GERMAN PROBABILITY & STATISTICS DAYS,
MANNHEIM, 27.09.2021 - 01.10.2021

GILLES BONNET
UNIVERSITY OF GRONINGEN, NETHERLANDS

WEAK CONVERGENCE OF THE INTERSECTION POINT PROCESS OF POISSON HYPERPLANES

Joint work with Anastas Baci and Christoph Thäle (arXiv:2007.06398)

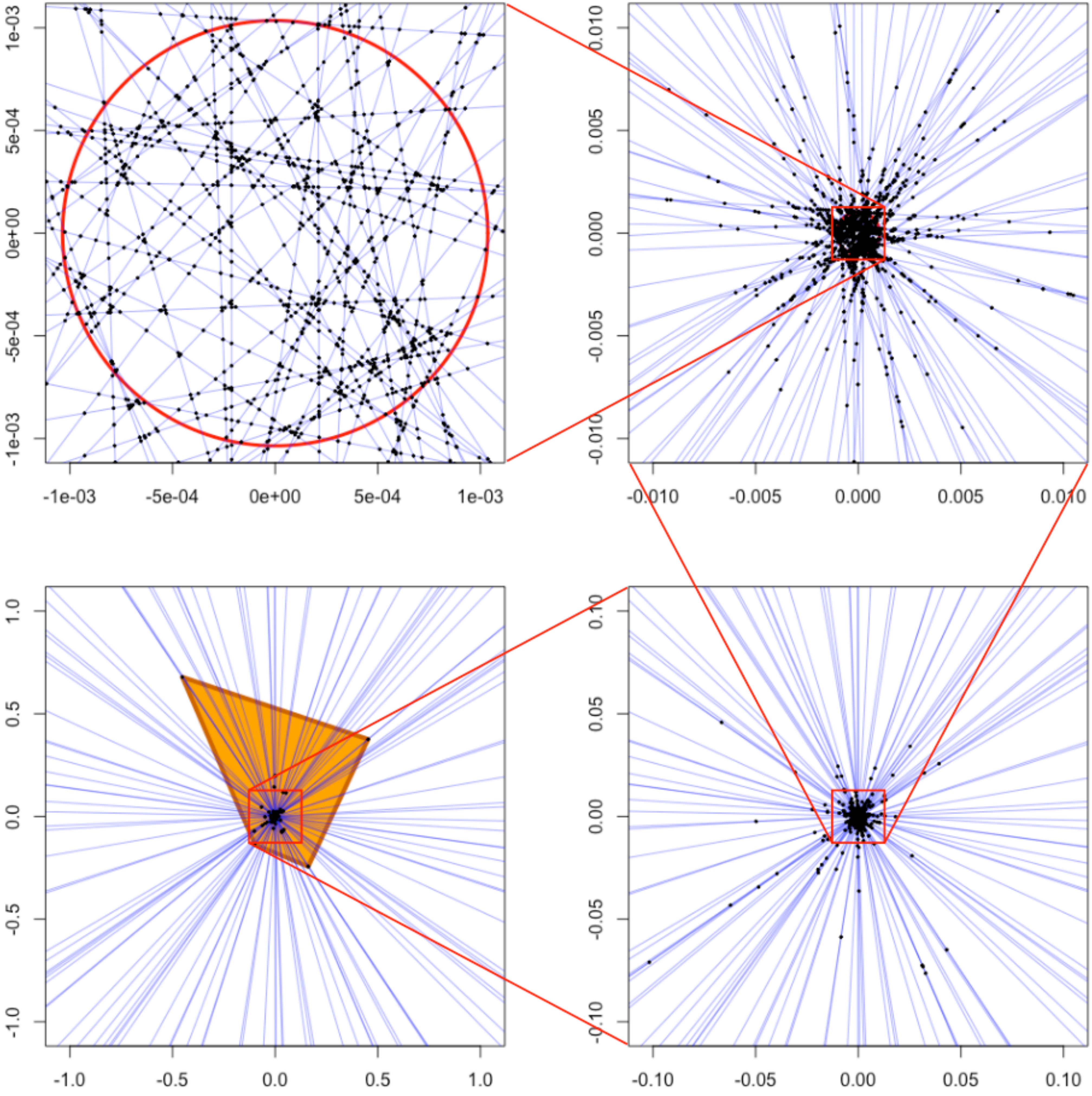
Accepted in Annales de l'Institut Henri Poincaré (B) Probabilités et Statistique

PHD POSITION

- Groningen, Netherlands
- Supervision: me
- In Stochastic Geometry:
  random polytopes/tessellations/graphs/…
- Start: anytime between now and September 2022
- 4 years

$t = 30\,000$

$R = t^{-2/3} \simeq 0.001$



POISSON HYPERPLANE PROCESS (STATIONARY, ISOTROPIC, RESTRICTED)

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3. Construct N i.i.d. hyperplanes intersecting the ball B_R :
    - $U_1, \ldots, U_N \in S^{d-1}$ i.i.d. unit vectors uniformly distributed on the unit sphere,
    - $s_1, \ldots, s_N \in [0,R]$ i.i.d. random numbers uniformly distributed between 0 and R,
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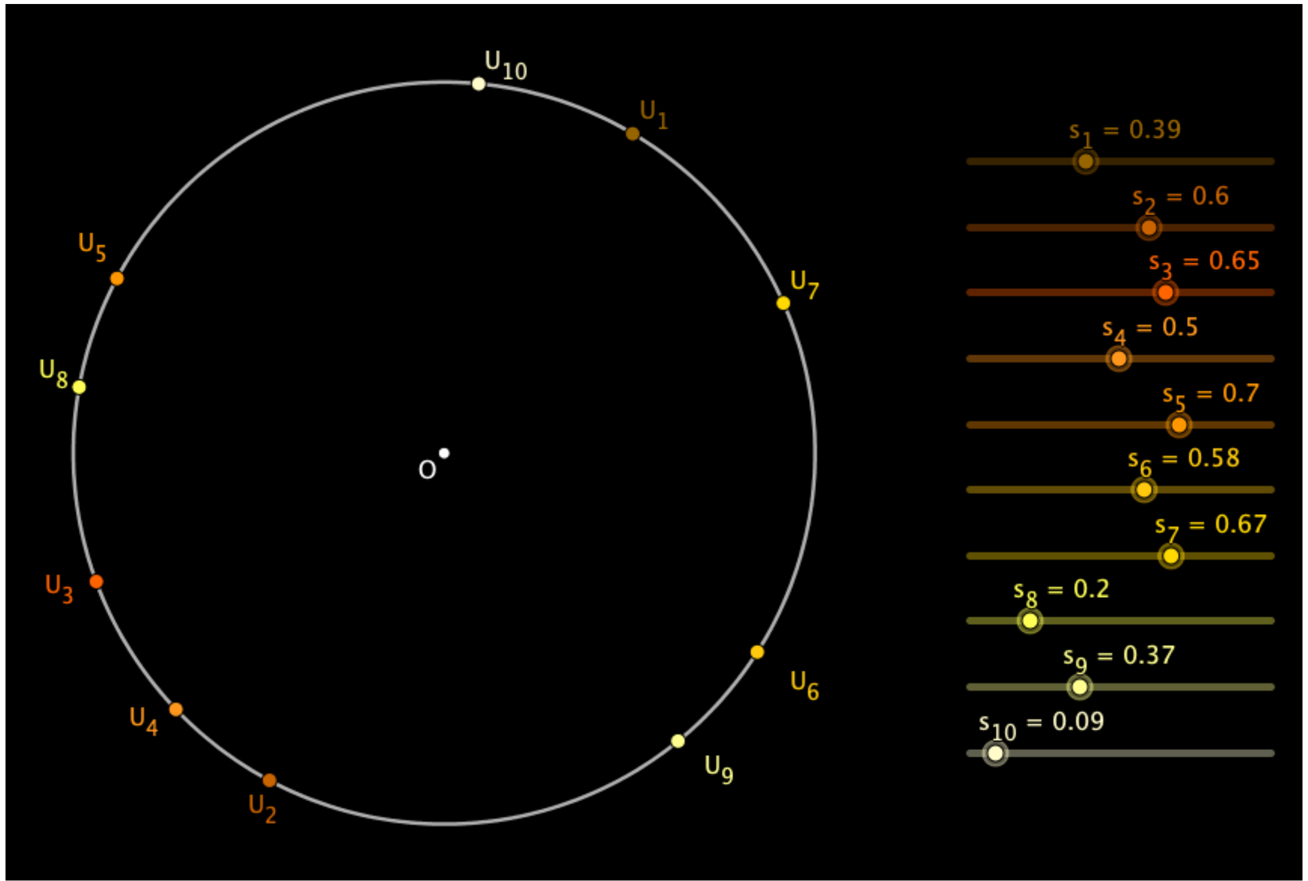

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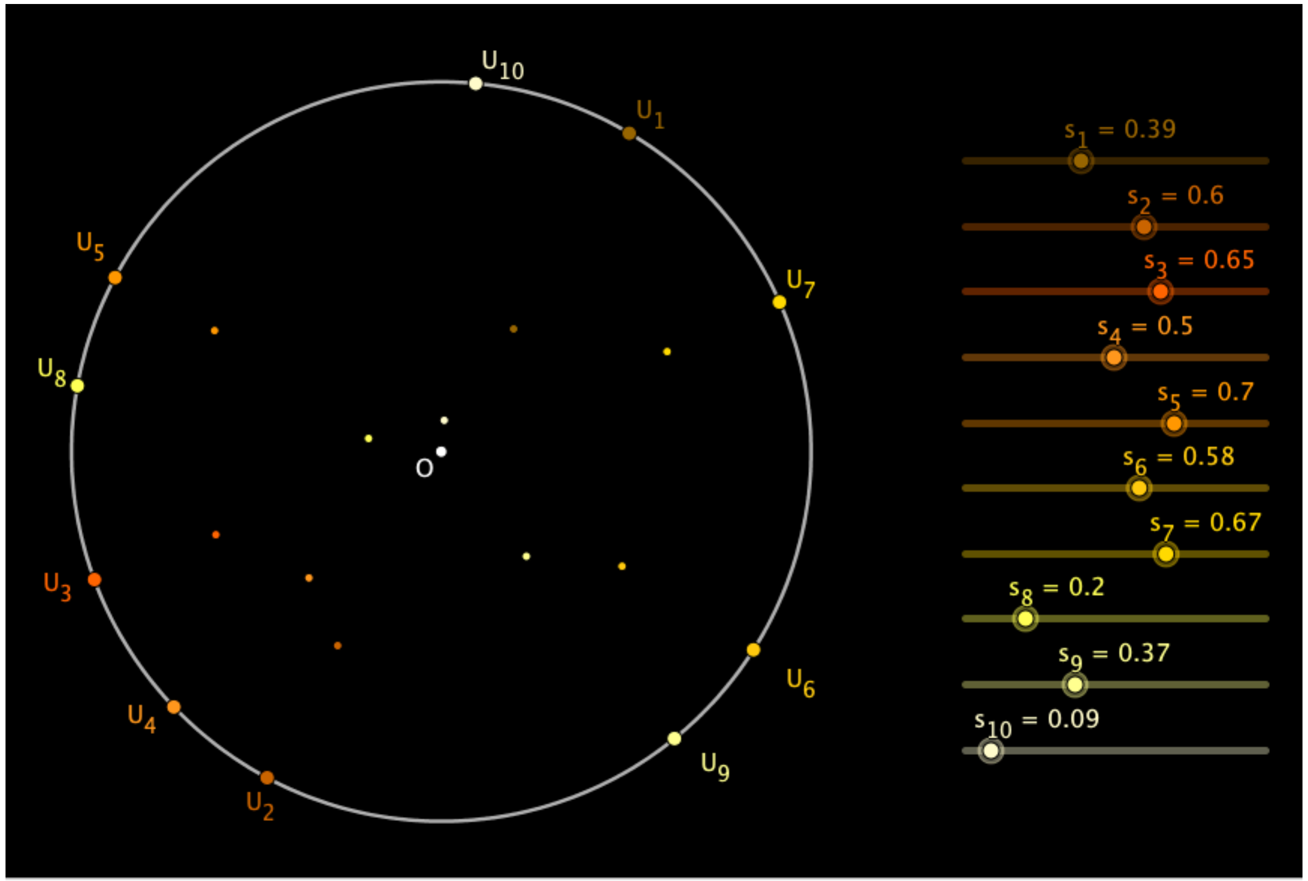

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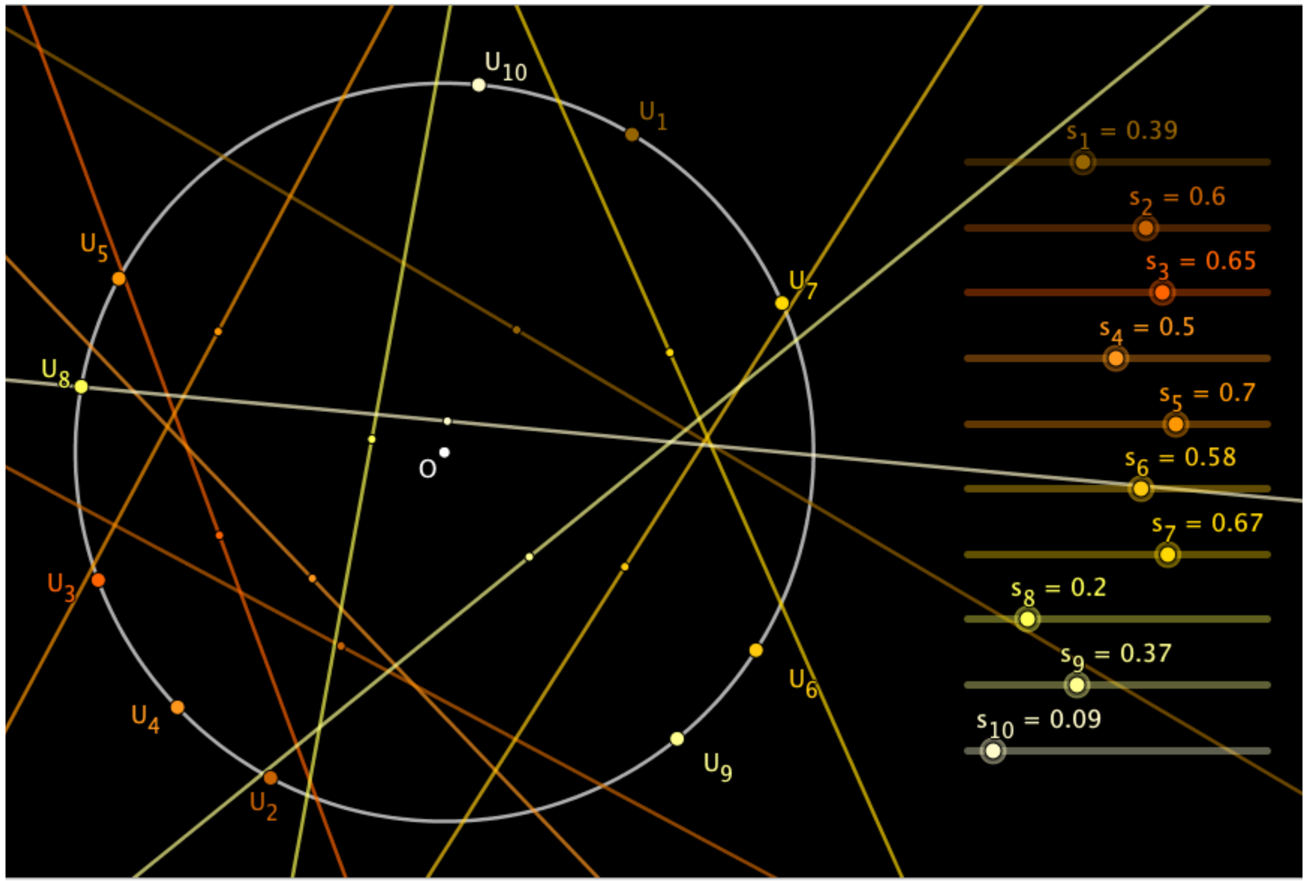

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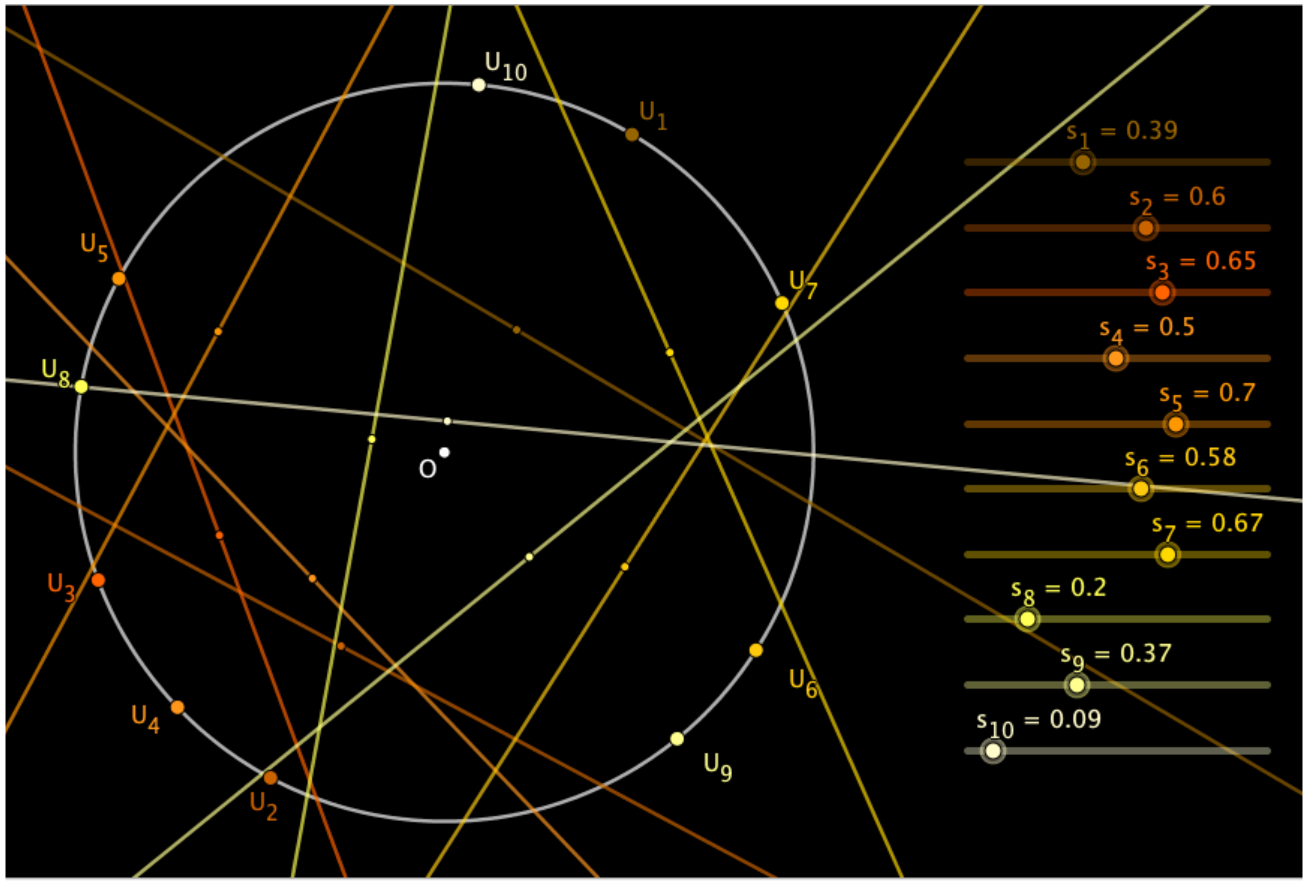
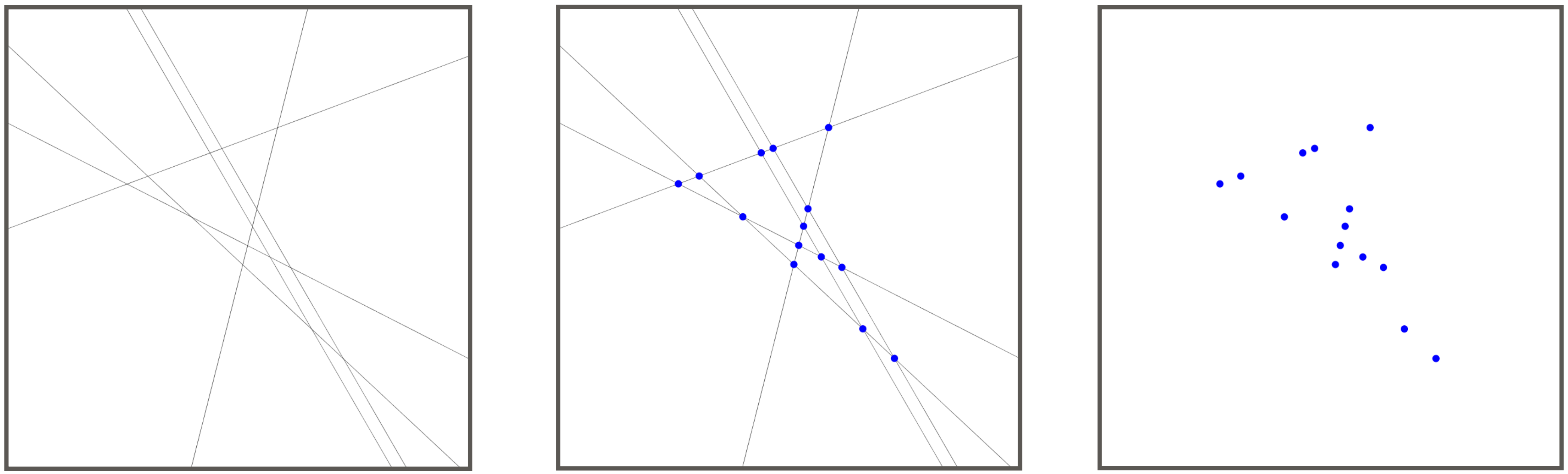


INTERSECTION PROCESS

Let $\eta_{t,R} := \{H_1, \ldots, H_N\}$ as defined earlier.
We now consider the corresponding intersection points :

$$\Xi_{t,R} = \{H_{i_1} \cap \cdots \cap H_{i_d} : 1 \leq i_1 < \cdots < i_d \leq N\}.$$



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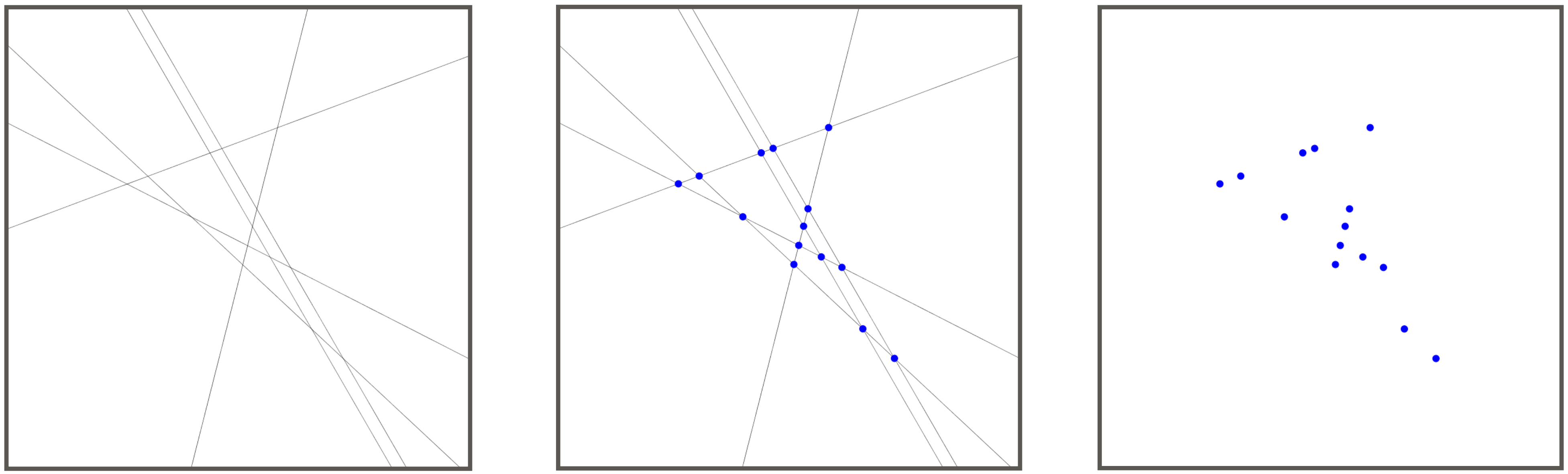

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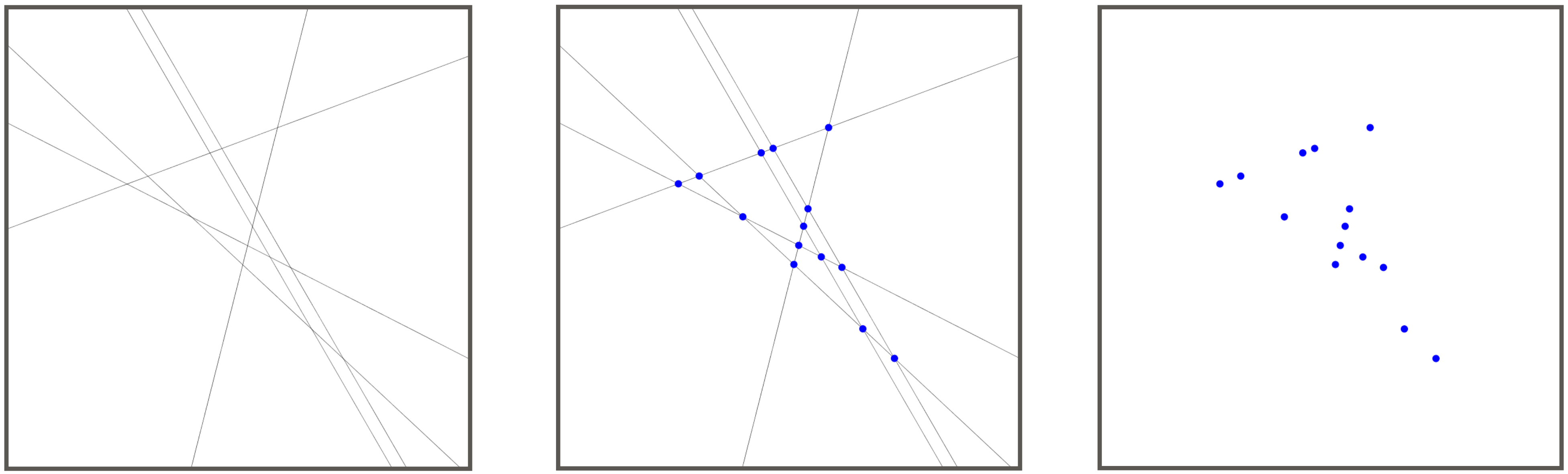

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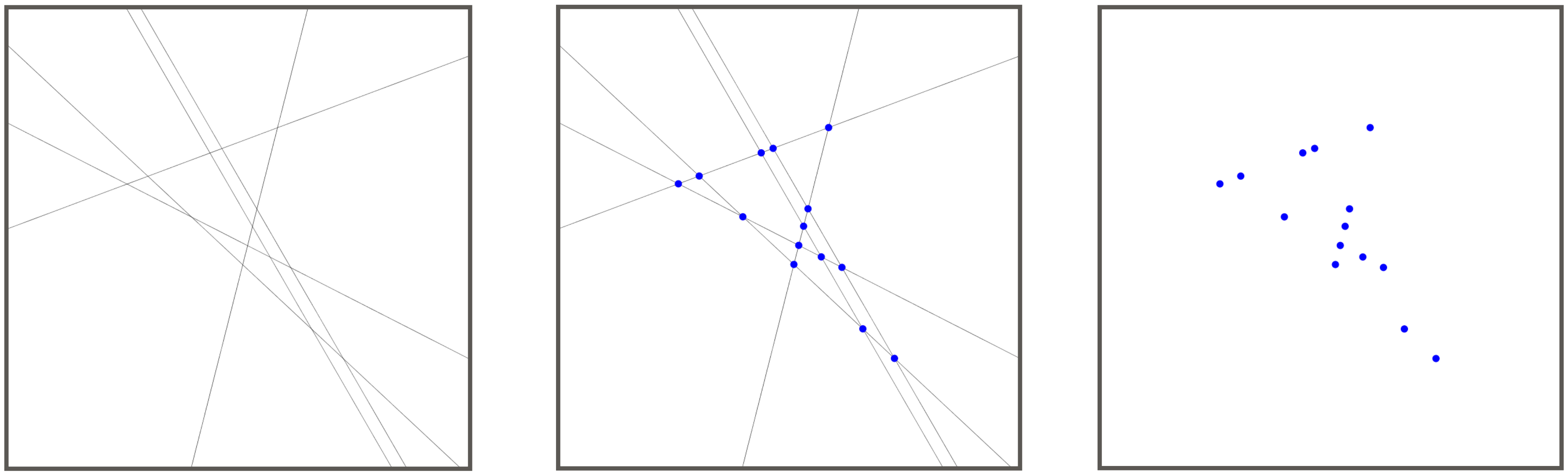

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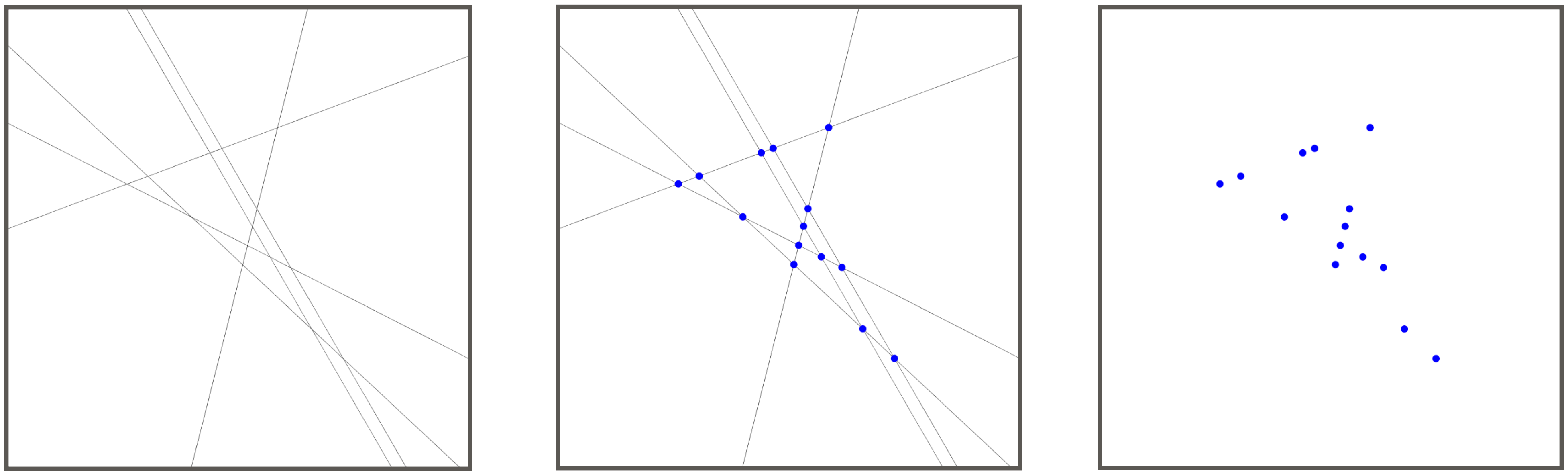

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2nd option (potentially) :

- $t \to \infty$,
- $R = R(t) \to 0$.

INTENSITY MEASURE

The intensity measure $L_{t,R}$ of the intersection process $\Xi_{t,R}$ is the measure on $\mathbb{R}^d$ defined by

$$L_{t,R}(S) = \mathbb{E}(\Xi_{t,R}(S)),$$

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In our paper with use tools from Integral Geometry. For instance, the multivariate Mecke formula gives

$$L_{t,R}(S) = \frac{t^d}{d!} \int_{A(d,d-1)^d} \mathbf{1}(H_1 \cap \ldots \cap H_d \in S) \prod_{i=1}^{d} \mathbf{1}(H_i \cap B_r \neq \emptyset)\, \mu_{d-1}^{\otimes d}(\mathrm{d}(H_1, \ldots, H_d)),$$

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LEMMA (Convergence of the density)

The density $f_{t,R}$ of the intensity measure $L_{t,R}$, with respect to the Lebesgue measure on $\mathbb{R}^d \backslash \{0\}$, satisfies

$$f_{t,R}(x) = \mathbf{1}\{\|x\| < R\} c_1 t^d + \mathbf{1}\{\|x\| \geq R\} R^{d+1} t^d \|x\|^{-(d+1)} \left[c_2 + O\left(\frac{R^2}{\|x\|^2} \right) \right],$$

where c_1 and c_2 are constant depending only on the dimension d.

In particular, if $R = t^{-\frac{d}{d+1}}$, it follows immediately that for any fixed $x \in \mathbb{R}^d \backslash \{0\}$,

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Let M be the measure on $\mathbb{R}^d \setminus \{0\}$ with density $c_2\|x\|^{-(d+1)}$.

We consider the total variation distance between the restricted $L_{t,R_{|(B_r)^c}}$ and $M_{|(B_r)^c}$. This is defined by

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Assume $R = t^{-\frac{d}{d+1}}$. For any $r > 0$,

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Candidate:

Let ζ be the Poisson point process on $\mathbb{R}^d \backslash \{0\}$ with intensity measure M.

DISTANCES BETWEEN SIMPLE POINT PROCESSES

Simple point process = random discrete subset of $\mathbb{R}^d$.

Total variation distance between discrete sets subsets $S_1, S_2 \subset \mathbb{R}^d$:

$$d_{\mathrm{TV}}(S_1, S_2) = \max(\#(S_1 \backslash S_2), \#(S_2 \backslash S_1)) \, .$$

Remark : This is the total variation distance between the counting measures of S_1 and S_2.

Kantorovich-Rubinstein distance between random discrete subsets X_1 and $X_2 \subset \mathbb{R}^2$:

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Recall : ζ is the Poisson point process on $\mathbb{R}^d \setminus \{0\}$ whose intensity measure M has density $x \mapsto c_2 \|x\|^{-(d+1)}$.

THEOREM (Bound on the K-R distance)

Assume that $R = t^{-\frac{d}{d+1}}$ and $0 < R < r < 1$. Then, we have that

$$d_{\mathrm{KR}}(\Xi_{t,R_{|(B_r)^c}}, \zeta_{|(B_r)^c}) \leq c t^{-\frac{1}{d+1}} \ln(t) r^{-3},$$

where c is a positive constant which depends only on d.

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As a corollary, we get :

THEOREM (Convergence in distribution of the point process)

Assume that $R = t^{-\frac{d}{d+1}}$. Then

$$\Xi_{t,R} \xrightarrow{d} \zeta, \quad t \to \infty ,$$

on $\mathbb{R}^d \backslash \{0\}$.

The convergence in distribution means here that $\Xi_{t,R}(B) \xrightarrow{d} \zeta(B)$ for all Borel sets $B \subset \mathbb{R}^d \backslash \{0\}$, relatively compact (in the space $\mathbb{R}^d \backslash \{0\}$) and with boundary of zero Lebesgue measure.

IDEA OF THE PROOF

By Theorem 3.1 of [Decreusefond, Schulte, Thäle, Ann. Probab. 2016], we bound the K-R distance

$$d_{\mathrm{KR}}(\Xi_{t,R_{|(B_r)^c}}, \zeta_{|(B_r)^c}) \leq d_{\mathrm{TV}}(L_{t,R_{|(B_r)^c}}, M_{|(B_r)^c}) + \frac{2^{d+1}}{d!}\rho_{t,R}(r),$$

where $\rho_{t,R}(r) = \max_{1 \leq \ell \leq d-1} I_{\ell,t,R}(r)$, with

$$I_{\ell,t,R}(r) = \int_{[B_r]^\ell} t^\ell \left(t^{d-\ell} \int_{[B_r]^{d-\ell}} \mathbf{1}\big(\|H_1 \cap \ldots \cap H_d\| \geq r \big)\ \mu_{d-1}^{\otimes(d-\ell)}(\mathrm{d}(H_{\ell+1}, \ldots, H_d)) \right)^2 \mu_{d-1}^{\otimes \ell}(\mathrm{d}(H_1, \ldots, H_\ell)),$$

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We have already seen that $d_{\mathrm{TV}}(L_{t,R_{|(B_r)^c}}, M_{|(B_r)^c}) \leq C t^{-\frac{2d}{d+1}} r^{-3}$.

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$$d_{\mathrm{KR}}(\Xi_{t,R_{|(B_r)^c}}, \zeta_{|(B_r)^c}) \leq d_{\mathrm{TV}}(L_{t,R_{|(B_r)^c}}, M_{|(B_r)^c}) + \frac{2^{d+1}}{d!}\rho_{t,R}(r),$$

where $\rho_{t,R}(r) = \max_{1 \leq \ell \leq d-1} I_{\ell,t,R}(r)$, with

$$I_{\ell,t,R}(r) = \int_{[B_r]^\ell} t^\ell \left(t^{d-\ell} \int_{[B_r]^{d-\ell}} \mathbf{1}\big(\|H_1 \cap \ldots \cap H_d\| \geq r \big) \ \mu_{d-1}^{\otimes(d-\ell)}(\mathrm{d}(H_{\ell+1}, \ldots, H_d)) \right)^2 \mu_{d-1}^{\otimes \ell}(\mathrm{d}(H_1, \ldots, H_\ell)),$$

and where $[B_r] = \{H \in A(d, d-1) : H \cap B_r \neq 0\}$.

We have already seen that $d_{\mathrm{TV}}(L_{t,R_{|(B_r)^c}}, M_{|(B_r)^c}) \leq Ct^{-\frac{2d}{d+1}} r^{-3}$.

It remains to deal with the integrals integrals $I_{\ell,t,R}(r)$.

$I_{\ell,t,R}(r) = \ldots = \ldots = \ldots = \ldots = \ldots = \ldots \leq \ldots = \ldots \leq \ldots = \ldots \leq Ct^{-\frac{1}{d+1}} \ln(t) r^{-2}$. ∎

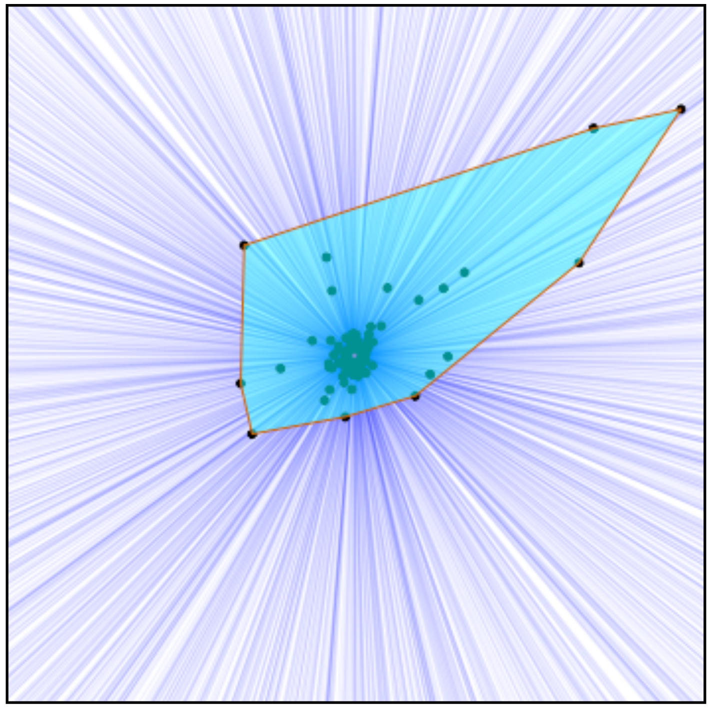
CONVEX HULL

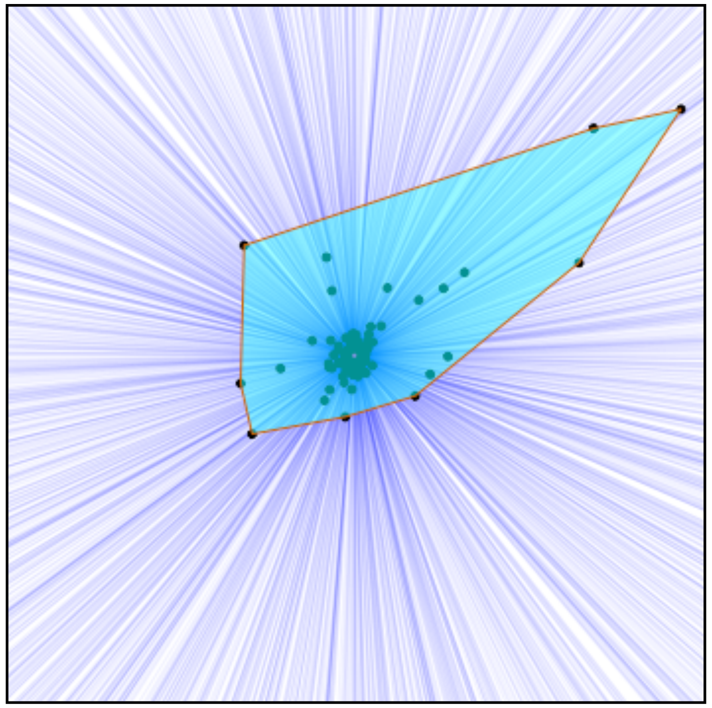
For a set $X \subset \mathbb{R}^d$, we denote :

- $\mathrm{conv}\, X$ its convex hull

For a polytope $P \subset \mathbb{R}^d$ we denote :

- $f_k(P)$ the number of its k-dimensional faces, $k \in \{0, \ldots, d\}$.





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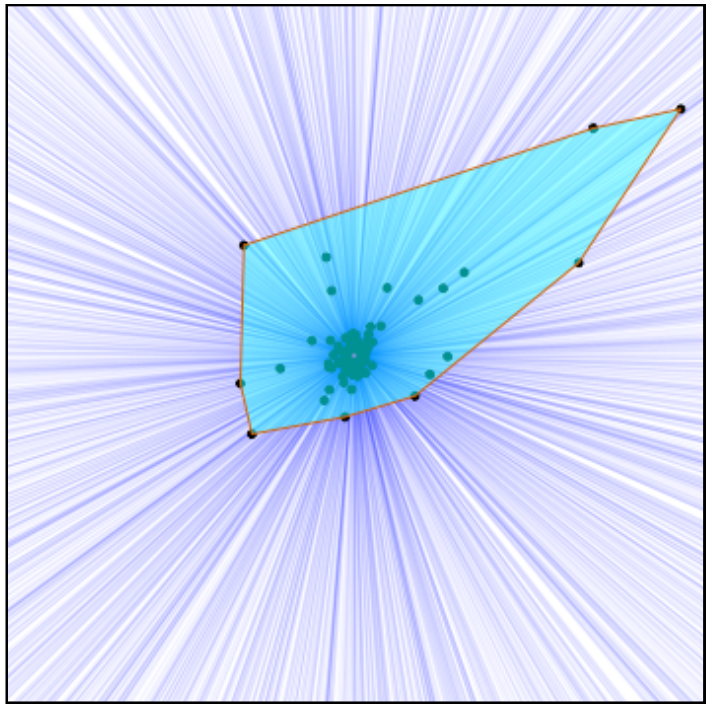
COROLARRY (Convergence of the convex hull)

Assume that $R = t^{-\frac{d}{d+1}}$. Then

$$\mathrm{conv}\, \Xi_{t,R} \xrightarrow{d} \mathrm{conv}\, \zeta, \quad t \to \infty.$$

and, for any $k \in \{0,\ldots,d\}$,

$$f_k(\mathrm{conv}\, \Xi_{t,R}) \xrightarrow{d} f_k(\mathrm{conv}\, \zeta), \quad t \to \infty.$$



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With Fatou's lemma and a result from [Kabluchko, Marynych, Temesvari and Thäle, PTRF 2019] we get

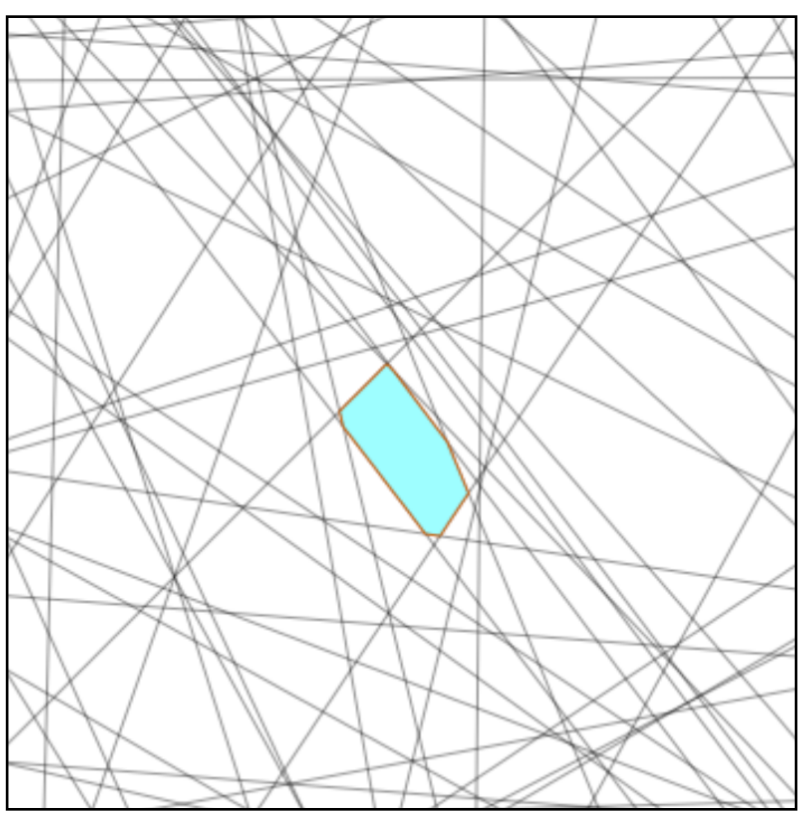
COROLLARY (Disproval of a conjecture from [Devroye,Toussaint, J. Algorithms 1993])
Assume that $d = 2$ and $R = t^{-\frac{d}{d+1}}$. Then

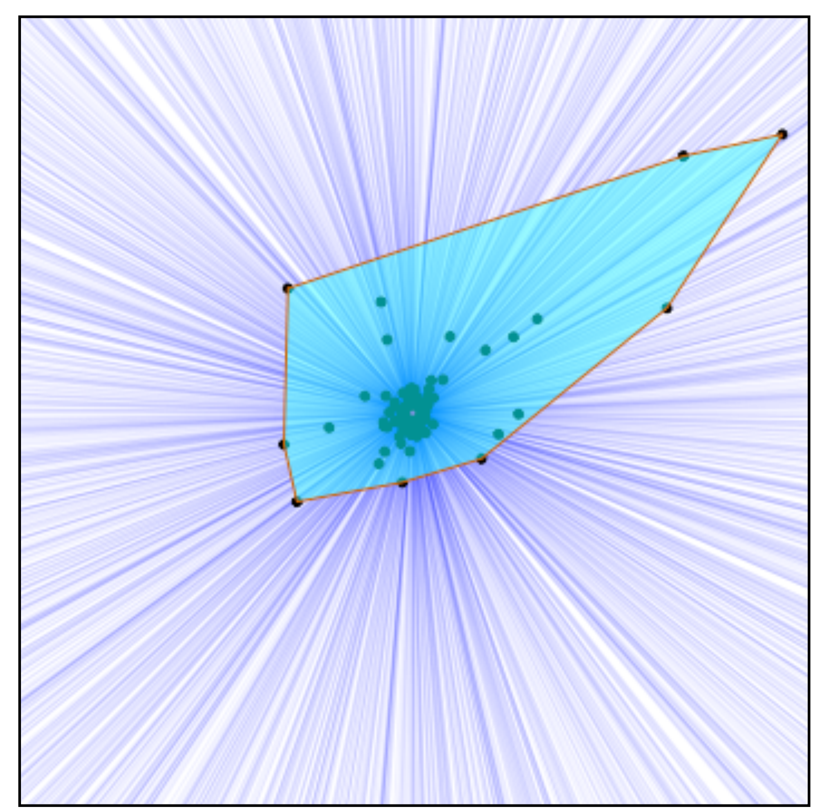
$$\liminf_{t\to\infty} \mathbb{E} f_0(\text{conv } \Xi_{t,R}) \geq \mathbb{E} f_0(\text{conv } \zeta) = \frac{\pi^2}{2} > 4.$$

ZERO CELL OF A POISSON HYPERPLANE PROCESS

- Z_γ : Zero cell of a stationary and isotropic Poisson hyperplane process of intensity,
- $Z_\gamma^\circ = \{x \in \mathbb{R}^d : \langle x, y \rangle \leq 1 \text{ for all } y \in Z_\gamma\}$ its dual.

$$C_d := \frac{1}{d!\,(\omega_d)^d} \int_{[-1,1]^d} \int_{((S^{d-1}\cap e_d^\perp)^d} \left[u_1 + z_1 e_d, \ldots, u_d + z_d e_d\right] \sigma_{d-2}^{\otimes d}(\mathrm{d}(u_1, \ldots, u_d))\, \mathrm{d}(z_1, \ldots, z_d)\,.$$





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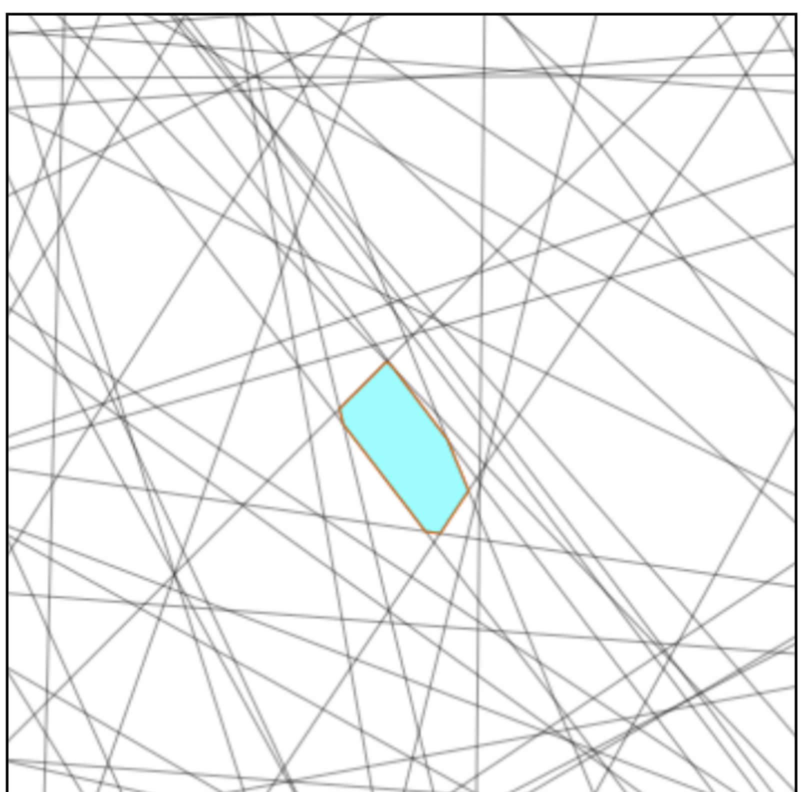
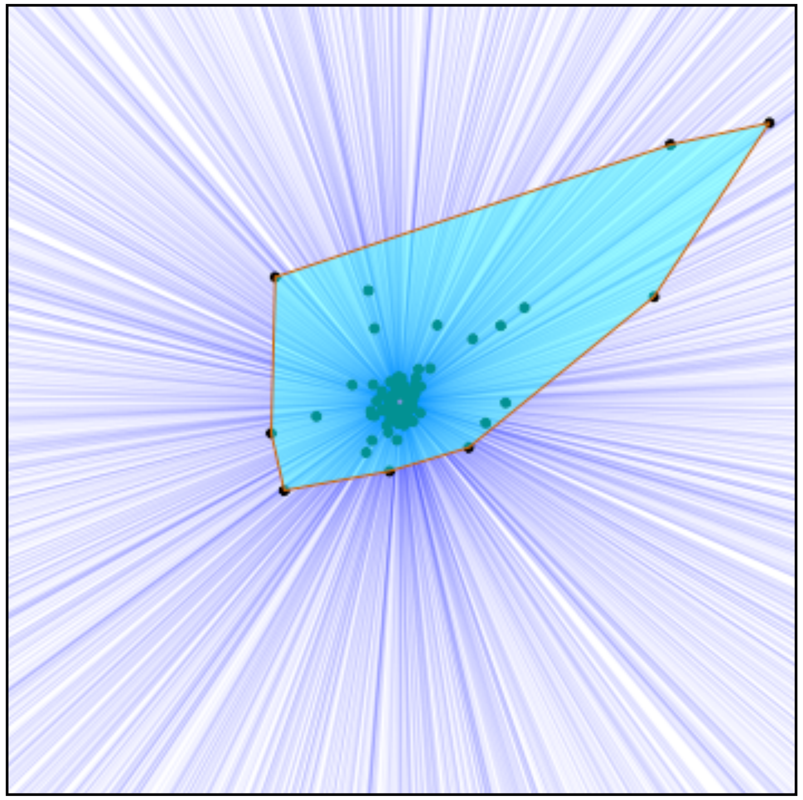
COROLARRY (Convergence of the convex hull to the polar of the zero cell)

Let $R = t^{-\frac{d}{d+1}}$ and put $\gamma_d := \frac{1}{2} C_d \omega_d$. Then $\mathrm{conv}\ \Xi_{t,R} \xrightarrow{d} Z_\gamma^\circ$, as $t \to \infty$.

Moreover, for all $k \in \{0,1,\ldots,d-1\}$, $f_k(\mathrm{conv}\ \Xi_{t,R}) \xrightarrow{d} f_{d-k-1}(Z_{\gamma_d})$, as $t \to \infty$.





THANK YOU FOR WATCHING !

REFERENCE

- Weak convergence of the intersection point process of Poisson hyperplanes, arXiv:2007.06398
  Anastas Baci, **Gilles Bonnet** and Christoph Thäle

IF YOU KNOW SOMEONE INTERESTED LET ME KNOW!

- 4 years PhD
- Groningen, Netherlands
- Supervision: me
- In Stochastic Geometry: random polytopes/tessellations/graphs/…
- Start: anytime between now and September 2022