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Bounds on the Combinatorial Diameter of Random Polytopes

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- 1 Diameter of polytopes: Hirsch conjecture**
- 2 Bounds for certain classes of deterministic polytopes**
- 3 Bounds for certain classes of random polytopes**
- 4 Proof of our upper bound**
- 5 Proof of our lower bound**



1 Diameter of polytopes: Hirsch conjecture

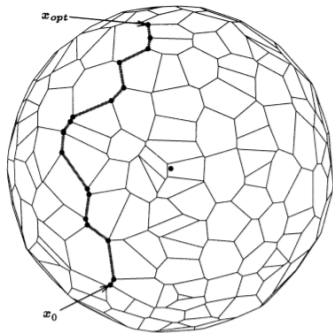
2 Bounds for certain classes of deterministic polytopes

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Linear programming:



- Polytope
= set of linear constraints $\langle a_i, x \rangle \leq b_i$
= set of *feasible solution*,
- One wants to maximize a linear objective function.
- The *simplex algorithm* searches the optimal vertex by exploring the *Vertices/Edges graph* of the polytope.
- What is the longest distance between vertices?



Let $P \subseteq \mathbb{R}^n$ be a polytope with m facets.

$$\text{distance}(x, y) = \min\{k \in \mathbb{N} : \exists [x, v_1], [v_1, v_2], \dots, [v_{k-1}, y] \in \text{Edges}(P)\},$$

$$\text{diam}(P) = \max\{\text{distance}(x, y) : x, y \in \text{Vertices}(P)\},$$

Hirsch conjecture (1957)

$$\text{diam}(P) \leq m - n$$



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Examples:

■ Simplex: $m = n + 1$ $\text{diam} = 1 = m - n$



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Examples:

- Simplex: $m = n + 1$ $\text{diam} = 1 = m - n$
- Cube: $m = 2n$ $\text{diam} = n = m - n$
- Cross-polytope: $m = 2^n$ $\text{diam} = 2 \ll m - n$



Santos' counter-example (2012, Annales of Mathematics)

There exists a polytope $P \subseteq \mathbb{R}^{43}$ with 86 facets and $\text{diam}(P) > 43 = 86 - 43$.



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Small improvement [Matschke, Santos, Weibel] (2012, Proc. London Math. Soc.)

There exists a polytope $P \subseteq \mathbb{R}^{20}$ with 40 facets and $\text{diam}(P) = 21 > 40 - 20$.



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There exists a polytope $P \subseteq \mathbb{R}^{43}$ with 86 facets and $\text{diam}(P) > 43 = 86 - 43$.

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There exists a polytope $P \subseteq \mathbb{R}^{20}$ with 40 facets and $\text{diam}(P) = 21 > 40 - 20$.

Elements of the proof (for both results):

- They construct first a “spindle” (intersection of two polyhedral cones) with diameter greater than the dimension (diameter 6, dimension 5).
- Then they show that it implies the existence of a counter-example.





Santos' construction leads also to the existence of family of polytope violating the conjecture by $\epsilon \geq 5\%$.

Santos families of counter-examples (2012)

There exists $n \in \mathbb{N}$, $\epsilon > 0$ and an infinite collection of polytopes $(P_k)_k$, such that

1. $P_k \subseteq \mathbb{R}^n$
2. $\text{diam}(P_k) \geq (1 + \epsilon)m_k$.

In particular, $\text{diam}(P_k) > m_k - n$.



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Polynomial Hirsch Conjecture

The diameter of a polytope is bounded by a polynomial of its dimension and number of facets.



Exponential in n , linear in m

$$\text{diam}(P) \leq 2^{n-3}m. \quad [\text{Barnette ('69, '74)}] \text{ and } [\text{Larman ('70)}]$$

Quasi-polynomial

$$\text{diam}(P) \leq m^{\log_2 n + 1}, \quad [\text{Kalai and Kleitman ('92)}]$$

$$\text{diam}(P) \leq (m - n)^{\log_2 n}, \quad [\text{Todd ('14)}]$$

$$\text{diam}(P) \leq (m - n)^{\log_2 O(n/\log n)}. \quad [\text{Sukewaga ('19)}]$$



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Similar results for graph induced by certain classes of simplicial polytopes:

- Barnette-Larman and Kalai-Kleitman bounds hold for *connected-layer families* [Eisanbrand et al. '10]
- Barnette-Larman bound hold for *pure, normal, pseudo-manifolds without boundary* [Labbé et al. '17].



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- 0-1 polytopes [Nadef '89]
- Leontief substitution systems [Grinold '71]
- Transportation polyhedra and their duals [Balinski '84] [Brightwell, Heuvel, Stougie '06] [Borgwardt, De Loera, Finhold, '18]
- Fractional stable-set and perfect matching polytopes [Michini, Sassano '14] [Sanità '18]



- "Well-conditioned" polytopes:

If P is defined by an integral matrix $A \in \mathbf{Z}^{m \times n}$ with minors less than Δ , then $\text{diam}(P) = O(n^3 \Delta^2 \log \Delta)$. [Dadush, Hähnle, '16]

- Polytopes with vertices in $\{0, 1, \dots, k\}^n$

- $\text{diam}(P) \leq nk$, [Kleinschmidt, Onn '92]

- $\text{diam}(P) \leq nk - \lceil n/2 \rceil$ for $k \geq 2$, [Del Pia, Michini '16]

- $\text{diam}(P) \leq nk - \lceil 2n/3 \rceil - (k - 2)$ for $k \geq 4$, [Deza, Pournin '18]



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Let $A \in \mathbb{R}^{m \times n}$ a random matrix.

Set $P = \{x \in \mathbb{R}^n : Ax \leq \mathbf{1}\} \subseteq \mathbb{R}^n$. \rightsquigarrow at most m facets.

Let $P'(W)$ = orthogonal projection on a fixed plane W .

Shadow-bound = Expected number of edges of $P'(W)$.

Remark: $\text{dist}(v_1, v_2) \leq$ number of edges of $P'(\text{span}(v_1, v_2))$, for any vertices $v_1, v_2 \in P$.



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i.i.d. rows [Borgwardt '87, '99]

Assume the rows of A are i.i.d.

Shadow-bound = $O(n^2 m^{1/(n-1)})$ for any rotational symmetric distribution.

Shadow-bound = $\Theta(n^2 m^{1/(n-1)})$ for the uniform distribution on the sphere.



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Smoothed analysis [Dadush, Huiberts, '19]

Assume that $A = \bar{A} + \sigma G$, where \bar{A} is fixed with rows of ℓ_2 norm at most 1, and G has i.i.d. $\mathcal{N}(0, 1)$ entries and $\sigma > 0$.

Shadow-bound = $O(n^2 \sqrt{\log m} / \sigma^2)$, when $\sigma \leq 1 / \sqrt{n \log m}$.

Our main theorem.

Suppose that $n, m \in \mathbb{N}$ satisfy $n \geq 2$ and $m \geq 2^{\Omega(n)}$.

Let $A^\top := (a_1, \dots, a_M) \in \mathbb{R}^{n \times M}$, where $M \sim \text{Poisson}(m)$, and a_1, \dots, a_M are sampled independently and uniformly from \mathbb{S}^{n-1} .

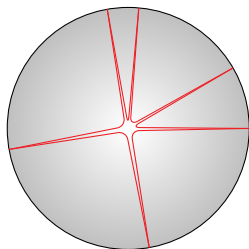
Set $P(A) := \{x \in \mathbb{R}^n : Ax \leq 1\}$.

Then, with probability at least $1 - m^{-n}$, we have that

$$\Omega(nm^{\frac{1}{n-1}}) \leq \text{diam}(P(A)) \leq O(n^2 m^{\frac{1}{n-1}} + n^5 4^n).$$

For $m = 2^{\Omega(n^2)}$, $\Omega(nm^{\frac{1}{n-1}}) \leq \text{diam}(P(A)) \leq O(n^2 m^{\frac{1}{n-1}})$.

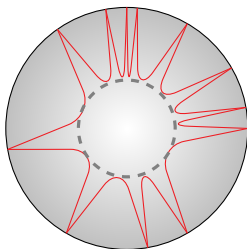
For the convex hull of m i.i.d. uniform unit vector in \mathbb{R}^n the following picture holds:



(a) sub-exponential regime

$$(\ln m)/n \rightarrow 0.$$

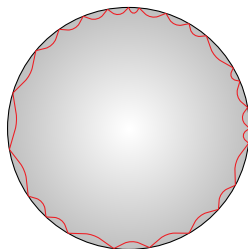
Facets' heights $\simeq O(1/\sqrt{n})$.



(b) exponential regime

$$(\ln m)/n \rightarrow \alpha.$$

Facets' heights $\simeq \sqrt{1 - e^{-\alpha}}$.



(c) super-exponential regime

$$(\ln m)/n \rightarrow \infty.$$

Facets' heights $\simeq 1$.



Let $A \in \mathbb{R}^{m \times n}$ a random matrix with i.i.d. β -distributed rows,
i.e. with density proportional to $\mathbf{1}(a \in \mathbb{B}^n)(1 - \|a\|^2)^\beta$.

Set $P = \{x \in \mathbb{R}^n : Ax \leq \mathbf{1}\} \subseteq \mathbb{R}^n$.

[Bogwardt and Huhn '99] provide a lower bound on the Diameter of $P(A)$.



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For $\beta \rightarrow -1$ that gives:

Best previous lower bound [Bogwardt and Huhn '99]

Assume that the rows of A are i.i.d. uniform unit vectors, then

$$\Omega \left(\frac{m^{1/(n-1)}}{(m^{1/(n-1)}n)^{\delta(n)}} \right) \leq \mathbb{E} \text{diam}(P(A)),$$

where $\delta(n) \rightarrow 0$ as $n \rightarrow \infty$.



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where $\delta(n) \rightarrow 0$ as $n \rightarrow \infty$.

Our results is a triple improvement (when $n \geq 2^{\Omega(n)}$):

1. we **remove the denominator** $(m^{1/(n-1)}n)^{\delta(n)}$,
2. we **add a factor n** (Bogwardt and Huhn wrote this would be difficult),
3. we show that **it holds with high probability** rather than only in expectation.

**1-neighbourly random polytope [Bárány, Füredi '88]**

Let $Q(A) = \text{conv}(a_1, \dots, a_m)$ with a_i i.i.d. uniformly distributed in the unit ball.

With probability $1 - o(1)$

$$\begin{array}{ll} \text{diam}(Q(A)) = 1, & \text{if } m \leq 1.125^n, \\ \text{diam}(Q(A)) > 1, & \text{if } m \geq 1.4^n. \end{array}$$

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Shadows of 3-dimensional convex hull [Glisse et al. '16]

Let $Q(A) = \text{conv}(a_1, \dots, a_M)$ with $a_i \in \mathbb{S}^2$ i.i.d. uniformly distributed in the unit sphere, and M a Poisson random variable with $\mathbb{E}(M) = m$.

1. With high probability, the maximum number of edges in any 2-dimensional projection is $\Theta(\sqrt{m})$.
2. In particular $\text{diam}(Q(A)) = O(\sqrt{m})$.



$$Q(A) = \text{conv}(a_1, \dots, a_m) = \{x : \langle x, y \rangle \leq 1, \forall y \in Q(A)\} = P(A)^\circ$$

$$P(A) = Q(A)^\circ$$

The proof of our main theorem (for $P(A)$) leads to similar bounds for $Q(A)$, smaller by a factor n .

Theorem (Convex hull)

Suppose that $n, m \in \mathbb{N}$ satisfy $n \geq 2$ and $m \geq 2^{\Omega(n)}$. Let M be Poisson distributed with $\mathbb{E}[M] = m$, and a_1, \dots, a_M i.i.d. uniform unit vectors. Then, letting $Q(A) := \text{conv}(a_1, \dots, a_M)$, with probability at least $1 - m^{-n}$, we have that

$$\Omega(m^{\frac{1}{n-1}}) \leq \text{diam}(Q(A)) \leq O(nm^{\frac{1}{n-1}} + n^5 4^n).$$



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$$\Omega(m^{\frac{1}{n-1}}) \leq \text{diam}(Q(A)) \leq O(nm^{\frac{1}{n-1}} + n^5 4^n).$$

Improvement we hope to achieve... soon

With the same setting as above

$$\Omega(\sqrt{nm}^{\frac{1}{n-1}}) \leq \text{diam}(Q(A)) \leq O(\sqrt{nm}^{\frac{1}{n-1}}).$$



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Theorem (Eisenbrand, Hähnle, Razborov and Rothvoß '10)

Let $G = (V, E)$ be a connected graph, where the vertices V of G are subsets of $\{1, \dots, k\}$ of cardinality n and the edges E of G are such that for each $u, v \in V$ there exists a path connecting u and v whose intermediate vertices all contain $u \cap v$. Then the following upper bounds on the diameter of G hold:

$$2^{n-1} \cdot k - 1 \text{ (Barnette–Larman),}$$

$$k^{1+\log n} - 1 \text{ (Kalai–Kleitman).}$$

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Let $A = \{a_1, \dots, a_m\} \subseteq \mathbb{S}^{n-1}$ be in general position. For a vertex $x \in P(A)$, we denote $A_x = \{a \in A : \langle a, x \rangle = 1\}$. Consider the following sets

$$V = \{A_x : x \text{ is a vertex of } P(A)\}, \quad E = \{\{A_x, A_y\} : [x, y] \text{ is an edge of } P(A)\}.$$

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$$\text{diam}(P(A)) \leq 2^{n-1}m - 1, \quad \text{diam}(P(A)) \leq m^{1+\log n} - 1.$$

These are almost the bounds presented earlier:

$$\text{diam}(P(A)) \leq 2^{n-3} \cdot m, \quad \text{diam}(P(A)) \leq m^{\log_2 n + 1}.$$





1. Choose a *scale* $\varepsilon = \varepsilon(n, m)$ at which, with high probability,
 - $\{a_1, \dots, a_M\} \subseteq \mathbb{S}^{n-1}$ is **sufficiently dense**:
 $C(v, \varepsilon) \cap A \neq \emptyset$ for any $v \in \mathbb{S}^{n-1}$,
 - $\{a_1, \dots, a_M\} \subseteq \mathbb{S}^{n-1}$ is **not too dense**:
 $\#(C(v, t\varepsilon) \cap A) \leq 45 \log(p)t^{n-1}$ for any $t \geq 1$ and $0 < p < m^{-n}$.



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2. **Bound the diameter of the shadow** on an arbitrary plane W
 - Let $w_1, w_2, \dots, w_{1/\varepsilon} = w_1 \in \mathbb{S}^{n-1} \cap W$.
 - (**localization**) Show that the shadow path “between” w_i and w_{i+1} is determined (whp) by $A \cap C(w_i, 6\varepsilon)$.
 - **Use the abstract diameter bound** to bound the length of the local shadow.
 - Sum the local contributions



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3. **Bound the diameter**
 - Let $N \subseteq \mathbb{S}^{n-1}$ be a fixed minimal ε -net.
 - Bound the shadow diameter for any plane $W = \text{span}(e_1, v)$, $v \in N$.
 - Connect the a_i to the closest shadow path.



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Quantify carefully everything... and you are done!



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- 4 Proof of our upper bound
- 5 Proof of our lower bound**

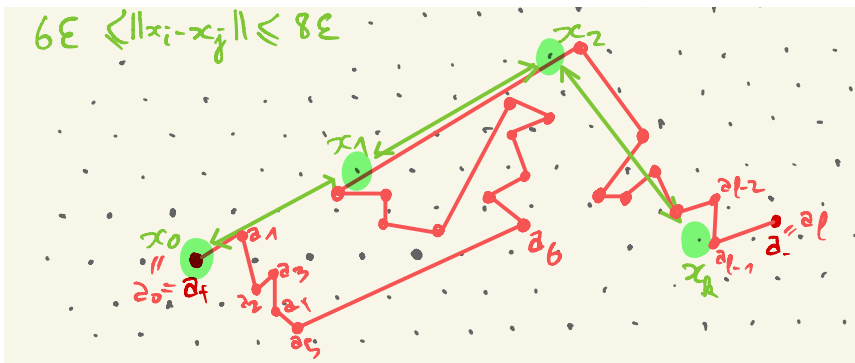


Lemma

For $n \geq 2$, let $P \subseteq \mathbb{R}^n$ be a simple bounded polytope containing the origin in its interior and let $Q = P^\circ := \{x \in \mathbb{R}^n : \langle x, y \rangle \leq 1, \forall y \in P\}$ denote the polar of P . Then,

$$\text{diam}(P) \geq (n - 1)(\text{diam}(Q) - 2).$$

Thus, we only need a lower bound for $\text{diam}(Q(A))$.



- Let $a_+, a_- \in A$ such that $\|a_+ - a_-\| \geq 1$.
- Let $a_+ \in N \subseteq \mathbb{S}^{n-1}$ be a minimal ϵ -net, $\epsilon = cm^{-1/(n-1)}$.
- Set $x_0 = a_+$ and for any $k \in \mathbb{N}$,

$$X_k = \{\mathbf{x} \in N^k : x_i \neq x_j \text{ and } 6\epsilon \leq \|x_i - x_{i+1}\| \leq 8\epsilon\}$$



1. Setting

- Let $a_+, a_- \in A$ such that $\|a_+ - a_-\| \geq 1$.
- Let $a_+ \in N \subseteq \mathbb{S}^{n-1}$ be a minimal ε -net, $\varepsilon = cm^{-1/(n-1)}$.
- Set $x_0 = a_+$ and for any $k \in \mathbb{N}$,

$$X_k = \{\mathbf{x} \in N^k : x_i \neq x_j \text{ and } 6\varepsilon \leq \|x_i - x_{i+1}\| \leq 8\varepsilon\}$$

2. Intermediate lemmas:

- $\#X_k \leq (17^{n-1})^k$
- For any path $[a_0, a_1], [a_1, a_2], \dots, [a_{\ell-1}, a_\ell]$ with $a_0 = a_+$ and $a_\ell = a_-$, there exists $k \geq k_0 = \Omega(1/\varepsilon)$ and $\mathbf{x} \in X_k$ such that, for any $i \in [k]$, there exists $j \in [\ell]$ and $x \in [a_{j-1}, a_j]$ such that $x/\|x\| \in C(x_i, \varepsilon)$
- It implies

$$\text{diam}(Q(A)) \geq \min_{k \geq k_0} \min_{\mathbf{x} \in X_k} \sum_{0 \leq i \leq k-1} \mathbf{1}(C(x_i, \varepsilon/2) \cap A \neq \emptyset) \mathbf{1}(C(x_{i+1}, \varepsilon/2) \cap A \neq \emptyset).$$

3. Conclude with the union bound.



THANK YOU!