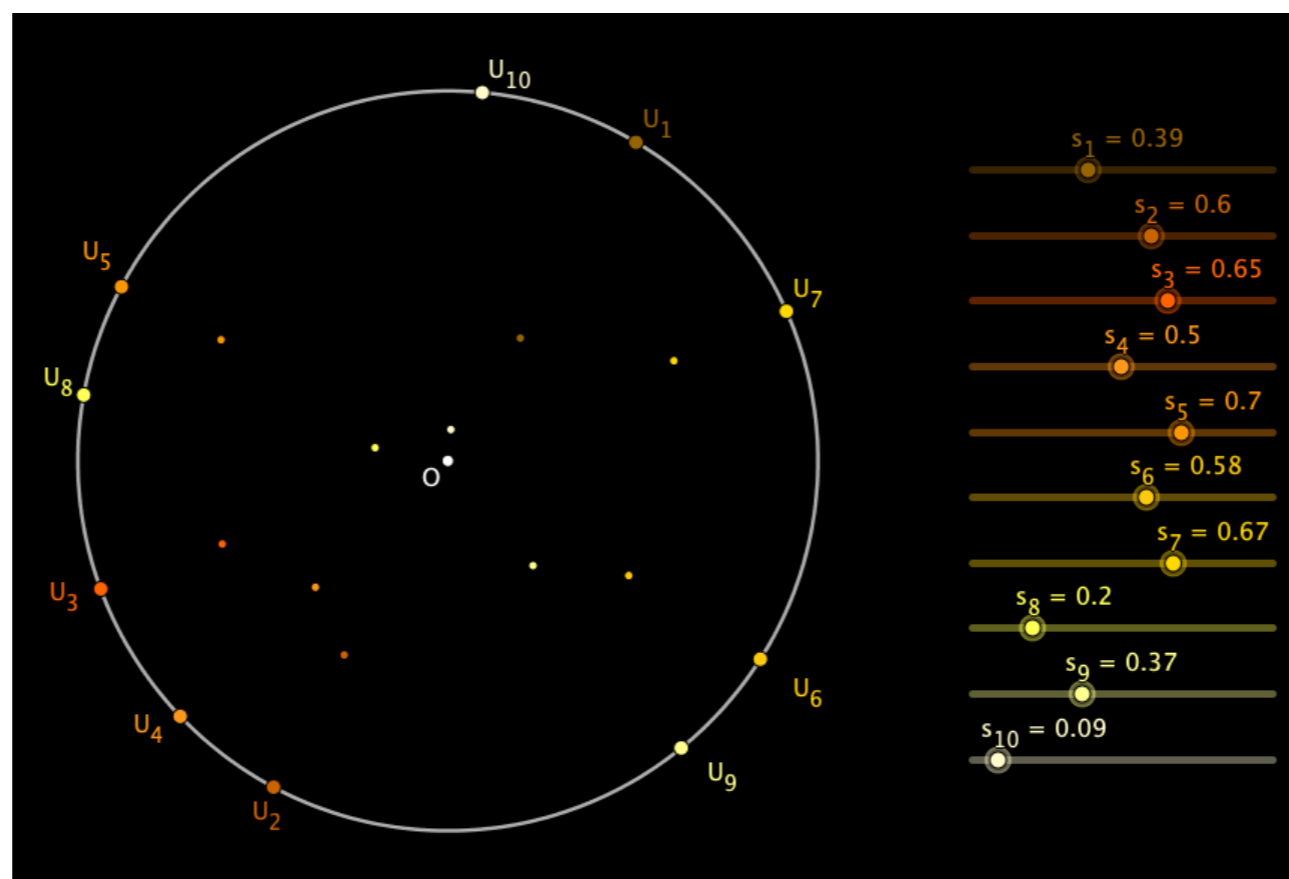
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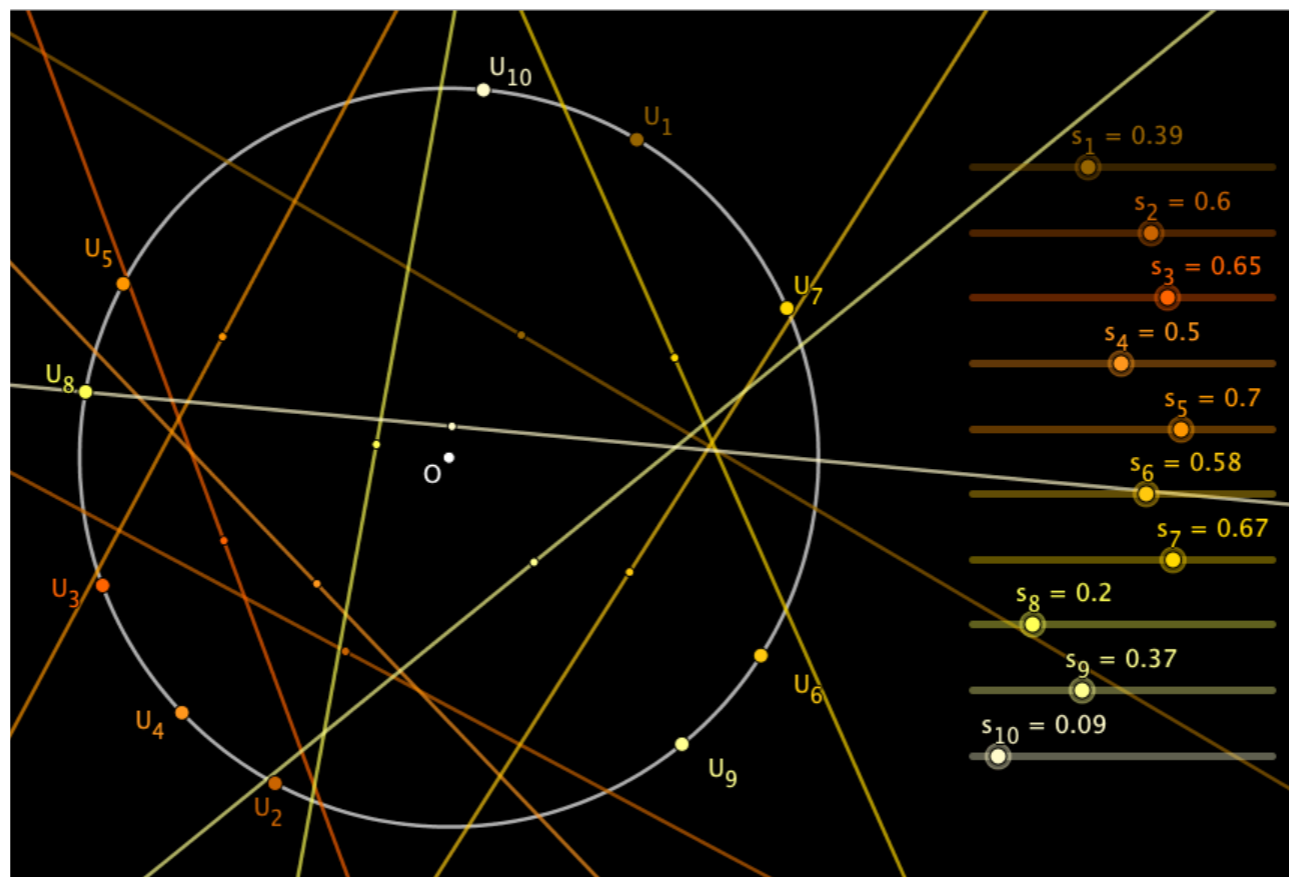
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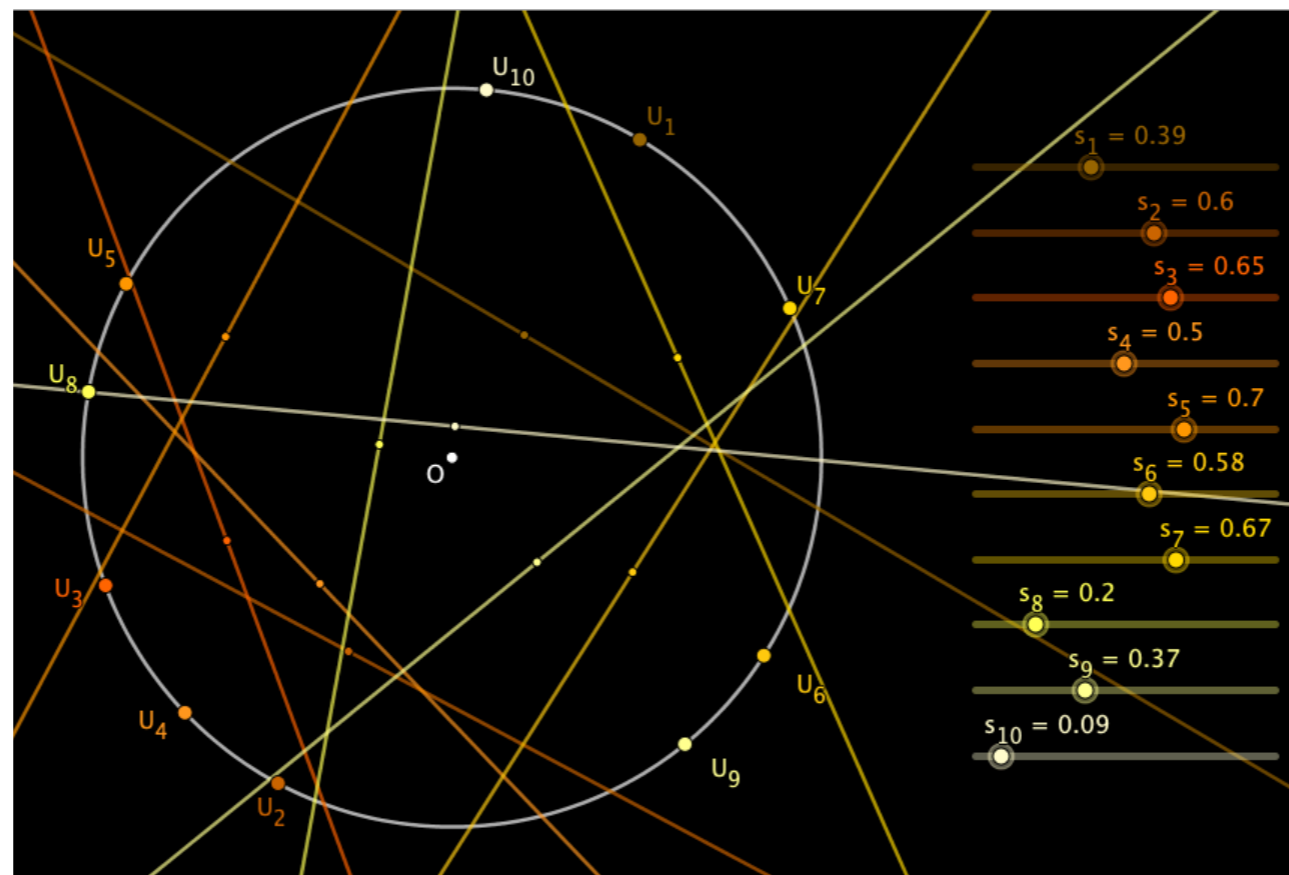
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Construction of the hyperplane process $\eta_{t,R}$

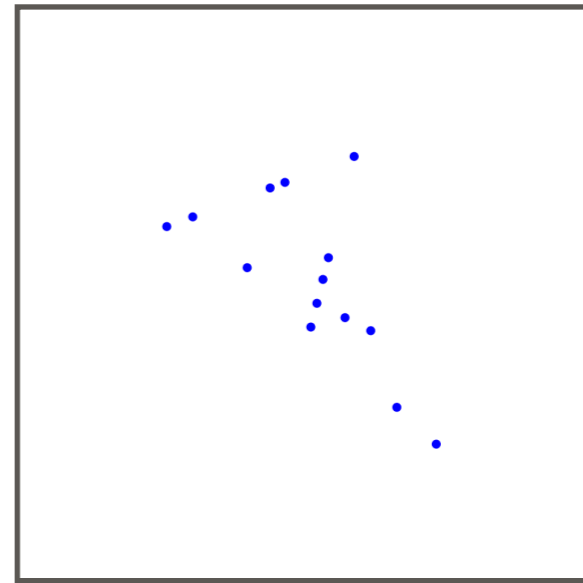
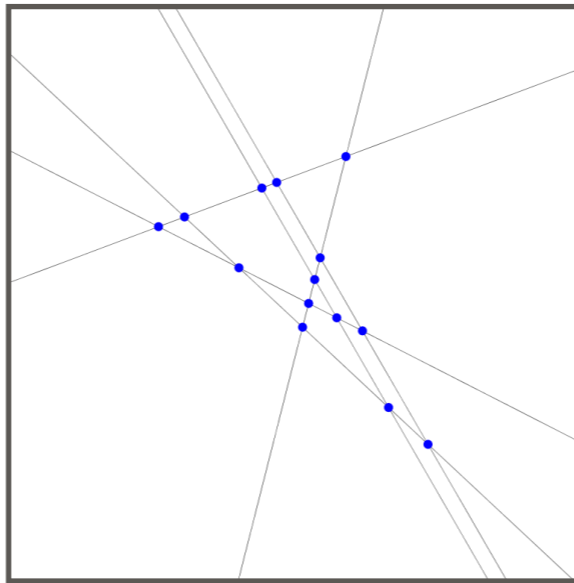
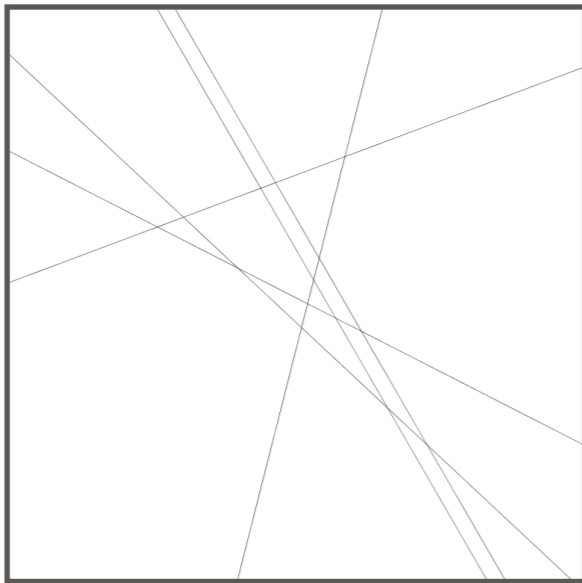


INTERSECTION PROCESS

Let $\eta_{t,R} := \{H_1, \dots, H_N\}$ as defined earlier.

We now consider the corresponding intersection points :

$$\Xi_{t,R} = \{H_{i_1} \cap \dots \cap H_{i_d} : 1 \leq i_1 < \dots < i_d \leq N\}.$$

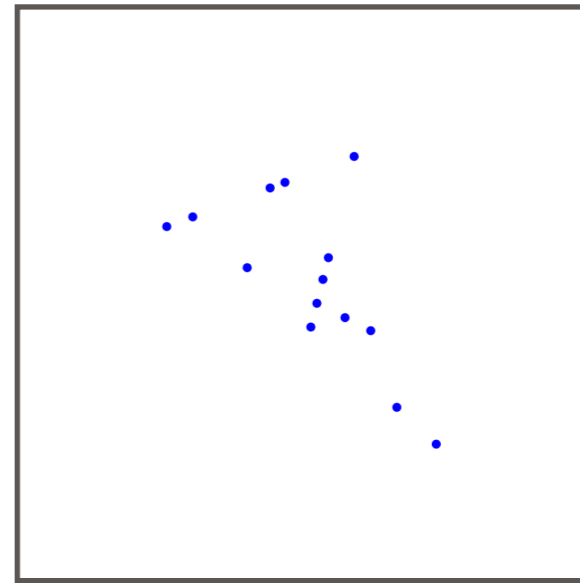
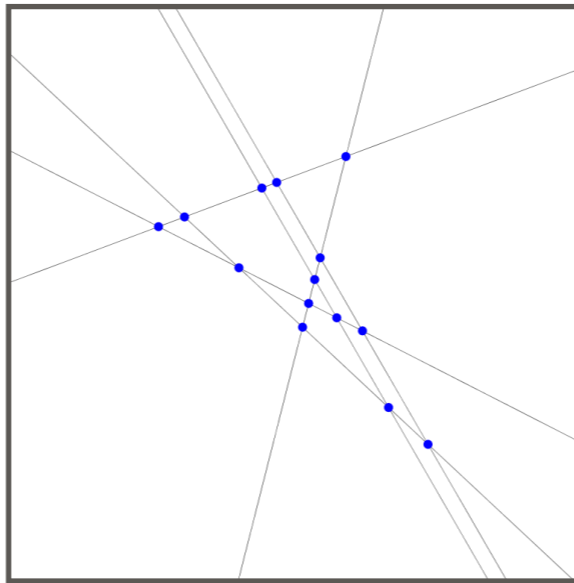
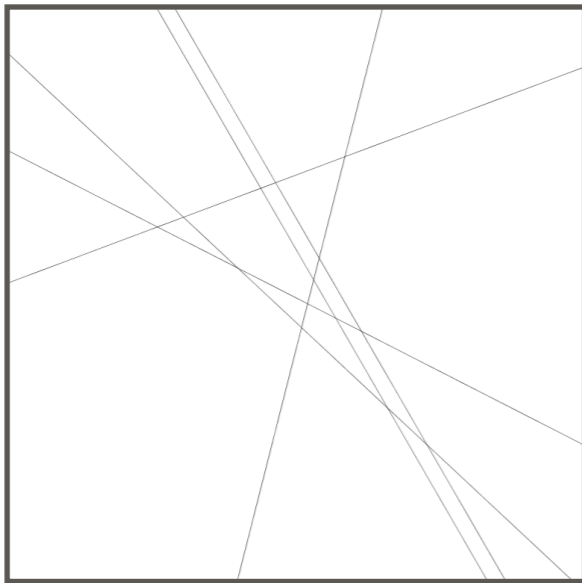


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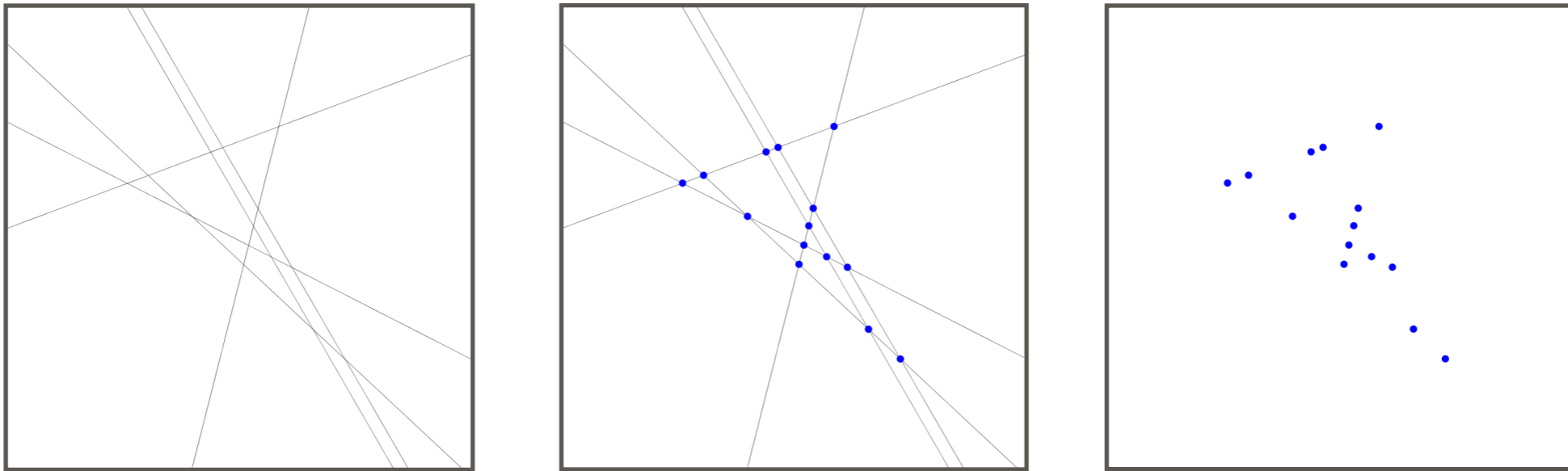
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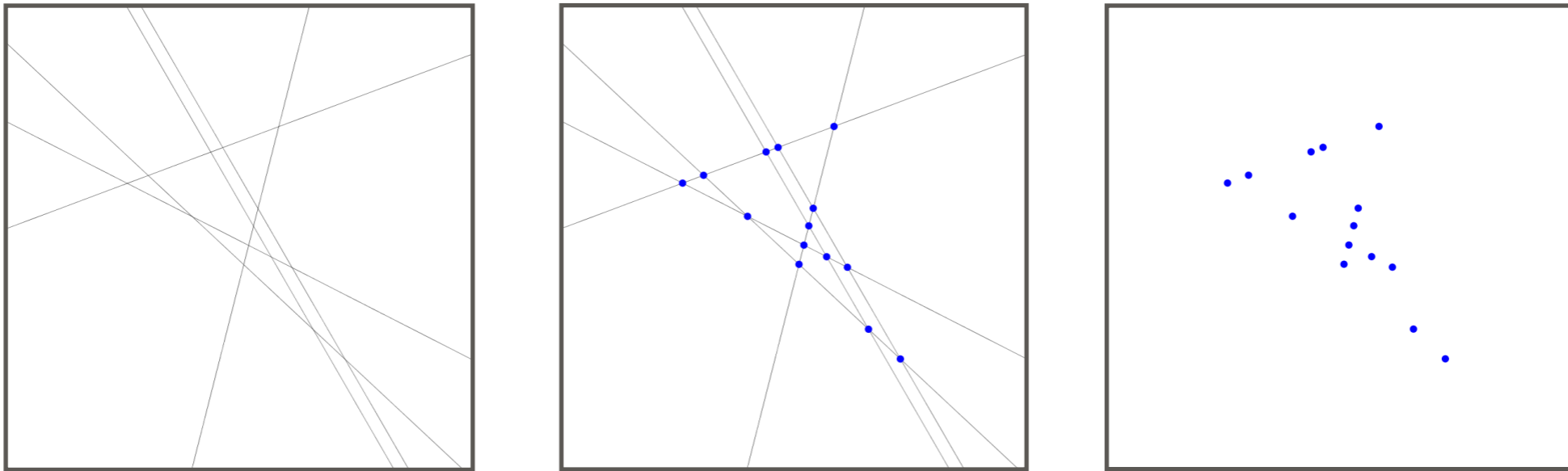
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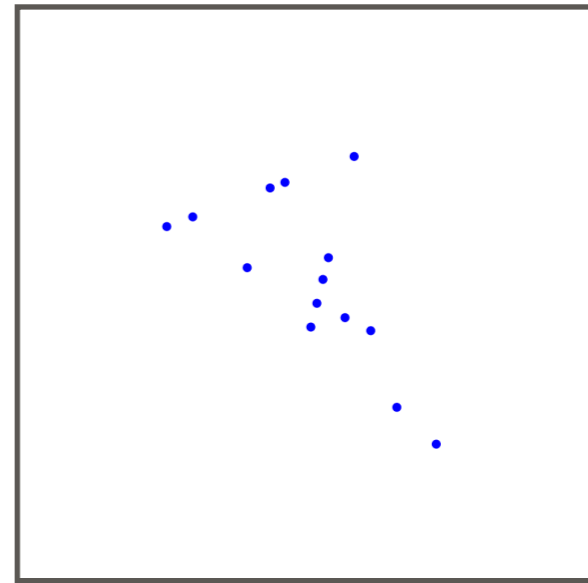
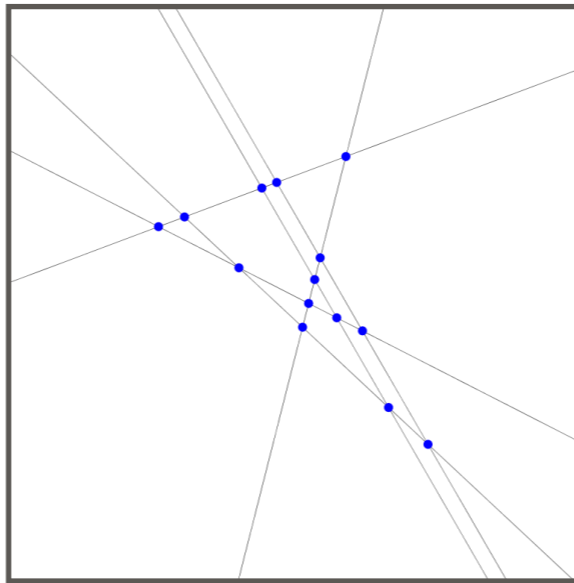
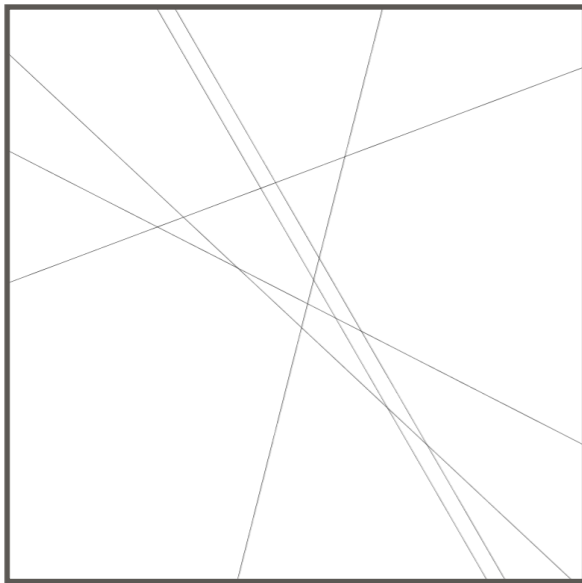


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2nd option (potentially) :

- $t \rightarrow \infty$,
- $R = R(t) \rightarrow 0$.

INTENSITY MEASURE

The **intensity measure** $L_{t,R}$ of the intersection process $\Xi_{t,R}$ is the measure on \mathbb{R}^d defined by

$$L_{t,R}(S) = \mathbb{E}(\Xi_{t,R}(S)),$$

for any $S \in \mathcal{B}(\mathbb{R}^d)$. Let $f_{t,R}$ be **its density**.

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LEMMA (Convergence of the density)

$$f_{t,R}(x) = \mathbf{1}\{\|x\| < R\}c_1t^d + \mathbf{1}\{\|x\| \geq R\}R^{d+1}t^d\|x\|^{-(d+1)} \left[c_2 + O\left(\frac{R^2}{\|x\|^2}\right) \right]$$

In particular, if $R = t^{-\frac{d}{d+1}}$ and $x \in \mathbb{R}^d \setminus \{0\}$ is fixed, $\lim_{t \rightarrow \infty} f_{t,R}(x) = c_2 \|x\|^{-(d+1)}$.

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The proof relies on the **Integral Geometry** toolbox.

$$\begin{aligned} L_{t,R}(S) &= \frac{t^d}{d!} \int_{A(d,d-1)^d} \mathbf{1}(H_1 \cap \dots \cap H_d \in S) \prod_{i=1}^d \mathbf{1}(H_i \cap B_R \neq \emptyset) \mu_{d-1}^{\otimes d}(d(H_1, \dots, H_d)), && \text{(multivariate Mecke)} \\ &= \int_S \frac{t^d}{d!} \int_{(A(x,d-1) \cap [B_R^d])^d} [H_1, \dots, H_d] \mu_{d-1}^x(d(H_1, \dots, H_d)) dx, && \text{(Blaschke Petkanschin)} \end{aligned}$$

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And...

<p>Lemma 3.3. Let $d \geq 2$, $k \in \{0, \dots, d-2\}$, $a \geq k+1$. Consider the function $J_{d,k,a}: A(d,k) \times [0, \infty) \rightarrow [0, \infty)$ defined by</p> $J_{d,k,a}(E, R) := \int_{(A(E,d-1) \cap [B_R^d])^a} [H_1, \dots, H_{d-k}]^a (\nu_{d-k}^E)^{\otimes(a-k)}(d(H_1, \dots, H_{d-k})).$ <p>Then, if we denote by s_E the distance between the origin and a flat $E \in A(d,k)$, we have that</p> <ol style="list-style-type: none"> the value $J_{d,k,a}(E, R)$ is a function of the ratio R/s_E, and $J_{d,k,a}(E, R) = \mathbf{1}\{1 < R/s_E\} C_{d,k,a}^{(1)} + \mathbf{1}\{1 \geq R/s_E\} \left(\frac{R}{s_E}\right)^{d-k+a} \left[C_{d,k,a}^{(2)} + O\left(\frac{R^2}{s_E^2}\right)\right],$ <p>where the constants $C_{d,k,a}^{(i)}$, $i=1,2$, depend only on the dimension difference $d-k$ and a, and the constant involved in bounding the big O term depends only on d, k and a.</p> <p>Remark 3.4. The constants $C_{d,k,a}^{(1)}$ and $C_{d,k,a}^{(2)}$ are represented in the proof below by the integrals (3.5) and (3.6) respectively.</p> <p>Proof. In this proof we will drop the indices d, k and a appearing in $J_{d,k,a}$ and write only J instead. Also we define $E_0 \in \mathcal{G}(d,k)$ and $x_E \in E$ to be, respectively, the $(d-k)$-dimensional linear subspace parallel to E and the orthogonal projection of the origin onto E. In particular $E = E_0 + x_E$ and $\ x_E\ = s_E$.</p> <p>Let $\theta \in SO_k$ be an arbitrary rotation and $\alpha > 0$. By considering the substitutions $H_i = \alpha \theta H_i$, $i \in \{1, \dots, d-k\}$, we see that $J(E_0 + x_E, R) = J(\theta E_0 + \alpha \theta x_E, \alpha R)$. In particular, by taking $\alpha = s_E^{-1} = s_E^{-1}$, we see that the first claim of the lemma holds.</p> <p>The rest of the proof is dedicated to show the second claim. First, we represent the affine hyperplanes H_i, $i \in \{1, \dots, d-k\}$, as sums $L_i + x_E$, where the L_i's are the linear hyperplanes parallel to the H_i's. Note that through this rewriting we can replace the subpace determinant in the definition of J by $[L_1, \dots, L_{d-k}]$, since it is invariant under translations of its components. Also we recall that, by definition, $\mu_{d-k}^E(dH_1, \dots, dH_{d-k}) = \nu_{d-k}^E(dL_1, \dots, dL_{d-k})$.</p> $J(E_0 + x_E, R) = \int_{(A(E_0, d-1) \cap [B_R^d])^a} \prod_{i=1}^a \mathbf{1}\{L_i + x_E \in [B_R^d]\} [L_1, \dots, L_{d-k}]^a (\nu_{d-k}^{E_0})^{\otimes(a-k)}(d(L_1, \dots, L_{d-k})). \quad (3.3)$ <p>Next, we parametrise the hyperplanes L_i by their unit orthonormal vectors u_i. By duality these vectors belong to E_0^\perp. Indeed, for any of the u_i's, the corresponding orthogonal hyperplanes $u_i^\perp \subset E_0$ contain E_0. More precisely, we use the transformation</p> $\int_{(A(E_0, d-1) \cap [B_R^d])^a} f(L) \nu_{d-k}^E(dL) = \int_{(S^{d-1} \cap E_0^\perp)^a} f(u) \frac{\sigma_{d-k-1}(du)}{\omega_{d-k}},$ <p>which holds for any non-negative measurable function $f: \mathcal{G}(E_0, d-1) \rightarrow \mathbb{R}$ and follows from the invariance and uniqueness of the Haar measure on $\mathcal{G}(E_0, d-1)$ similarly to (2.2). Iterating this formula over the $(d-k)$-fold integral in (3.3) and using the fact that</p> $[u_1, \dots, u_{d-k}] = [u_1, \dots, u_{d-k}],$ <p>we get</p> $J(E_0 + x_E, R) = \frac{1}{(\omega_{d-k})^a} \int_{(S^{d-1} \cap E_0^\perp)^a} \prod_{i=1}^a \mathbf{1}\{u_i + x_E \in [B_R^d]\} [u_1, \dots, u_{d-k}]^a (\nu_{d-k}^{E_0})^{\otimes(a-k)}(d(u_1, \dots, u_{d-k})).$ <p>Now, we will simplify the indicator functions. For this we observe that a hyperplane of the form $u + x_E$ can be written as $u + (u, x_E)u$. Indeed, these two expressions represent the same hyperplane characterised by being parallel</p>	<p>to u^\perp and containing the point x_E. It makes clear that such a hyperplane intersects the ball B_R^d if and only if the scalar product (u, x_E) is between $-R$ and R. Therefore,</p> $J(E_0 + x_E, R) = \frac{1}{(\omega_{d-k})^a} \int_{(S^{d-1} \cap E_0^\perp)^a} \prod_{i=1}^a \mathbf{1}\{ (u_i, x_E) \leq R\} [u_1, \dots, u_{d-k}]^a (\nu_{d-k}^{E_0})^{\otimes(a-k)}(d(u_1, \dots, u_{d-k})). \quad (3.4)$ <p>At this point, we split the proof by considering whether $s_E < R$ (see part) or $s_E \geq R$ (more intricate part).</p> <p>Case where $s_E < R$: Thanks to the Cauchy-Schwarz inequality we see that, in this case, the conditions in the indicator functions in (3.4) are always satisfied and therefore</p> $J(E_0 + x_E, R) = \frac{1}{(\omega_{d-k})^a} \int_{(S^{d-1} \cap E_0^\perp)^a} [u_1, \dots, u_{d-k}]^a (\nu_{d-k}^{E_0})^{\otimes(a-k)}(d(u_1, \dots, u_{d-k})). \quad (3.5)$ <p>The later expression is a constant which depends only on a and $d-k$, and therefore the lemma is proved for the case where $s_E < R$.</p> <p>Case where $s_E \geq R$: Now we will adapt the slice integration formula [2, Corollary A.5] to transform integrals over the $(d-k-1)$-dimensional unit sphere $S^{d-1} \cap E_0^\perp$ of the space E_0^\perp into integrals over the $(d-k-2)$-dimensional slices obtained by cutting that sphere by hyperplanes parallel to E_0. In this set-up, the slice integration formula can be written as</p> $\int_{S^{d-1} \cap E_0^\perp} f(u) \sigma_{d-k}(du) = \int_{-1}^1 (1-y^2)^{\frac{d-k-3}{2}} \int_{(S^{d-1} \cap E_0^\perp) \cap y E_0} f(\sqrt{1-y^2} u' + \frac{y x_E}{s_E}) \sigma_{d-k-2}(du') dy,$ <p>for any non-negative measurable function $f: S^{d-1} \cap E_0^\perp \rightarrow \mathbb{R}$. Iterating this formula over the $(d-k)$-fold integral in (3.4), we get</p> $J(E_0 + x_E, R) = \frac{1}{(\omega_{d-k})^a} \int_{-1}^1 \prod_{i=1}^a (1-y_i^2)^{\frac{d-k-3}{2}} \int_{(S^{d-1} \cap E_0^\perp) \cap y_1 E_0} \dots \int_{(S^{d-1} \cap E_0^\perp) \cap y_a E_0} f(\sqrt{1-y_1^2} u'_1 + \frac{y_1 x_E}{s_E}, \dots, \sqrt{1-y_a^2} u'_a + \frac{y_a x_E}{s_E}) \sigma_{d-k-2}^{\otimes a}(d(u'_1, \dots, u'_a)) dy_1, \dots, dy_a.$ <p>We will now simplify the condition in the indicator functions. Using that, for any $i \in \{1, \dots, d-k\}$, the vectors u'_i and x_E are orthogonal, we see that</p> $\left \sqrt{1-y_i^2} u'_i + \frac{y_i x_E}{s_E} \right = \left(\frac{y_i x_E}{s_E} \right)^2 + y_i^2 = y_i^2 s_E^2,$ <p>and thus we get the simplification</p> $\prod_{i=1}^a \mathbf{1}\left\{ \left \sqrt{1-y_i^2} u'_i + \frac{y_i x_E}{s_E} \right \leq R \right\} = \prod_{i=1}^a \mathbf{1}\left\{ y_i \leq \frac{R}{s_E} \right\}.$ <p>Since we are in the case where $s_E \geq R$, we have that the interval $[-\frac{R}{s_E}, \frac{R}{s_E}]$ is a subset of $[-1, 1]$ and therefore we can pass the latterly discussed conditions into the domain of the outer integral as follows.</p> $J(E_0 + x_E, R) = \frac{1}{(\omega_{d-k})^a} \int_{[-\frac{R}{s_E}, \frac{R}{s_E}]^a} \prod_{i=1}^a (1-y_i^2)^{\frac{d-k-3}{2}} \int_{(S^{d-1} \cap E_0^\perp) \cap y_1 E_0} \dots \int_{(S^{d-1} \cap E_0^\perp) \cap y_a E_0} f(\sqrt{1-y_1^2} u'_1 + \frac{y_1 x_E}{s_E}, \dots, \sqrt{1-y_a^2} u'_a + \frac{y_a x_E}{s_E}) \sigma_{d-k-2}^{\otimes a}(d(u'_1, \dots, u'_a)) dy_1, \dots, dy_a.$	<p>Weak convergence of the intersection point process of Poisson hyperplanes</p> $\times \left[\sqrt{1-y_1^2} u'_1 + \frac{y_1 x_E}{s_E}, \dots, \sqrt{1-y_{d-k}^2} u'_{d-k} + \frac{y_{d-k} x_E}{s_E} \right]^a \times \sigma_{d-k-2}^{\otimes a}(d(u'_1, \dots, u'_{d-k})) d(y_1, \dots, y_{d-k}).$ <p>In the next step we apply for each $i \in \{1, \dots, d-k\}$ the linear substitution $z_i = \frac{y_i}{s_E} u'_i$, which has the effect of bringing the integration domain of the outer integral back to the unit cube $[-1, 1]^{d-k}$, i.e.,</p> $J(E_0 + x_E, R) = \left(\frac{R}{s_E}\right)^{d-k} \int_{[-1, 1]^{d-k}} \prod_{i=1}^a \left(1 - \frac{R^2}{s_E^2} z_i^2\right)^{\frac{d-k-3}{2}} \int_{(S^{d-1} \cap E_0^\perp) \cap z_1 E_0} \dots \int_{(S^{d-1} \cap E_0^\perp) \cap z_a E_0} \left[\sqrt{1 - \left(\frac{R z_1}{s_E}\right)^2} u'_1 + \frac{R z_1 x_E}{s_E s_E}, \dots, \sqrt{1 - \left(\frac{R z_{d-k}}{s_E}\right)^2} u'_{d-k} + \frac{R z_{d-k} x_E}{s_E s_E} \right]^a \times \sigma_{d-k-2}^{\otimes a}(d(u'_1, \dots, u'_{d-k})) d(z_1, \dots, z_{d-k}).$ <p>Using that the vectors u'_i's are orthogonal to x_E and multilinearity we can take out of the subspace determinant the factor $\frac{x_E}{s_E}$, which appears in front of each term $z_i \frac{x_E}{s_E}$. In geometric terms the $(d-k)$-dimensional volume of the corresponding parallelepiped (spanned by the vectors $\sqrt{1 - \left(\frac{R z_1}{s_E}\right)^2} u'_1 + \frac{R z_1 x_E}{s_E s_E}$ and $\sqrt{1 - \left(\frac{R z_{d-k}}{s_E}\right)^2} u'_{d-k} + \frac{R z_{d-k} x_E}{s_E s_E}$) is transformed linearly by a map which reduces to a homothety of ratio s_E/R on the line $\text{span}\{x_E\}$ and the identity on the complementary subspace x_E^\perp. This gives the equality</p> $J(E_0 + x_E, R) = \frac{1}{(\omega_{d-k})^a} \int_{[-1, 1]^{d-k}} \prod_{i=1}^a \left(1 - \frac{R^2}{s_E^2} z_i^2\right)^{\frac{d-k-3}{2}} \int_{(S^{d-1} \cap E_0^\perp) \cap z_1 E_0} \dots \int_{(S^{d-1} \cap E_0^\perp) \cap z_a E_0} \left[\sqrt{1 - \left(\frac{R z_1}{s_E}\right)^2} u'_1 + z_1 \frac{x_E}{s_E}, \dots, \sqrt{1 - \left(\frac{R z_{d-k}}{s_E}\right)^2} u'_{d-k} + z_{d-k} \frac{x_E}{s_E} \right]^a \times \sigma_{d-k-2}^{\otimes a}(d(u'_1, \dots, u'_{d-k})) d(z_1, \dots, z_{d-k}).$ <p>In the final part of this proof we will approximate the product and subspace determinant above by terms independent from R and s_E, and bound the approximation error in terms of the ratio $\frac{R^2}{s_E^2}$. This will eventually allow us to conclude the proof.</p> <p>Since in the last integral the variables z_i all belong to the bounded interval $[-1, 1]$, we have that the product is arbitrarily close to 1 as long as the ratio $\frac{R^2}{s_E^2}$ is small enough. More precisely one can check that</p> $\prod_{i=1}^a \left(1 - \frac{R^2}{s_E^2} z_i^2\right)^{\frac{d-k-3}{2}} = 1 - O\left(\frac{R^2}{s_E^2}\right),$ <p>where the constant in the big O term is a bounded positive number depending only on d and k. Similarly as above we have that, for $z_i \in [-1, 1]$,</p> $\left \sqrt{1 - \left(\frac{R z_i}{s_E}\right)^2} u'_i + z_i \frac{x_E}{s_E} \right = 1 - O\left(\frac{R^2}{s_E^2}\right).$ <p>In particular, for any given u'_i's and z_i's the subspace determinant in the last integral tends to $[u'_1 + z_1 \frac{x_E}{s_E}, \dots, u'_{d-k} + z_{d-k} \frac{x_E}{s_E}]$, as the ratio $\frac{R^2}{s_E^2}$ goes to 0. We still need to bound the error involved in this approximation. To do this we observe that the (subspace) determinant is a locally Lipschitz function and that each of the involved vectors $\sqrt{1 - \left(\frac{R z_i}{s_E}\right)^2} u'_i + z_i \frac{x_E}{s_E}$ belong to the bounded cylinder $(S^{d-1} \cap E_0^\perp) \cap z_i E_0$ and $[-1, 1] \frac{x_E}{s_E}$. Therefore,</p> $\left \sqrt{1 - \left(\frac{R z_1}{s_E}\right)^2} u'_1 + z_1 \frac{x_E}{s_E}, \dots, \sqrt{1 - \left(\frac{R z_{d-k}}{s_E}\right)^2} u'_{d-k} + z_{d-k} \frac{x_E}{s_E} \right = [u'_1 + z_1 \frac{x_E}{s_E}, \dots, u'_{d-k} + z_{d-k} \frac{x_E}{s_E}] + O\left(\frac{R^2}{s_E^2}\right).$	<p>Weak convergence of the intersection point process of Poisson hyperplanes</p> <p>Putting these approximations together and remembering that the constants in the big O terms depend only on d and k, we can take these error terms out of the integral of investigation, which gives</p> $J(E_0 + x_E, R) = \left(\frac{R}{s_E}\right)^{d-k+a} \left[C_{d,k,a}^{(2)} + O\left(\frac{R^2}{s_E^2}\right) \right],$ <p>where the constants involved in bounding the big O term depend now on a (and also d and k as before), and $C_{d,k,a}^{(2)}$ is the constant, which depends only on $d-k$ and a and is defined by</p> $C_{d,k,a}^{(2)} := \frac{1}{(\omega_{d-k})^a} \int_{[-1, 1]^{d-k}} \int_{(S^{d-1} \cap E_0^\perp) \cap z_1 E_0} \dots \int_{(S^{d-1} \cap E_0^\perp) \cap z_a E_0} [u'_1 + z_1 \frac{x_E}{s_E}, \dots, u'_{d-k} + z_{d-k} \frac{x_E}{s_E}]^a \times \sigma_{d-k-2}^{\otimes a}(d(u'_1, \dots, u'_{d-k})) d(z_1, \dots, z_{d-k}). \quad (3.6)$ <p>This concludes the proof. \square</p> <h4>4. Convergence of the intensity measure</h4> <h5>4.1. Pointwise convergence of the density function</h5> <p>In this section we consider the convergence of the intensity measure $L_{t,R}$ of the intersection point process $\Xi_{t,R}$, as $t \rightarrow \infty$. We start with a lemma dealing with the convergence of the density function $f_{t,R}$ of $L_{t,R}$.</p> <p>Lemma 4.1. The density $f_{t,R}$ of the intensity measure $L_{t,R}$, with respect to the Lebesgue measure on $\mathbb{R}^d \setminus \{0\}$, satisfies</p> $f_{t,R}(x) = \mathbf{1}\{\ x\ < R\} \frac{C_{d,k,a}^{(1)}}{d!} t^d + \mathbf{1}\{\ x\ \geq R\} \left(\frac{R}{\ x\ }\right)^{d+1} \left[\frac{C_{d,k,a}^{(2)}}{d!} t^d + O\left(\frac{R^2}{\ x\ ^2}\right) \right], \quad (4.1)$ <p>where the constants $C_{d,k,a}^{(i)}$, $i \in \{1, 2\}$, are the same constants as in Lemma 3.3 applied with $k=0$ and $a=1$. In particular, if $R = t^{-\frac{d}{d+1}}$, it follows immediately that for any fixed $x \in \mathbb{R}^d \setminus \{0\}$,</p> $\lim_{t \rightarrow \infty} f_{t,R}(x) = C_2 \ x\ ^{-(d+1)},$ <p>with</p> $C_2 := \frac{1}{d!} \int_{A(d,d-1)} \int_{(S^{d-1} \cap E_0^\perp)^d} [u_1 + z_1 \frac{x_E}{s_E}, \dots, u_d + z_d \frac{x_E}{s_E}]^d \sigma_{d-2}^{\otimes d}(d(u_1, \dots, u_d)) d(z_1, \dots, z_d). \quad (4.2)$ <p>Proof. Applying the Blaschke-Petkanschin formula (2.1) to the integral representation (2.4) of the intensity measure we get, for any Borel set $B \subset \mathbb{R}^d \setminus \{0\}$,</p> $L_{t,R}(B) = \int_B \int_{A(d,d-1) \cap [B_R^d]} [H_1, \dots, H_d] \mu_{d-1}^{\otimes d}(d(H_1, \dots, H_d)) dx, \quad (4.3)$ <p>and therefore the density function $f_{t,R}$ of $L_{t,R}$ is given by</p> $f_{t,R}(x) = \frac{t^d}{d!} \int_{A(d,d-1) \cap [B_R^d]} [H_1, \dots, H_d] \mu_{d-1}^{\otimes d}(d(H_1, \dots, H_d)), \quad x \in \mathbb{R}^d \setminus \{0\}.$ <p>Applying Lemma 3.3 with $E = \{x\}$, $a=1$ and $k=0$ yields the result. \square</p> <h5>4.2. Convergence in total variation of the (restricted) intensity measure</h5> <p>Recall that $L_{t,R}$ is the intensity measure of the intersection point process $\Xi_{t,R}$ where $R = t^{-\frac{d}{d+1}}$ and M is the intensity measure of the limiting Poisson point process ζ which has density $C_2 \ x\ ^{-(d+1)}$, where C_2 is the constant defined in</p>
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INTENSITY MEASURE

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Let \mathbf{M} be the **measure** on $\mathbb{R}^d \setminus \{0\}$ with **density** $c_2 \|x\|^{-(d+1)}$.

We consider the **total variation distance** between the restricted $L_{t,R}|_{(B_r)^c}$ and $M|_{(B_r)^c}$. This is defined by

$$d_{\text{TV}}(L_{t,R}|_{(B_r)^c}, M|_{(B_r)^c}) = \sup\{ |L_{t,R}(A) - M(A)| : A \subset (B_r)^c \}.$$

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Assume $R = t^{-\frac{d}{d+1}}$. For any $r > 0$,

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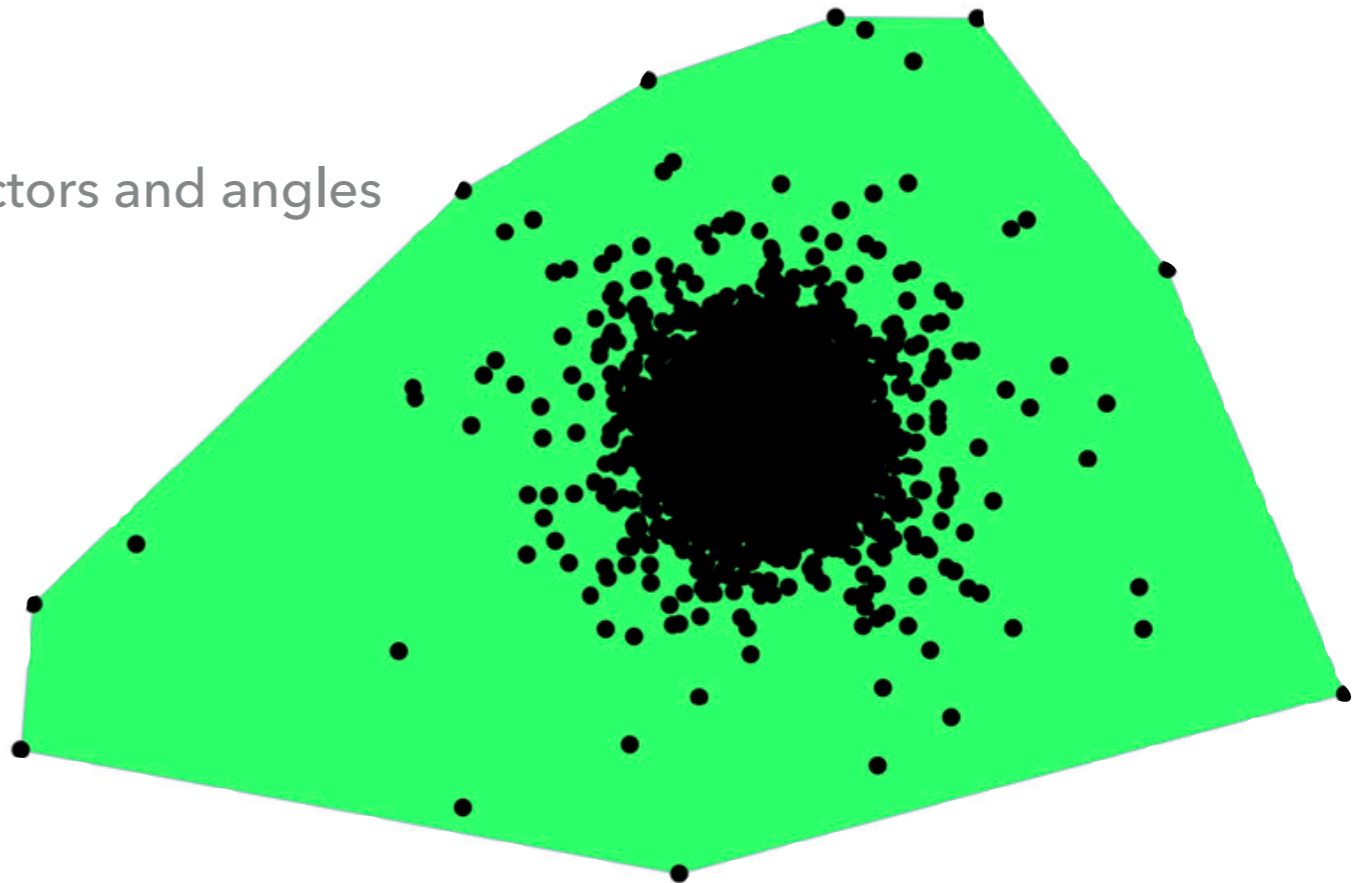
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Illustration:

Beta polytopes and **Poisson polyhedra**: f -vectors and angles

Kabluchko, Thäle, Zaporozhets

Advances in Mathematics (2020)



DISTANCES BETWEEN SIMPLE POINT PROCESSES

Simple point process = random discrete subset of \mathbb{R}^d .

Total variation distance between discrete sets subsets $S_1, S_2 \subset \mathbb{R}^d$:

$$d_{\text{TV}}(S_1, S_2) = \max(\#(S_1 \setminus S_2), \#(S_2 \setminus S_1)) .$$

Remark : This is the total variation distance between the counting measures of S_1 and S_2 .

Kantorovich-Rubinstein distance between random discrete subsets X_1 and $X_2 \subset \mathbb{R}^2$:

$$d_{\text{KR}}(X_1, X_2) = \inf_{(Y_1, Y_2) \in \Sigma(X_1, X_2)} \mathbb{E} d_{\text{TV}}(Y_1, Y_2),$$

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Recall : ζ is the Poisson point process on $\mathbb{R}^d \setminus \{0\}$ whose intensity measure M has density $x \mapsto c_2 \|x\|^{-(d+1)}$.

THEOREM (Bound on the K-R distance)

Assume that $R = t^{-\frac{d}{d+1}}$ and $0 < R < r < 1$. Then, we have that

$$d_{\text{KR}}(\Xi_{t, R|_{(B_r)^c}}, \zeta_{|(B_r)^c}) \leq ct^{-\frac{1}{d+1}} \ln(t)r^{-3},$$

where c is a positive constant which depends only on d .

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where c is a positive constant which depends only on d .

As a corollary, we get :

THEOREM (Convergence in distribution of the point process)

Assume that $R = t^{-\frac{d}{d+1}}$. Then

$$\Xi_{t,R} \xrightarrow{d} \zeta, \quad t \rightarrow \infty,$$

on $\mathbb{R}^d \setminus \{0\}$.

The convergence in distribution means here that $\Xi_{t,R}(B) \xrightarrow{d} \zeta(B)$ for all Borel sets $B \subset \mathbb{R}^d \setminus \{0\}$, relatively compact (in the space $\mathbb{R}^d \setminus \{0\}$) and with boundary of zero Lebesgue measure.

IDEA OF THE PROOF

By Theorem 3.1 of [Decreusefond, Schulte, Thäle, Ann. Probab. 2016], we bound the K-R distance

$$d_{\text{KR}}(\Xi_{t,R}|_{(B_r)^c}, \zeta|_{(B_r)^c}) \leq d_{\text{TV}}(L_{t,R}|_{(B_r)^c}, M|_{(B_r)^c}) + \frac{2^{d+1}}{d!} \rho_{t,R}(r),$$

where $\rho_{t,R}(r) = \max_{1 \leq \ell \leq d-1} I_{\ell,t,R}(r)$, with

$$I_{\ell,t,R}(r) = \int_{[B_r]^\ell} t^\ell \left(t^{d-\ell} \int_{[B_r]^{d-\ell}} \mathbf{1}(\|H_1 \cap \dots \cap H_d\| \geq r) \mu_{d-1}^{\otimes(d-\ell)}(d(H_{\ell+1}, \dots, H_d)) \right)^2 \mu_{d-1}^{\otimes \ell}(d(H_1, \dots, H_\ell)),$$

and where $[B_r] = \{H \in A(d, d-1) : H \cap B_r \neq \emptyset\}$.

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We have already seen that $d_{\text{TV}}(L_{t,R|_{(B_r)^c}}, M_{|(B_r)^c}) \leq Ct^{-\frac{2d}{d+1}} r^{-3}$.

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We have already seen that $d_{\text{TV}}(L_{t,R|_{(B_r)^c}}, M_{|(B_r)^c}) \leq Ct^{-\frac{2d}{d+1}} r^{-3}$.

It remains to deal with the integrals $I_{\ell,t,R}(r)$.

$$I_{\ell,t,R}(r) = \dots = \dots = \dots = \dots = \dots = \dots \leq \dots = \dots \leq \dots = \dots \leq Ct^{-\frac{1}{d+1}} \ln(t) r^{-2}. \quad \blacksquare$$

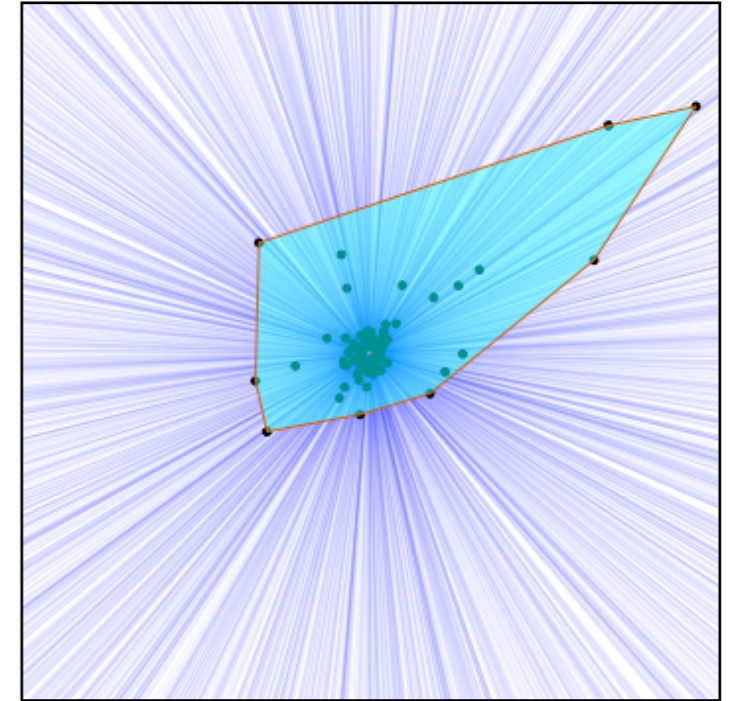
CONVEX HULL

For a set $X \subset \mathbb{R}^d$, we denote :

- $\text{conv } X$ its convex hull

For a polytope $P \subset \mathbb{R}^d$ we denote :

- $f_k(P)$ the number of its k -dimensional faces, $k \in \{0, \dots, d\}$.



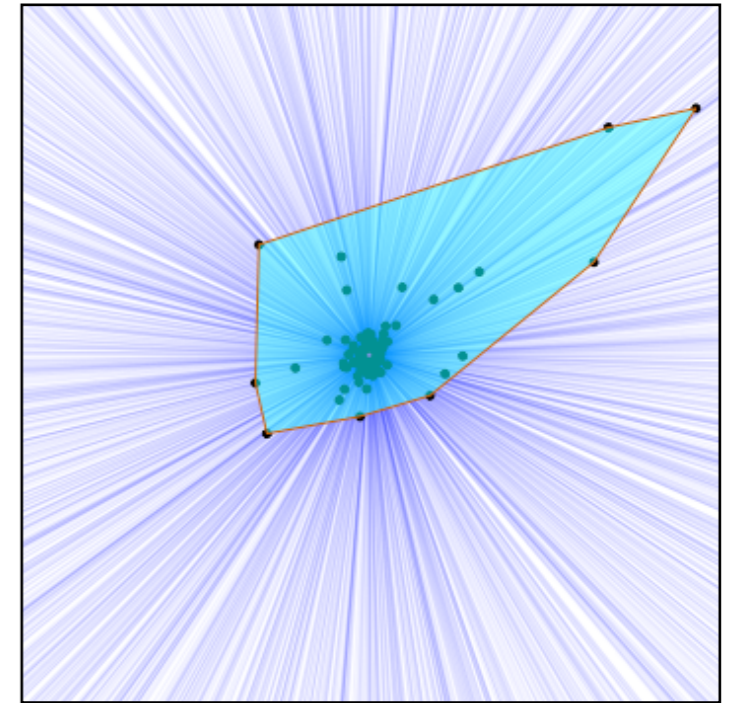
CONVEX HULL

For a set $X \subset \mathbb{R}^d$, we denote :

- $\text{conv } X$ its convex hull

For a polytope $P \subset \mathbb{R}^d$ we denote :

- $f_k(P)$ the number of its k -dimensional faces, $k \in \{0, \dots, d\}$.



COROLLARY (Convergence of the convex hull)

Assume that $R = t^{-\frac{d}{d+1}}$. Then

$$\text{conv } \Xi_{t,R} \xrightarrow{d} \text{conv } \zeta, \quad t \rightarrow \infty.$$

and, for any $k \in \{0, \dots, d\}$,

$$f_k(\text{conv } \Xi_{t,R}) \xrightarrow{d} f_k(\text{conv } \zeta), \quad t \rightarrow \infty.$$

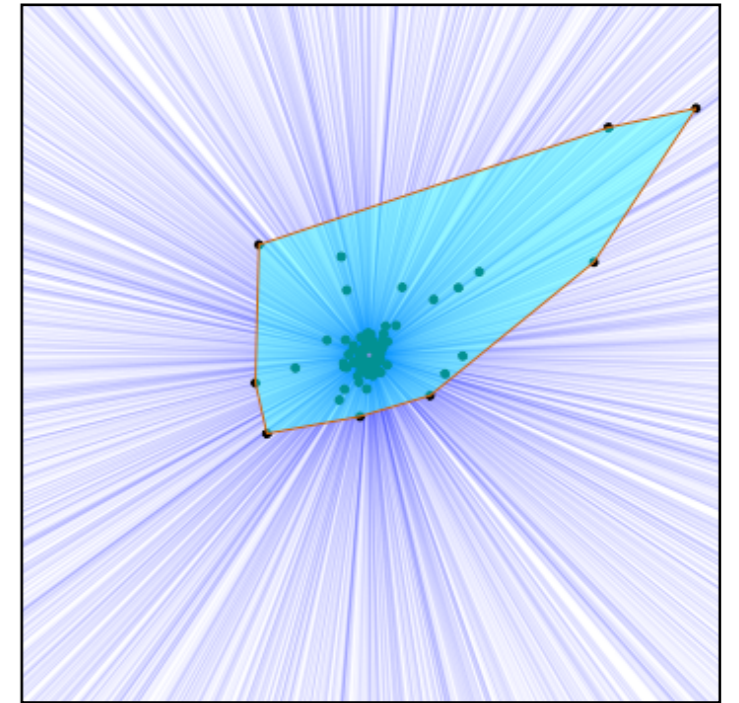
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With Fatou's lemma and a result from [Kabluchko, Marynych, Temesvari and Thäle, PTRF 2019] we get

COROLLARY (Disproof of a conjecture from [Devroye, Toussaint, J. Algorithms 1993])

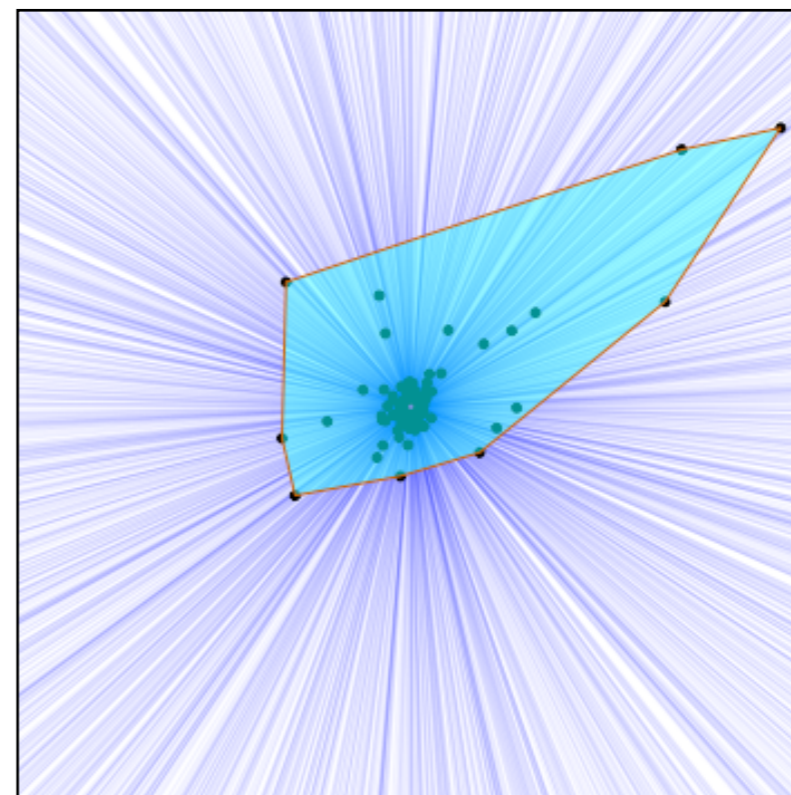
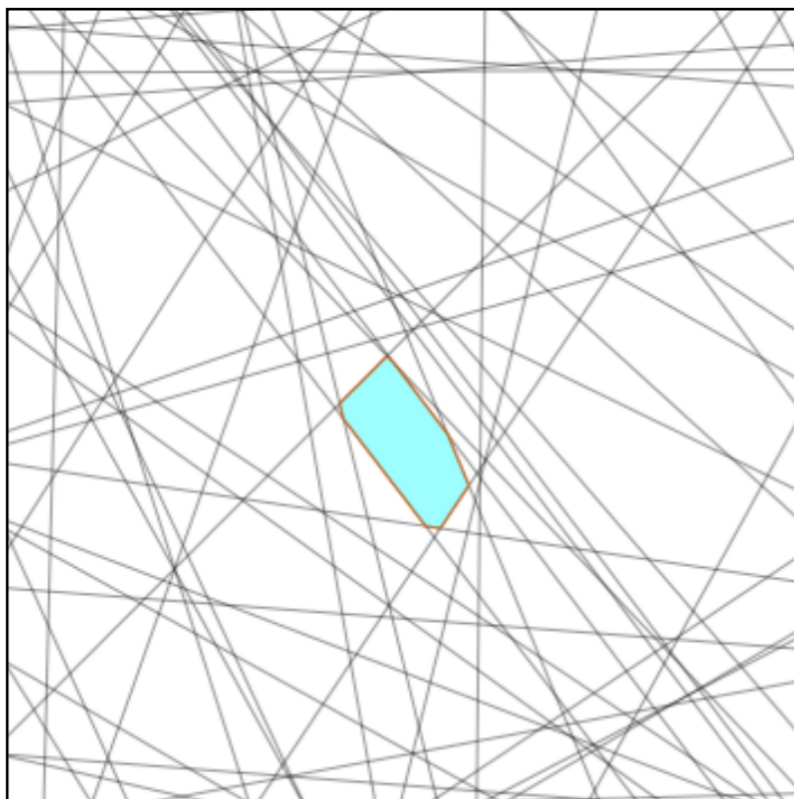
Assume that $d = 2$ and $R = t^{-\frac{d}{d+1}}$. Then

$$\liminf_{t \rightarrow \infty} \mathbb{E}f_0(\text{conv } \Xi_{t,R}) \geq \mathbb{E}f_0(\text{conv } \zeta) = \frac{\pi^2}{2} > 4.$$

ZERO CELL OF A POISSON HYPERPLANE PROCESS

- ▶ Z_γ : Zero cell of a stationary and isotropic Poisson hyperplane process of intensity,
- ▶ $Z_\gamma^\circ = \{x \in \mathbb{R}^d : \langle x, y \rangle \leq 1 \text{ for all } y \in Z_\gamma\}$ its dual.

$$C_d := \frac{1}{d! (\omega_d)^d} \int_{[-1,1]^d} \int_{((S^{d-1} \cap e_d^+)^\circ)^d} [u_1 + z_1 e_d, \dots, u_d + z_d e_d] \sigma_{d-2}^{\otimes d}(d(u_1, \dots, u_d)) d(z_1, \dots, z_d).$$



ZERO CELL OF A POISSON HYPERPLANE PROCESS

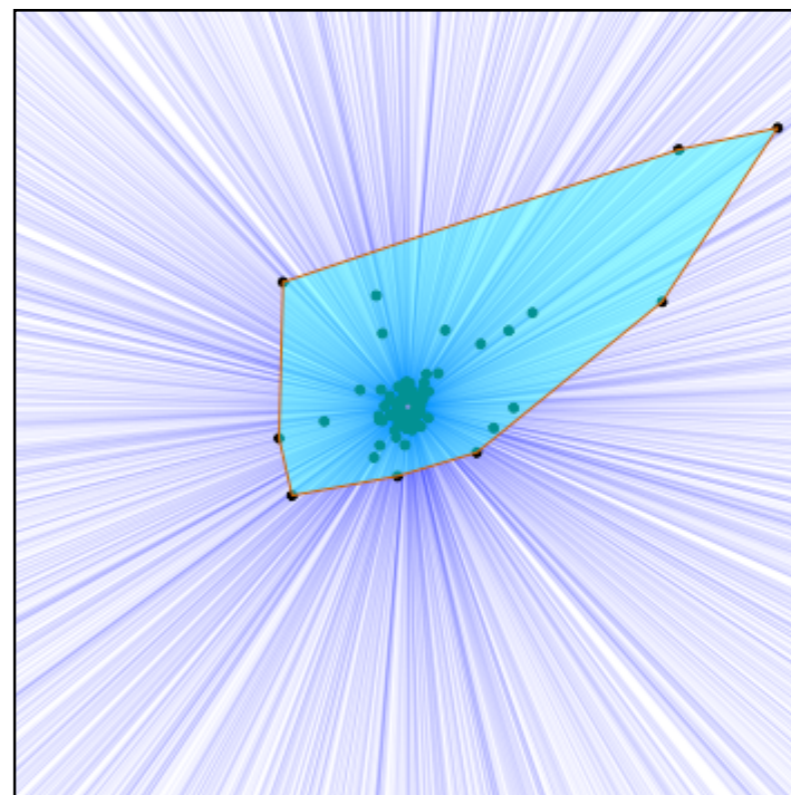
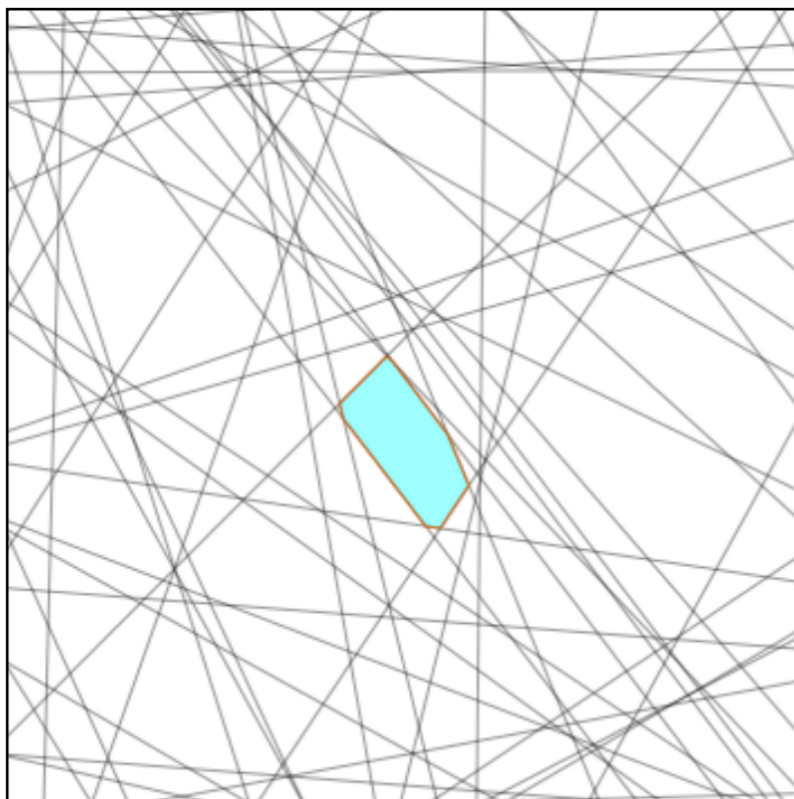
- ▶ Z_γ : Zero cell of a stationary and isotropic Poisson hyperplane process of intensity, γ
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COROLLARY (Convergence of the convex hull to the polar of the zero cell)

Let $R = t^{-\frac{d}{d+1}}$ and put $\gamma_d := \frac{1}{2} C_d \omega_d$. Then $\Xi_{t,R} \xrightarrow{d} Z_{\gamma_d}^\circ$, as $t \rightarrow \infty$.

Moreover, for all $k \in \{0, 1, \dots, d-1\}$, $f_k(\text{conv } \Xi_{t,R}) \xrightarrow{d} f_{d-k-1}(Z_{\gamma_d}^\circ)$, as $t \rightarrow \infty$.



MERCI POUR VOTRE ATTENTION !

REFERENCE

Weak convergence of the intersection point process of Poisson hyperplanes,
Annales de l'Institut Henri Poincaré (B) Probabilités et Statistique (accepté)
A. Baci, G. B. and C. Thäle

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