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Asymptotic Bounds on the Combinatorial Diameter of Random Polytopes

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Probability & Statistics seminar, Groningen, 26 Nov 2021

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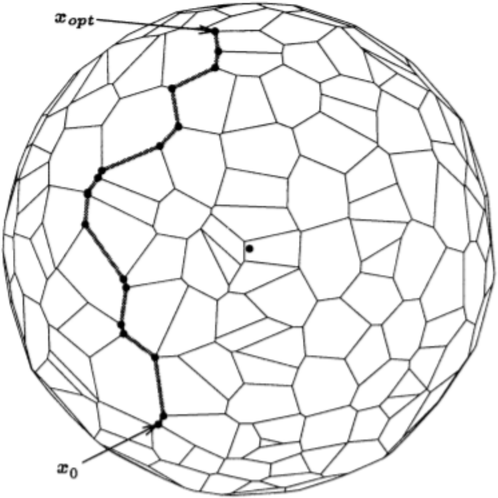
Illustration from [Bogwardt, Huhn '99]



Let $P \subseteq \mathbb{R}^n$ be a polytope with m facets.

$$\text{distance}(x, y) = \min\{k \in \mathbb{N} : \exists [x, v_1],\ [v_1, v_2], \ldots, [v_{k-1}, y] \in \text{Edges}(P)\},$$
$$\text{diam}(P) = \max\{\text{distance}(x, y) : x, y \in \text{Vertices}(P)\},$$

Hirsch conjecture (1957)

$$\text{diam}(P) \leq m - n.$$

Examples:

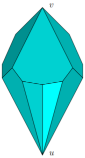
- Simplex: $m = n + 1$ and $\text{diam} = 1 = m - n$,
- Cube: $m = 2n$ and $\text{diam} = n = m - n$,
- Cross-polytope: $m = 2^n$ and $\text{diam} = 2 \ll m - n$.

Santos' counter-example (2012, Annales of Mathematics)

There exists a polytope $P \subseteq \mathbb{R}^{43}$ with 86 facets and $\mathrm{diam}(P) > 43$.

Small improvement [Matschke, Santos, Weibel] (2012, Proc. London Math. Soc.)

There exists a polytope $P \subseteq \mathbb{R}^{20}$ with 40 facets and $\mathrm{diam}(P) = 21$.



Elements of the proof (for both results):

- They construct first a "spindle" (intersection of two polyhedral cones) with diameter greater than the dimension (diameter 6, dimension 5).
- Then they show that it implies the existence of a counter-example.

Santos' construction leads also to the existence of family of polytope violating the conjecture by $\epsilon \geq 5\%$.

Santos famillies of counter-examples (2012)

There exists $n \in \mathbb{N}$, $\epsilon > 0$ and an infinite collection of polytopes $P_k \subseteq \mathbb{R}^n$ with m_k facets and $\operatorname{diam}(P_k) \geq (1+\epsilon)m_k$.
In particular $m_k \to \infty$ and, for k large enough, $\operatorname{diam}(P_k) > m_k - n$.

Polynomial Hirsch Conjecture

The diameter of a polytope is bounded by a polynomial of its dimension and number of facets.

Exponential in n, linear in m

$$\operatorname{diam}(P) \leq 2^{n-3} m.$$ [Barnette ('69, '74) and Larman ('70)]

Quasi-polynomial

$$\operatorname{diam}(P) \leq m^{\log_2 n+1},$$ [Kalai and Kleitman ('92)]

$$\operatorname{diam}(P) \leq (m-n)^{\log_2 n},$$ [Todd ('14)]

$$\operatorname{diam}(P) \leq (m-n)^{\log_2 O(n/\log n)}.$$ [Sukewaga ('19)]

Similar results for graph induced by certain classes of simplicial polytopes:

- Barnette-Larman and Kalai-Kleitman bounds hold for *connected-layer families* [Eisanbrand et al. '10]
- Barnette-Larman bound hold for *pure, normal, pseudo-manifolds without boundary* [Labbé et al. '17].

Classes of polytopes for which the Hirsch conjecture holds

- 0-1 polytopes [Nadef '89]
- Leontief substitution systems [Grinold '71]
- Transportation polyhedra and their duals [Balinski '84] [Brightwell, Heuvel, Stougie '06] [Borgwardt, De Loera, Finhold, '18]
- Fractional stable-set and perfect matching polytopes [Michini, Sassano '14] [Sanità '18]

- ”Well-conditioned” polytopes:
  If P is defined by an integral matrix $A \in \mathbf{Z}^{m \times n}$ with minors less than Δ, then $\mathrm{diam}(P) = O(n^3 \Delta^2 \log \Delta)$. [Dadush, Hähnle, ’16]
- Polytopes with vertices in $\{0, 1, \ldots, k\}^n$
  - $\mathrm{diam}(P) \leq nk$, [Kleinschmidt, Onn ’92]
  - $\mathrm{diam}(P) \leq nk - \lceil n/2 \rceil$ for $k \geq 2$, [Del Pia, Michini ’16]
  - $\mathrm{diam}(P) \leq nk - \lceil 2n/3 \rceil - (k-2)$ for $k \geq 4$, [Deza, Pournin ’18]

Let $A \in \mathbb{R}^{m \times n}$ a random matrix.
Set $P = \{x \in \mathbb{R}^n : Ax \leq \mathbf{1}\} \subseteq \mathbb{R}^n$.
Let $P'(W)$ = orthogonal projection on a fixed plane W.
Shadow-bound = Expected number of edges of $P'(W)$.
Remark: $\mathrm{dist}(a_i, a_j) \leq$ number of edges of $P'(\mathrm{span}(v_1, v_2))$.

i.i.d. rows [Borgwardt '87, '99]

Assume the rows of A are i.i.d.
Shadow-bound $= O(n^2 m^{1/(n-1)})$ for any rotational symmetric distribution.
Shadow-bound $= \Theta(n^2 m^{1/(n-1)})$ for the uniform distribution on the sphere.

Smoothed analysis [Dadush, Huiberts, '19]

Assume that $A = \bar{A} + \sigma G$, where $\bar{A}$ is fixed with rows of ℓ_2 norm at most 1, and G has i.i.d. $\mathcal{N}(0,1)$ entries and $\sigma > 0$.
Shadow-bound $= O(n^2 \sqrt{\log m}/\sigma^2)$, when $\sigma \leq 1/\sqrt{n \log m}$.

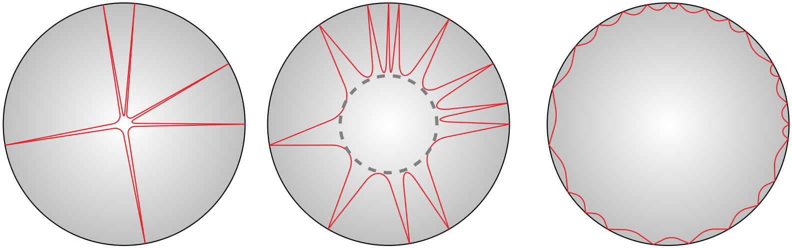
Our main theorem.

Suppose that $n, m \in \mathbb{N}$ satisfy $n \geq 2$ and $m \geq 2^{\Omega(n)}$. Let $A^{\mathsf{T}} := (a_1, \ldots, a_M) \in \mathbb{R}^{n \times M}$, where M is Poisson distributed with $\mathbb{E}[M] = m$, and $a_1, \ldots, a_M$ are sampled independently and uniformly from $\mathbb{S}^{n-1}$. Then, letting $P(A) := \{x \in \mathbb{R}^n : Ax \leq 1\}$, with probability at least $1 - m^{-n}$, we have that

$$\Omega(nm^{\frac{1}{n-1}}) \leq \mathrm{diam}(P(A)) \leq O(n^2 m^{\frac{1}{n-1}} + n^5 4^n).$$

For $m = 2^{\Omega(n^2)}$, $\Omega(nm^{\frac{1}{n-1}}) \leq \mathrm{diam}(P(A)) \leq O(n^2 m^{\frac{1}{n-1}})$.

For the convex hull of m i.i.d. uniform unit vector in $\mathbb{R}^n$ the following picture holds:



(a) sub-exponential regime
$(\ln m)/n \to 0$.
Facets' heights $\simeq O(1/\sqrt{n})$.

(b) exponential regime
$(\ln m)/n \to \alpha$.
Facets' heights $\simeq \sqrt{1 - e^{-\alpha}}$.

(c) super-exponential regime
$(\ln m)/n \to \infty$.
Facets' heights $\simeq 1$.

Let $A \in \mathbb{R}^{m \times n}$ a random matrix with i.i.d. β-distributed rows,

i.e. with density proportional to $\mathbf{1}(a \in \mathbb{B}^n)(1 - \|a\|^2)^\beta$.

Set $P = \{x \in \mathbb{R}^n : Ax \leq \mathbf{1}\} \subseteq \mathbb{R}^n$.

[Bogwardt and Huhn '99] provide a lower bound on the Diameter of $P(A)$.

For $\beta \to -1$ that gives:

Best previous lower bound [Bogwardt and Huhn '99]

Assume that the rows of A are i.i.d. uniform unit vectors, then

$$\Omega\left(\frac{m^{1/(n-1)}}{(m^{1/(n-1)}n)^{\delta(n)}}\right) \leq \mathbb{E}\,\mathrm{diam}(P(A)),$$

where $\delta(n) \to 0$ as $n \to \infty$.

Our results is a triple improvement (when $n \geq 2^{\Omega(n)}$):

1. we remove the denominator $(m^{1/(n-1)}n)^{\delta(n)}$,
2. we add a factor n (Borgwardt and Huhn wrote this would be difficult),
3. we show that it holds with high probability rather then only in expectation.

1-neighbourly random polytope [Bárány, Füredi ’88]

Let $Q(A) = \operatorname{conv}(a_1, \dots, a_m)$ with a_i i.i.d. uniformly distributed in the unit ball. With probability $1 - o(1)$

$$\operatorname{diam}(Q(A)) = 1, \qquad \text{if } m \leq 1.125^n,$$
$$\operatorname{diam}(Q(A)) > 1, \qquad \text{if } m \geq 1.4^n.$$

Shadows of 3-dimensional convex hull [Glisse et al. ’16]

Let $Q(A) = \operatorname{conv}(a_1, \dots, a_M)$ with $a_i \in \mathbb{S}^2$ i.i.d. uniformly distributed in the unit sphere, and M a Poisson random variable with $\mathbb{E}(M) = m$.

1. With high probability, the maximum number of edges in any 2-dimensional projection is $\Theta(\sqrt{m})$.
2. In particular $\operatorname{diam}(Q(A)) = O(\sqrt{m})$.

$Q(A) = \text{conv}(a_1, \ldots, a_m) \text{``}=\text{''} \{x : \langle x, y\rangle \leq 1, \forall y \in Q(A)\} = P(A)^\circ$
$P(A) = Q(A)^\circ$

The proof of our main theorem (for $P(A)$) leads to similar bounds for $Q(A)$, smaller by a factor n.

Theorem (Convex hull)

Suppose that $n, m \in \mathbb{N}$ satisfy $n \geq 2$ and $m \geq 2^{\Omega(n)}$. Let M be Poisson distributed with $\mathbb{E}[M] = m$, and $a_1, \ldots, a_M$ i.i.d. uniform unit vectors. Then, letting $Q(A) := \text{conv}(a_1, \ldots, a_M)$, with probability at least $1 - m^{-n}$, we have that

$$\Omega(m^{\frac{1}{n-1}}) \leq \text{diam}(Q(A)) \leq O(nm^{\frac{1}{n-1}} + n^5 4^n).$$

Improvement we hope to achieve... soon

With the same setting as above

$$\Omega(\sqrt{n}m^{\frac{1}{n-1}}) \leq \text{diam}(Q(A)) \leq O(\sqrt{n}m^{\frac{1}{n-1}}).$$

Theorem (Eisenbrand, Hähnle, Razborov and Rothvoß '10)

Let $G = (V, E)$ be a connected graph, where the vertices V of G are subsets of $\{1, \dots, k\}$ of cardinality n and the edges E of G are such that for each $u, v \in V$ there exists a path connecting u and v whose intermediate vertices all contain $u \cap v$. Then the following upper bounds on the diameter of G hold:

$2^{n-1} \cdot k - 1$ *(Barnette–Larman),* $k^{1+\log n} - 1$ *(Kalai–Kleitman).*

Let $A = \{a_1, ..., a_m\} \subseteq \mathbb{S}^{n-1}$ be in general position. For a vertex $x \in P(A)$, we denote $A_x = \{a \in A : \langle a, x \rangle = 1\}$. Consider the following sets

$$V = \{A_x : x \text{ is a vertex of } P(A)\}, \quad E = \{\{A_x, A_y\} : [x, y] \text{ is an edge of } P(A)\}.$$

$$\operatorname{diam}(P(A)) \leq 2^{n-1} m - 1, \qquad \operatorname{diam}(P(A)) \leq m^{1+\log n} - 1.$$

These are almost the bounds presented earlier:

$$\operatorname{diam}(P(A)) \leq 2^{n-3} \cdot m, \qquad \operatorname{diam}(P(A)) \leq m^{\log_2 n + 1}.$$

1. Choose a *scale* $\varepsilon = \varepsilon(n, m)$ at which, with high probability,
   - $\{a_1, \ldots, a_M\} \subseteq \mathbb{S}^{n-1}$ is sufficiently dense: $C(v, \varepsilon) \cap A \neq \emptyset$ for any $v \in \mathbb{S}^{n-1}$,
   - $\{a_1, \ldots, a_M\} \subseteq \mathbb{S}^{n-1}$ is not too dense: $\#(C(v, t\varepsilon) \cap A) \leq 45 \log(p) t^{n-1}$ for any $t \geq 1$ and $0 < p < m^{-n}$.
2. Bound the diameter of the shadow on an arbitrary plane W
   - Let $w_1, w_2, \ldots, w_{1/\varepsilon} = w_1 \in \mathbb{S}^{n-1} \cap W$.
   - (localization) Show that the shadow path "between" w_i and w_{i+1} is determined (whp) by $A \cap C(w_i, 6\varepsilon)$.
   - Use the abstract diameter bound to bound the length of the local shadow.
   - Sum the local contributions
3. Bound the diameter
   - Let $N \subseteq \mathbb{S}^{n-1}$ be a fixed minimal ε-net.
   - Bound the shadow diameter for any plane $W = \mathrm{span}(e_1, v)$, $v \in N$.
   - Connect the a_i to the closest shadow path.

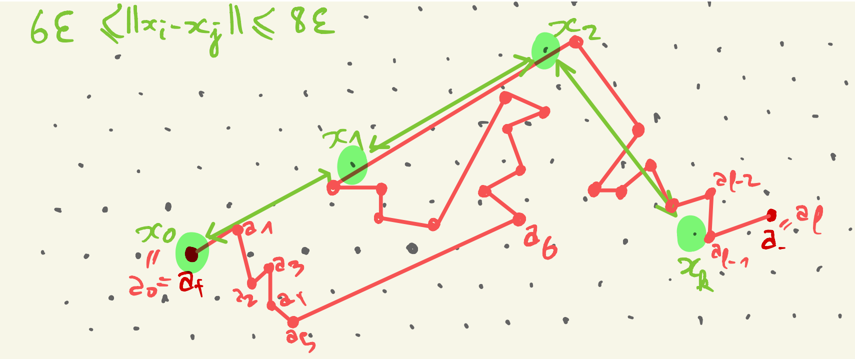
Quantify carefully everything... and you are done!

Lemma

For $n \geq 2$, let $P \subseteq \mathbb{R}^n$ be a simple bounded polytope containing the origin in its interior and let $Q = P^\circ := \{x \in \mathbb{R}^n : \langle x, y \rangle \leq 1, \forall y \in P\}$ denote the polar of P. Then,

$$\mathrm{diam}(P) \geq (n-1)(\mathrm{diam}(Q) - 2).$$

Thus, we only need a lower bound for $\mathrm{diam}(Q(A))$.



- Let $a_+, a_- \in A$ such that $\|a_+ - a_-\| \geq 1$.
- Let $a_+ \in N \subseteq \mathbb{S}^{n-1}$ be a minimal ε-net, $\varepsilon = cm^{-1/(n-1)}$.
- Set $x_0 = a_+$ and for any $k \in \mathbb{N}$,

$$X_k = \{\mathbf{x} \in N^k : x_i \neq x_j \text{ and } 6\varepsilon \leq \|x_i - x_{i+1}\| \leq 8\varepsilon\}$$

1. Setting
   - Let $a_+, a_- \in A$ such that $\|a_+ - a_-\| \geq 1$.
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2. Intermediate lemmas:
   - $\#X_k \leq (17^{n-1})^k$
   - For any path $[a_0, a_1], [a_1, a_2], \ldots, [a_{\ell-1}, a_\ell]$ with $a_0 = a_+$ and $a_\ell = a_-$, there exists $k \geq k_0 = \Omega(1/\varepsilon)$ and $\mathbf{x} \in X_k$ such that, for any $i \in [k]$, there exists $j \in [\ell]$ and $x \in [a_{j-1}, a_j]$ such that $x/\|x\| \in C(x_i, \varepsilon)$
   - It implies

$$\operatorname{diam}(Q(A)) \geq \min_{k \geq k_0} \min_{\mathbf{x} \in X_k} \sum_{0 \leq i \leq k-1} \mathbf{1}(C(x_i, \varepsilon/2) \cap A \neq \emptyset) \mathbf{1}(C(x_{i+1}, \varepsilon/2) \cap A \neq \emptyset).$$

3. Conclude with the union bound.

THANK YOU!