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# Random polytopes in fixed and high dimension

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*Mark Kac seminar*, Utrecht, June 03 2022

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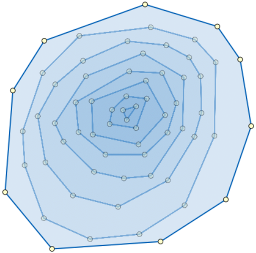
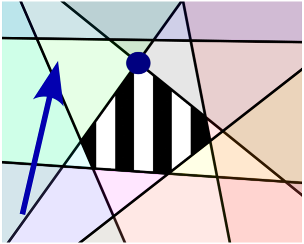
**Convex hull** of random points.



*Example*

Generalization of extreme values

**Intersection** of half spaces.

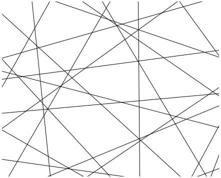


*Example*

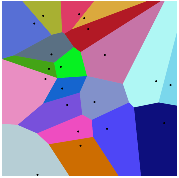
Feasible solutions of a set of (random) linear inequality constraints.

$$\mathrm{conv}(X_1,\dots X_n) \overset{\textbf{convex duality}}{\longleftrightarrow} \bigcap_i \{x \in \mathbb{R}^d : \langle x, X_i \rangle \leq 1\}$$

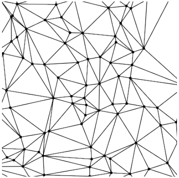
- **Cells in a random tessellation**:



**(a)** Hyperplane tessellation



**(b)** Voronoi tessellation



**(c)** Delaunay tessellation

*Example*: Signal processing/compression in high-dimension.

- Even more options:
  - (random) **projection** of a higher dimensional (random) polytope
  - (random) **intersection** of a higher dimensional (random) polytope
  - . . .

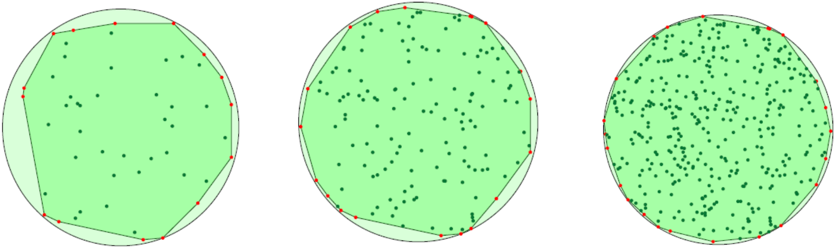
- Generalization of **extreme values** from one to multi dimensional data,
- **Probabilistic method**:
  The **Banach-Mazure compactum** over $\mathbb{R}^d$ has size $\Theta(d)$.
  Proof using **Gaussian polytopes** [Gluskin] '81, 1.5 pages long
- **Signal processing/data compression**,
- **Approximation** of convex set by a polytope,
- **Linear programming** (complexity of the simplex algorithm),
- **High dimensional convex geometry**,
- ...

- **Metric quantities**
  - volume,
    surface area,
    mean-width,
    ... (intrinsic volumes $V_i(P)$, $i = 0, 1, \ldots, d$)
  - in-radius (radius of the largest ball inside),
    out-radius (radius of the smallest ball containing $P$),
  - distance with a fixed convex body (Hausdorff distance, volume difference,...),
  - ...
- **Combinatorial quantities**
  - vertices number $f_0(P)$,
    facets number $f_{d-1}(P)$,
    ... (more generally, number of $k$-dimensional faces $f_k(P)$),
  - combinatorial diameter,
  - ...

$K \subseteq \mathbb{R}^d$ a convex body

$X_1, X_2, \ldots \in K$ i.i.d. **uniformly distributed** in $K$

$K_n = \mathrm{conv}(X_1, \ldots, X_n)$



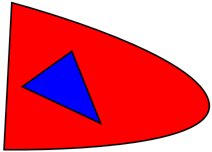
## Three questions

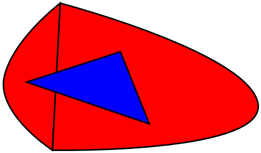
1. Is $n \mapsto \mathbb{E}\,\mathrm{Vol}\, K_n$ increasing?
2. Is $K \mapsto \mathbb{E}\,\mathrm{Vol}\, K_n$ increasing?
3. Is $n \mapsto \mathbb{E} f_0 K_n$ increasing?

1. YES! $K_n \subseteq K_{n+1} \Rightarrow \mathbb{E}\,\mathrm{Vol}\, K_n \subseteq \mathbb{E}\,\mathrm{Vol}\, K_{n+1}$.

## Second question

2. Is $K \mapsto \mathbb{E}\operatorname{Vol} K_n$ increasing?





Trivially true in dimension $d = 1$

**'06** Question asked by Meckes

**'12** → True in dimension $2$, [Rademacher]
→ Counterexample in dimensions $d \geq 4$.

**'16** Counterexamples also for higher moments. [Reichenwallner, Reitzner]

**'16** Counterexample in dimension 3. [Kunis, Reichenwallner, Reitzner]

**'18** Counterexamples for lower dimensional simplices. [Reichenwallner]

## Third question

**3.** Is $n \mapsto \mathbb{E} f_0 K_n$ increasing?

**Problem:** Adding a point might kill many vertices.

**Hopes:** $\mathbb{E} f_0 K_n \geq c \log^{d-1} n$ consequence of [Bárány, Larman, '88]

**But:** There exists a convex body $K$ and a sequence $(n_i)_i$ with

$\mathbb{E} f_o(K_{n_{2i}}) \gg n_{2i} n_{2i}^{\frac{d-1}{d+1}}$ and $\mathbb{E} f_0(K_{n_{2i+1}}) \ll \log(n_{2i})^{d-1}(n_{2i})$. (also [BL88])

- **'05** V.H. Vu calls it a *tantalizing conjecture*
- **'13** True for $d = 2$ [Devillers, Glisse, Goaoc, Moroz, Reitzner]
  True for $n$ large enough in any dimension
- **'14** True for the **facet number** and $K = \mathbb{B}^d$ [Beermann]
  True for the **facet number** if the points are **normally distributed**

### Definition: Beta and Beta-prime distributions

Both distributions are defined on $\mathbb{R}^d$ and depend on the value of a parameter $\beta$.

**Beta distribution:**

- if $\beta > -1$, density proportional to $\mathbb{1}(x \in \mathbb{B}^d)(1 - \|x\|)^\beta$;
- if $\beta = -1$, = uniform distribution on $\mathbb{S}^{d-1}$.

**Beta-prime distribution:**

- if $\beta > d/2$, density proportional to $(1 + \|x\|)^{-\beta}$;

Special cases: *uniform distributions on* $\mathbb{S}^{d-1}$ *and* $\mathbb{B}^d$, and the *normal distribution*.

$$P_{n,d}^{\beta} = \mathrm{conv}(X_1, \ldots, X_n) \text{ with } X_1, \ldots, X_n \text{ i.i.d.}$$

### Theorem [B., Grote, Temesvari, Thäle, Turchi, Wespi] '17

If $X_1, X_2, \ldots$ are beta or beta prime distributed, then $\mathbb{E} f_{d-1}(P_n) > \mathbb{E} f_{d-1}(P_{n-1})$.

(also: random spherical polytopes.)

## Third question

**3.** Is $n \mapsto \mathbb{E} f_0 K_n$ increasing?

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**'14** True for the **facet number** and $K = \mathbb{B}^d$ [Beermann]

True for the **facet number** if the points are **normally distributed**

**'17** True for the **facet number** and if the points are **beta** or **beta-prime** distributed

[**B.**, Grote, Temesvari, Thäle, Turchi, Wespi]

**'18** True for **the whole $f$-vector** for random projections of regular polytopes

[Kabluchko, Thäle]

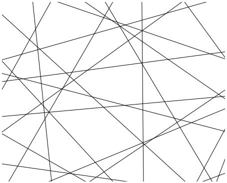
## Open problems

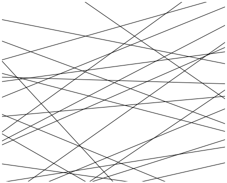
1. Is $n \mapsto \mathbb{E} f_0(K_n)$ increasing, for arbitrary $K \subseteq \mathbb{R}^d$ in dimension $\geq 3$?
2. Monotonicity of the distribution: $\mathbb{P}(f_0(K_n) \geq \ell) \leq \mathbb{P}(f_0(K_{n+1}) \geq \ell)$?
3. Monotonicity of other combinatorial quantities: e.g. combinatorial diameter?

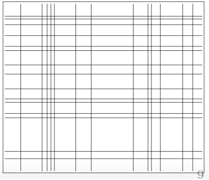
$\eta$ stationary Poisson hyperplane process in $\mathbb{R}^d$

$\rightarrow \varphi$ directional distribution

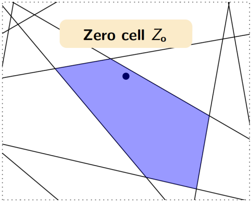
$\rightarrow \gamma$ intensity

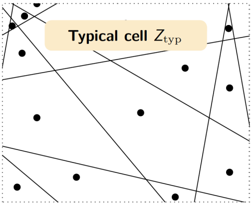






Zero cell $Z_0$ // Typical cell $Z$




**Distributions of $Z$ and $Z_0$?**

- $\mathbb{E}\mathrm{Vol}(Z) = ?$
- $\mathbb{E}f_{d-1}(Z) = ?$
- $\mathbb{P}(f_{d-1}(Z) = \ell) = ?$
- ...

## The zero-cell is volume-biased

$$\mathbb{E} g(Z_0) = \mathbb{E}\left[ \frac{\operatorname{Vol} Z}{\mathbb{E} \operatorname{Vol} Z} g(Z) \right]$$

... where $g$ is any *stationary invariant* positive measurable function.

More generally, for $g$ non stationary invariant: $\mathbb{E} g(Z_0 - c(Z_0)) = \mathbb{E}\left[ \frac{\operatorname{Vol} Z}{\mathbb{E} \operatorname{Vol} Z} g(Z) \right]$

## The typical cell is contained in the zero-cell

$$Z \subseteq Z_0$$

... for an appropriate coupling.

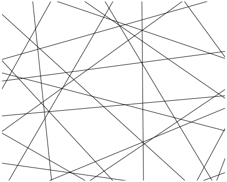
## typical cell, first moment faces number

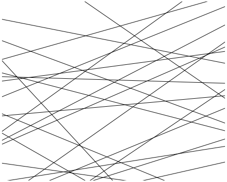
$f_k(Z) = \#$ faces of dimension $k$.

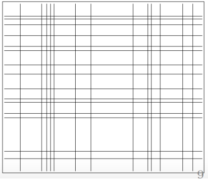
**First moment of the faces numbers of the typical cell**

$\mathbb{E} f_k(Z)$ is independent of $\gamma$ and $\varphi$.

$\Rightarrow \mathbb{E} f_k(Z) = \mathbb{E} f_k(Z_{\text{cuboid}}) = f_k(\text{Cube}) = 2^{d-k}\binom{d}{k}$.

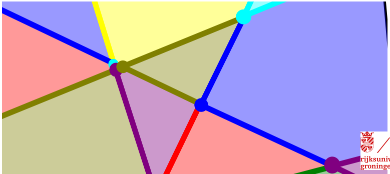






*Neat and simple proof:*

Each vertex takes the edges and cells that contain it and are on its right.



**Theorem** [Mecke] '95 Adv. Appl. Prob.

$$\mathbb{E}\operatorname{Vol} Z_\varphi \geq \mathbb{E}\operatorname{Vol} Z_{\text{isotropic}}$$

- equality holds only in the isotropic setting.
- extend to any (integer) moment;
- also true for the zero cell.

**Open problem (Is it true for intrinsic volumes?)**

$$\mathbb{E} V_i(Z_\varphi) \overset{?}{\geq} \mathbb{E} V_i(Z_{\text{isotropic}}), \quad i \in \{1, \ldots, d-i\}$$

**Theorem** [Wiecker] '86 Prob. Th. Rel. Fields

$$\mathbb{E}\operatorname{Vol} Z_{0,\text{isotropic}} = d!\kappa_d \left(\frac{2\kappa_{d-1}}{d\kappa_d}\gamma\right)^{-d}$$

Wiecker proved much more

- $\mathbb{E}\operatorname{Vol} Z_0 = 2^{-d} d! \operatorname{Vol}(\Pi^o_{\varphi,\gamma})$ (no isotropy assumption)
  $\Rightarrow \mathbb{E}\operatorname{Vol} Z_0 \geq \mathbb{E}\operatorname{Vol} Z_{0,\text{isotropic}}$
  (with equality only in the isotropic case)
- $\mathbb{E}\mathscr{H}^k(\operatorname{skel}_k Z_0) = 2^{-d} d! V_{d-k}(\Pi_{\varphi,\gamma}) V_k(\Pi^o_{\varphi,\gamma})$ $k = 0, \dots, d-1$
  $\Rightarrow 2^d = \mathbb{E} f_0(Z_{0,\text{cuboid}}) \leq \mathbb{E} f_0(Z_0) \leq \mathbb{E} f_0(Z_{0,\text{isotropic}}) = 2^{-d} d! \kappa_d^2$
  (with equalities only in the cuboid and isotropic case)

**Notation**:
$\kappa_d = \operatorname{Vol}\mathbb{B}^d$, $\operatorname{skel}_k = k$-dimensional skeleton, $V_k = k$-th intrinsic volume,
$\Pi_{\varphi,\gamma}$ and $\Pi^o_{\varphi,\gamma}$ are convex bodies determined by $\varphi$ and $\gamma$. They are called *zonotopes*.

### Challenge

Find $\mathbb{P}(f_{d-1}(Z) = k)$.

Even in the simplest setting ($d = 2$ and isotropy) this is answered only for a few values of $k$.

$\eta$ *stationary* and *isotropic* Poisson hyperplane process in $\mathbb{R}^d$

$Z$ typical cell

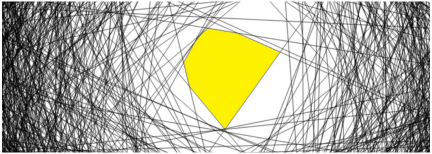
**Theorem** **[Calka, Hilhorst] '08 J. Stat. Phys.**

If $d = 2$, then

$$\mathbb{P}(f_1(Z) = n) \sim 16\pi^{\frac{5}{2}} n^{-1/2} \left(\frac{n}{\pi e}\right)^{-2n}.$$

In the same article, similar results about:

- the zero cell of Poisson hyperplane tessellation which are *non-stationary* (but scale homogeneous)
- the typical Poisson Voronoi cell.



Picture credit: [Kabluchko, Thäle, Zaporozhets] Adv. in Math. (2020)

Beta polytopes and Poisson polyhedra: $f$-vectors and angles

## Theorem [B., Calka, Reitzner] '18 Adv. in Math.

$$\mathbb{P}(f_{d-1}(Z) = n) = \Theta(n)^{-\frac{2}{d-1}n}$$

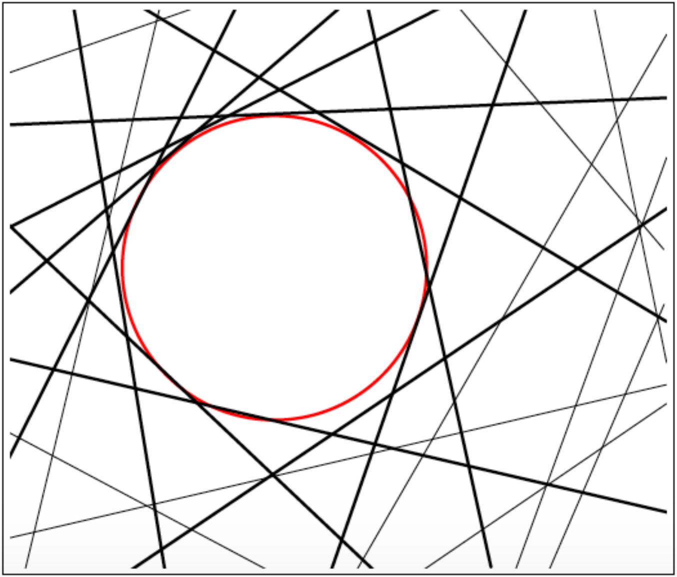
We also show:

- $n \mapsto \mathbb{P}(f_{d-1}(Z) = n)$ decreases for $n$ large enough;
- it extends to non-isotropic distributions.

Foreword by **David Kendall** in *Stochastic geometry and its applications* (1st ed. '87) authored by his son **Wilfrid Kendall** and D. Stoyan and J. Mecke:

> My own first contact with *classical geometrical probability* occurred during the war, when the Superintendent of my group (Louis Rosenhead) asked me to investigate the following problem; for the sake of clarity I formulate it in the modern terminology.
>
> > A euclidean plane with a marked origin O carries a Poisson field of unsensed lines with a uniform intensity. Almost surely the point O will lie in the interior of a unique Crofton cell $C$, with (unlabelled) shape $\sigma(C)$ and area $a(C)$. What probability statements can be made about $C$ which convey information about the strength of a fabric ('a sheet of paper') consisting of the field of lines ('fibres')? Thus it would be useful to be able to calculate the rate of occurrence of splinter-shaped cells $C$, and the rate of occurrence of cells with large area $a(C)$.
>
> A few moments of the $a(C)$-distribution were already known, and I managed to add one more to these. One would have preferred to be able to say something about the asymptotics of the marginal $a(C)$-distribution valid for large areas, and to throw light one way or the other on my conjecture that the conditional law for $\sigma(C)|a(C)$ converges weakly, as $a(C) \to \infty$, to the degenerate law concentrated at the circular shape.

# Kendall's Problem



## Kendall's conjecture ('87 minus decades)

$$\mathbb{P}\left(\exists r>0 : r\mathbb{B}^2 \subseteq Z_0 \subseteq (1+\epsilon)r\mathbb{B}^2 \;\middle|\; \mathrm{Area}(Z_0) > a\right) \xrightarrow{a\to\infty} 1 \qquad \forall \varepsilon > 0.$$

- **'95** heuristic proof [Miles]
- **'98** strong support for the conjecture [Goldman]
- **'97** first proof (in russian) [Kovalenko]
- **'99** simplified proof (in english) [Kovalenko]
- **'04** higher dim, no isotropy, also typical cell — Ann Probab [Hug, Reitzner, Schneider]
- **'04** finite numbers of directions (and more) — Adv Appl Probab [Hug, Reitzner, Schneider]
- **'07** far reaching generalization (geometric stability...) — Geom Funt Anal [Hug, Schneider]
- **'18** small progress on remaining open cases — Adv Math [**B.**, Calka, Reitzner]
- **...** (also results about large faces of cells; Kendall's problem in the sphere, ...)

**Assumptions:** $d = 2$, stationary, isotropy, unit intensity.

For a convex body $K \subseteq \mathbb{R}^2$, set

$$\vartheta(K) := \frac{\text{Perimeter}(K)/\sqrt{\text{Area}(K)}}{\text{Perimeter}(\mathbb{B}^2)/\sqrt{\text{Area}(\mathbb{B}^2)}} - 1.$$

- $\vartheta(K) \geq 0$;
- $\vartheta(K) = 0 \Leftrightarrow K$ is a ball.

**Theorem** [Hug, Schneider] '07 GAFA

$$\limsup_{a\to\infty} \frac{1}{a^2} \log \mathbb{P}\left(\vartheta(Z_0) \geq \epsilon \mid \text{Area}(Z_0) > a\right) \leq -\epsilon,$$

**Theorem (Refinement: LDP)** [B.] PhD thesis '16

$$\limsup_{a\to\infty} \frac{1}{a^2} \log \mathbb{P}\left(\vartheta(Z_0) \geq \epsilon \mid \text{Area}(Z_0) > a\right) = -\epsilon,$$

Both results extend to a much larger setting.

## Open problems

What is the shape of the typical cell conditioned on having a

- large number of facets?
- large mean-width? (=perimeter in dimension 2)

For a convex body $K \subseteq \mathbb{R}^d$, and $j \in \{1, 2, \ldots, d-1\}$, set $\widehat{\vartheta}_j(K) := \frac{V_{j+1}(K)}{V_1(K)}$.

- $\widehat{\vartheta}_j \geq 0$
- $\widehat{\vartheta}_j = 0 \Leftrightarrow K$ is contained in a $j$-dimensional flat.

## Theorem (cells with large mean-width are not elongated) [B., Calka, Reitzner] '18

Assume $d \geq 4$, isotropy and stationarity.

Let $j \in \{1, 2, \ldots, \lceil (d-1)/2 \rceil - 1\}$ and $\epsilon > 0$ sufficiently small. Then

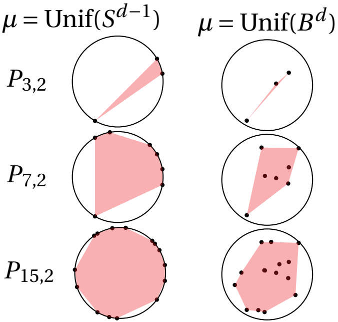
$$\lim_{a \to \infty} \mathbb{P}\left(\widehat{\vartheta}_j(Z) \geq \epsilon \mid V_1(Z) \geq a\right) = 0.$$

## What do we do now?

1. Jump to the next section? (Case study 3: High dimensional convex hulls.)
2. Discuss proofs' ideas? (on the board)

# Table of Contents

- $X_1, X_2, \ldots \in \mathbb{R}^d$ i.i.d. random vectors distributed according to a probability measure $\mu = \mu_d$
- $P_{n,d} = \text{conv}(X_1, \ldots, X_n)$




What does a **high dimensional** random polytope look like ?

**Distribution**

- Uniform on the sphere,
- Uniform in the ball,
- Gaussian,
- Beta distributions,
- ...

**Regime**

- $n \simeq \alpha d$ linear,
- $n \simeq d^\alpha$ polynomial,
- $n \simeq \alpha^d$ exponential,
- $n \simeq d^{\alpha d}$ super exponential,
- ...

**Characteristic**

- Number of facets,
- Height of the facets,
- Volume,
- $\mathbf{1}(0 \in P_{n,d})$,
- ...

| Distribution | Regime | Characteristic |
| --- | --- | --- |
| Uniform on the sphere, | $n \simeq \alpha d$ linear. | $\mathbf{1}(0 \in P_{n,d})$, |

## Theorem [Wendel] '62

$$\mathbb{P}(0 \notin P_{n,d}) = \frac{1}{2^{n-1}} \sum_{k=0}^{d-1} \binom{n-1}{k} = \mathbb{P}(B_{n-1} < d).$$

where $B_{n-1}$ is a Binomial random variable with parameters $n-1$ and $1/2$.

Since $[2B_{n-1} - (n-1)]/\sqrt{n-1} \xrightarrow{d} Z \sim \mathcal{N}(0,1)$ we have

## Corollary (Threshold phenomena)

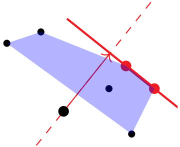
$$\lim_{d\to\infty} \mathbb{P}(0 \in P_{n,d}) = \begin{cases} 0 & \textit{if } n \le (2-\epsilon)d, \\ 1 & \textit{if } n \ge (2+\epsilon)d. \end{cases}$$

## Corollary (Phase transition)

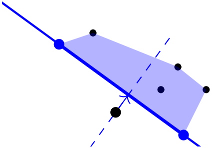
*If* $n = 2d + x\sqrt{d} + o(\sqrt{d})$ *then* $\lim_{d\to\infty} \mathbb{P}(0 \in P_{n,d}) = \Phi(x)$.

where $\Phi(x) = \mathbb{P}(Z \le x)$ is the CDF of a standard normal random variable.

Let $F$ be a facet of $P_{n,d}$.
Set $H = \operatorname{affineHull}(F)$ its supporting hyperplane
Let $u \in S^{d-1}$ and $h \in \mathbb{R}$ such that $H = \{x \in \mathbb{R}^d : \langle x, u \rangle = h\}$ and $P_{n,d} \subseteq \{x \in \mathbb{R}^d : \langle x, u \rangle \leq h\}$. We call $h$ the height of the facet $F$.



**(a)** A facet with positive height.



**(b)** A facet with negative height.

The **typical height** $H_{\text{typ}}$ is a random variable defined by

$$\mathbb{P}(H_{\text{typ}} \in \cdot) = \mathbb{P}([X_1, \dots, X_d] \text{ has height } \in \cdot \mid [X_1, \dots, X_d] \text{ is a facet of } P_{n,d}).$$

Distribution assumption: **uniform on the sphere**.

Let $[h_1, h_2] \subseteq [-1, 1]$.

$$\mathbb{P}(H_{\mathrm{typ}} \in [h_1, h_2]) = \frac{\mathbb{P}([X_1, \dots, X_d] \text{ is facet \& its height is } \in [h_1, h_2])}{\mathbb{P}([X_1, \dots, X_d] \text{ is a facet of } P_{n,d})} = \frac{I_{[h_1,h_2]}}{I_{[-1,1]}},$$

where

$$I_{[h_1,h_2]} = \mathbb{P}([X_1, \dots, X_d] \text{ is facet \& its height is } \in [h_1, h_2])$$

$\vdots$ Toolbox from integral geometry

$$= \int_{h_1}^{h_2} c_1 (1-h)^{\frac{d^2-2d-1}{2}} \left( c_2 \int_{-1}^{h} (1-s^2)^{\frac{d-3}{2}} \, \mathrm{d}s \right)^{n-d} \mathrm{d}h.$$

## Theorem

[B., O'Reilly] '22+

Distribution assumption: **Uniform on the sphere.**

- (sub-exponential) If $(\ln n)/d \to 0$, then $H_{\text{typ}} \xrightarrow{\mathbb{P}} 0$.
  - If $n-d = O(\sqrt{d})$, then $dH_{\text{typ}} - \frac{n-d}{\sqrt{d}} r_1 \xrightarrow{d} Z \sim \mathcal{N}(0,1)$.
  - If $\sqrt{d} \ll n-d \ll d$, then $\frac{d^{3/2}}{n-d} H_{\text{typ}} \xrightarrow{\mathbb{P}} r_2$.
  - If $n \sim \rho d$, then $\sqrt{d} H_{\text{typ}} \xrightarrow{\mathbb{P}} r_3(\rho)$.
  - If $\ln n \ll d \ll n$, then $\sqrt{\frac{d}{\ln(n/d)}} H_{\text{typ}} \xrightarrow{\mathbb{P}} r_4$.
- (exponential) If $(\ln n)/d \to \alpha$, then $H_{\text{typ}} \xrightarrow{\mathbb{P}} \sqrt{1-e^{-\alpha}}$.
- (super-exponential) If $(\ln n)/d \to \infty$, then $H_{\text{typ}} \xrightarrow{\mathbb{P}} 1$.
  - If $\ln n \gg d$, then $-\frac{d-1}{\ln n} \ln\left(1 - H_{\text{typ}}^2\right) \xrightarrow{\mathbb{P}} r_5$.
  - If $\ln n \gg d \ln d$, then $r_6 \frac{n}{d} (1 - H_{\text{typ}}^2)^{\frac{d}{2}} - \sqrt{d} \xrightarrow{\text{tv}} Z \sim \mathcal{N}(0,1)$.
  - If "$d$ is fixed", then $n c_d (1 - H_{\text{typ}}^2)^{\frac{d-1}{2}} \xrightarrow{\text{tv}} \Gamma_{d-1}$.

where $r_1 = \sqrt{\frac{2}{\pi}}$, $r_2 = \frac{2}{\pi}$, $r_3 = \operatorname{argmax}(r \mapsto (\rho-1)\ln\Phi(r) - r^2/2)$, $r_4 = \sqrt{2}$, $r_5 = 2$, $r_6 = \frac{1}{2\sqrt{\pi}}$.

$$H_{\min} := \min\{\text{height}(F) \mid F \in \text{Facets}(P_{n,d})\}$$
$$H_{\max} := \max\{\text{height}(F) \mid F \in \text{Facets}(P_{n,d})\}$$

These random variables have a similar asymptotic as the typical height:

**Theorem** [B., O'Reilly] '22+

Distribution assumption: **Uniform on the sphere.**

For $H \in \{H_{\min}, H_{\text{typ}}, H_{\max}\}$, we have

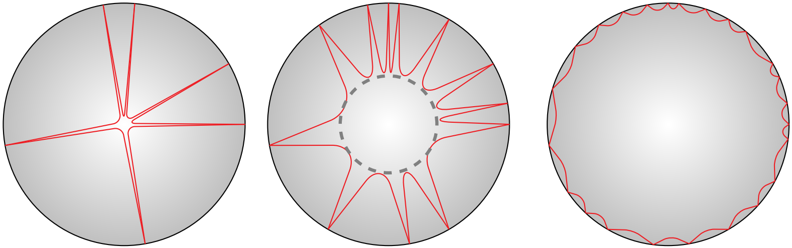
- (sub-exponential) If $(\ln n)/d \to 0$, then $H \xrightarrow{\mathbb{P}} 0$.
  - If $n - d \ll d$, then $\sqrt{d}H \xrightarrow{\mathbb{P}} 0$.
  - If $n \sim \rho d$, then $\mathbb{P}(r_7(\rho) \le \sqrt{d}H \le r_8(\rho)) \to 1$.
  - If $\ln n \ll d \ll n$, then $\sqrt{\frac{d}{\ln(n/d)}}H \xrightarrow{\mathbb{P}} r_4$.
- (exponential) If $(\ln n)/d \to \alpha$, then $H \xrightarrow{\mathbb{P}} \sqrt{1-e^{-\alpha}}$.
- (super-exponential) If $(\ln n)/d \to \infty$, then $H \xrightarrow{\mathbb{P}} 1$.
  - If $\ln n \gg d$, then $-\frac{d-1}{\ln n}\ln\left(1-H^2\right) \xrightarrow{\mathbb{P}} r_5$.

**Distribution**

- Uniform on the sphere,

**Characteristic**

- Height of the facets,



**(a) sub-exponential regime**
$(\ln n)/d \to 0$.
Facets' heights $\simeq O(1/\sqrt{d})$.

**(b) exponential regime**
$(\ln n)/d \to \alpha$.
Facets' heights $\simeq \sqrt{1-e^{-\alpha}}$.

**(c) super-exponential regime**
$(\ln n)/d \to \infty$.
Facets' heights $\simeq 1$.

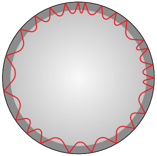
Distribution assumption: **Uniform on the sphere**

**Hausdorff distance:**

$$\mathrm{d_H}(P_{n,d}, B^d) = 1 - H_{\min}$$

**Corollary**

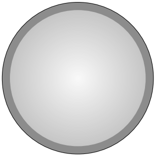
$\mathrm{d_H}(P_{n,d}, B^d) \to 0 \Leftrightarrow (\ln n)/d \to \infty.$



**Volume ratio:**

$$\frac{\mathbb{E}\, V_d(P_{n,d})}{V_d(B^d)}$$

⚠ The volume of the ball $B^d$ is concentrated in a thin shell of width of order $\frac{1}{d}$.





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**Theorem: Volume Threshold** [Pivovarov] '07 Studia Math.

Distribution assumption: **Uniform on the sphere/ball.**

Fix $\varepsilon \in (0,1)$.

Then
$$\lim_{d\to\infty} \frac{\mathbb{E}\, V_d(P_{n,d})}{V_d(B^d)} = \begin{cases} 0 & \text{if } n \le d^{(1-\varepsilon)(\frac{d}{2})}, \\ 1 & \text{if } n \ge d^{(1+\varepsilon)(\frac{d}{2})}. \end{cases}$$

**Theorem: Extension of Pivovarov's result** [B., Chasapis, Grote, Temesvari, Turchi] '18 Comm Cont Math

Distribution assumption: **Beta distribution**: density $\propto \mathbb{1}(x \in B^d)\,(1-\|x\|^2)^\beta$.

Fix $\varepsilon \in (0,1)$ and let $\beta = \beta(d) > -1$.

Then
$$\lim_{d\to\infty} \frac{\mathbb{E}\, V_d(P_{n,d})}{V_d(B^d)} = \begin{cases} 0 & \text{if } n \le d^{(1-\varepsilon)(\beta+\frac{d}{2})}, \\ 1 & \text{if } n \ge d^{(1+\varepsilon)(\beta+\frac{d}{2})}. \end{cases}$$

In the same paper we show similar results for **intrinsic volumes** and for **Gaussian distribution**.

## Theorem: Phase transition for random polytopes [B., Kabluchko, Turchi] '21 RSA

Distribution assumption: **Beta distribution**: density $\propto \mathbb{1}(x \in B^d)\,(1-\|x\|^2)^\beta$.

Let $\beta = \beta(d) > -1$. If $x \in \mathbb{R}$ is fixed and $n = \left(\frac{d}{2x+o(1)}\right)^{\frac{d}{2}+\beta}$, then

$$\lim_{d\to\infty} \frac{\mathbb{E}\,V_d(P_{n,d})}{V_d(B^d)} = e^{-x}$$

In the same paper we show similar results for:

- **intrinsic volumes**.
- asymptotic for the **number of vertices** for the uniform distribution on the ball.

## Current work with Kabluchko and Turchi

Asymptotic of $\frac{\mathbb{E}\,V_d(P_{n,d})}{V_d(B^d)}$ for arbitrary regime.

# What do we do now?

1. Jump to the next section? (Case study 4: Combinatorial diameter of random polyotpes.)
2. Discuss the front line of our research (slides)

Let $X_1, \dots X_n \in \mathbb{R}^d$ i.i.d uniformly distributed in $\mathbb{B}^d$.

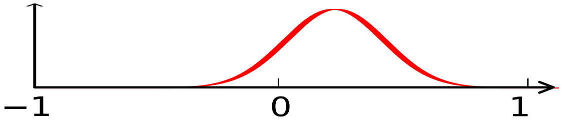
$P_{n,d} = \operatorname{conv}(X_1, \dots, X_n)$

$$\mathbb{E} V_d(P_{n,d}) = \cdots \boxed{\text{Toolbox from integral geometry}} \cdots = B_{n,d} \int_{-1}^{1} f_r^p(x) F_r^q(x) \, \mathrm{d}x$$

with $p \sim 2r \sim d, \quad q = n - d - 1$ and

$$f_r(x) = c_r (1 - x^2)^r \qquad \text{(density on } [-1, 1]\text{)},$$

$$F_r(x) = \int_{-1}^{x} f_r(t) \, \mathrm{d}t \qquad \text{(cumulative distribution function).}$$



Set $g = \log[f_r^p(x) F_r^q(x)]$
and $x_d = \operatorname{argmax} g(x)$.
Is it true that

$$\int_{-1}^{1} f_r(x)^p F_r(x)^q \, \mathrm{d}x = \int_{-1}^{1} e^{g(x)} \, \mathrm{d}x \overset{?}{\sim} \sqrt{\frac{2\pi}{-g''(x_d)}} \, g(x_d)?$$

**Theorem** [B., Turchi, Kabluchko] **in preparation**

**YES.** It is true.

If $g$ would be of the form $g(x) = d h(x)$ with $d \to \infty$, it would be trivial by Laplace's method.
But $g$ is much more complicated to handle, $g = g_{p,q,r}$.
It is **not a trivial problem**.

**Lemma (Generalized Laplace's method)** [B., Turchi, Kabluchko] **in preparation**

Let $(g_d\colon [a,b] \to \mathbb{R})_{d\in\mathbb{N}}$ a sequence of three times differentiable concave functions, such that each of them has a unique maximizer $x_d \in (a,b)$. Assume that

$$\sqrt{-g_d''(x_d)}\,\min(x_d - a, b - x_d) \to \infty \tag{1}$$

and for any $k \in \mathbb{N}$

$$\lim_{d\to\infty} \sup \left\{ \frac{|g_d'''(x)|^{\frac{1}{3}}}{|g_d''(x_d)|^{\frac{1}{2}}} : |x - x_d| \le \frac{k}{\sqrt{-g_d''(x_d)}},\ x \in [a,b] \right\} = 0. \tag{2}$$

Then it holds

$$\int_a^b e^{g_d(x)}\,\mathrm{d}x \sim \sqrt{\frac{2\pi}{-g_d''(x_d)}}\, e^{g_d(x_d)}, \qquad \text{as } d \to \infty.$$

$$\mathbb{E}V_d(P_{n,d}) \sim B_{n,d}\sqrt{\frac{2\pi}{-g''(x_d)}}\, f_r(x_d)^p F_r(x_d)^q, \qquad x_d = \operatorname{argmax} f_r^p(x) F_r^q(x),$$

with $p \sim 2r \sim d, \quad q = n - d - 1$ and

$$f_r(x) = c_r(1-x^2)^r \text{ is a density on } [-1,1], \qquad F_r(x) = \int_{-1}^{x} f_r(t)\,\mathrm{d}t \text{ is a cdf.}$$

## Lemma

1. $\left[\frac{-f_r' F_r}{f_r^2}\right](x_d) = \frac{q}{p}$ (because $[f^p F^q]'(x_d) = 0$)
2. $\frac{r+1}{r}\left[\frac{-f_r'(1-F_r)}{f_r^2}\right](x_d) = 1 + \sum_{k\geq 1} c_k \left(\frac{1-x^2}{rx^2}\right)^k$ (approx. $1 - F(x) = \int_x^1 f(t)\,\mathrm{d}t$)

## New goal

$f^p(x_d) \sim ?$ and $F^q(x_d) \sim ?$ given that

**1.** $\left[\frac{-f'F}{f^2}\right](x_d) = \frac{q}{p}$

**2.** $\left[\frac{-f'(1-F)}{f^2}\right](x_d) = 1+\epsilon_r$

1. ⊕ 2. ⇒ $\left[\frac{-f'}{f^2}\right](x_d) = \frac{q}{p} + 1 + \epsilon_r$

Plug this in 1. ⇒ $\left[\frac{q}{p} + 1 + \epsilon_r\right] F(x_d) = \frac{q}{p}$

$$\Leftrightarrow F(x_d)^q = \left(1 + \frac{p(1+\epsilon_r)}{q}\right)^q$$

$$\sim \exp\big(p(1+\epsilon_r)\big) \qquad \text{if } p \ll q^2 \ (\Leftrightarrow n-d \gg \sqrt{d}).$$

A bit of reorganization $\Rightarrow \underbrace{\left[\frac{c_r}{2}\left(\frac{q}{p}+1+\epsilon_r\right)\right]^2}_{=:T^R} \underbrace{(1-x_d^2)^{2(r+1)}}_{X^R} = \underbrace{x^2}_{=:1-X}$

Note that $X \propto f_r(x_d)^2$.

**New goal**

$X^{R^2} \sim ?$ given that $(TX)^R = 1-X, \quad X \in [0,1], \quad T = T(R) \geq 1, \quad R \to \infty.$

Fixed point iteration method with the function $f(x) = \frac{1}{T}(1-x)^{\frac{1}{R}}$ leads us to:

$$X^{R^2} \sim \frac{(T-1)^R}{T^{R^2+R}} \exp\left(\frac{1}{T-1}\log(\frac{1}{T-1})+1\right)$$

assuming that $\frac{\log(T-1)}{T-1} = o(\sqrt{R})$ (which seems to be satisfied for $n = e^{\Omega(\sqrt{d})}$).

... more to come soon...

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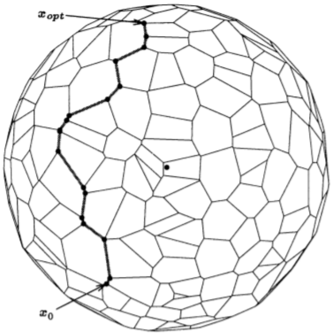



**Linear programming:**

- Polytope
  = set of linear constraints $\langle a_i, x\rangle \leq b_i$
  = set of *feasible solution*,
- One want to maximize a linear objective function.
- The *simplex algorithm* searches the optimal vertex by exploring the *Vertices/Edges graph* of the polytope.
- What is the longest distance between vertices?

Let $P \subseteq \mathbb{R}^n$ be a polytope with $m$ facets.

$$\text{distance}(x,y) = \min\{k \in \mathbb{N} : \exists [x,v_1],\ [v_1,v_2],\ldots,[v_{k-1},y] \in \text{Edges}(P)\},$$

$$\text{diam}(P) = \max\{\text{distance}(x,y) : x,y \in \text{Vertices}(P)\},$$

**Hirsch conjecture (1957)**

$$\text{diam}(P) \leq m - n$$

Examples:

- Simplex: $m = n+1$ $\quad$ $\text{diam} = 1 = m - n$
- Cube: $m = 2n$ $\quad$ $\text{diam} = n = m - n$
- Cross-polytope: $m = 2^n$ $\quad$ $\text{diam} = 2 \ll m - n$

**Counter-example** [Santos] '12 Annales of Mathematics

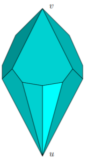
There exists a polytope $P \subseteq \mathbb{R}^{43}$ with $86$ facets and $\mathrm{diam}(P) > 43 = 86 - 43$.

**Small improvement** [Matschke, Santos, Weibel] '12 Proc. London Math. Soc.

There exists a polytope $P \subseteq \mathbb{R}^{20}$ with $40$ facets and $\mathrm{diam}(P) = 21 > 40 - 20$.

Elements of the proof (for both results):

- They construct first a "spindle" (intersection of two polyhedral cones) with diameter greater than the dimension (diameter 6, dimension 5).
- Then they show that it implies the existence of a counter-example.



Santos' construction leads also to the existence of family of polytope violating the conjecture by $\epsilon \geq 5\%$.

**Familly of counter-examples** [Santos] '12

There exists $n \in \mathbb{N}$, $\epsilon > 0$ and an infinite collection of polytopes $(P_k)_k$, such that

1. $P_k \subseteq \mathbb{R}^n$
2. $\operatorname{diam}(P_k) \geq (1+\epsilon) m_k$.

In particular, $\operatorname{diam}(P_k) > m_k - n$.

**Polynomial Hirsch Conjecture**

The diameter of a polytope is bounded by a polynomial of its dimension and number of facets.

**Exponential in $n$, linear in $m$**

$$\mathrm{diam}(P) \leq 2^{n-3} m.$$

[Barnette ('69, '74)] and [Larman ('70)]

**Quasi-polynomial**

$$\mathrm{diam}(P) \leq m^{\log_2 n+1},$$ [Kalai, Kleitman ('92)]

$$\mathrm{diam}(P) \leq (m-n)^{\log_2 n},$$ [Todd ('14)]

$$\mathrm{diam}(P) \leq (m-n)^{\log_2 O(n/\log n)}.$$ [Sukewaga ('19)]

Similar results for graph induced by certain classes of simplicial polytopes:

- Barnette-Larman and Kalai-Kleitman bounds hold for *connected-layer families* [Eisanbrand et al. '10]
- Barnette-Larman bound hold for *pure, normal, pseudo-manifolds without boundary* [Labbé et al. '17].

# Classes of polytopes for which the Hirsch conjecture holds

- 0-1 polytopes [Nadef '89]
- Leontief substitution systems [Grinold '71]
- Transportation polyhedra and their duals [Balinski '84] [Brightwell, Heuvel, Stougie '06] [Borgwardt, De Loera, Finhold, '18]
- Fractional stable-set and perfect matching polytopes [Michini, Sassano '14] [Sanità '18]

- "Well-conditioned" polytopes:
  If $P$ is defined by an integral matrix $A \in \mathbf{Z}^{m\times n}$ with minors less than $\Delta$, then $\operatorname{diam}(P) = O(n^3\Delta^2\log\Delta)$. [Dadush, Hähnle, '16]
- Polytopes with vertices in $\{0,1,\dots,k\}^n$
  - $\operatorname{diam}(P) \le nk$, [Kleinschmidt, Onn '92]
  - $\operatorname{diam}(P) \le nk - \lceil n/2\rceil$ for $k \ge 2$, [Del Pia, Michini '16]
  - $\operatorname{diam}(P) \le nk - \lceil 2n/3\rceil - (k-2)$ for $k \ge 4$, [Deza, Pournin '18]

Let $A \in \mathbb{R}^{m \times n}$ a random matrix.

Set $P = \{x \in \mathbb{R}^n : Ax \leq \mathbf{1}\} \subseteq \mathbb{R}^n$. $\rightsquigarrow$ at most $m$ facets.

Let $P'(W)$ = orthogonal projection on a fixed plane $W$.

**Shadow-bound** = Expected number of edges of $P'(W)$.

Remark: $\mathrm{dist}(v_1, v_2) \leq$ # edges of $P'(\mathrm{span}(v_1, v_2))$, for any vertices $v_1, v_2 \in P$.

## i.i.d. rows [Borgwardt] '87, '99

Assume the rows of $A$ are i.i.d.

Shadow-bound $= O(n^2 m^{1/(n-1)})$ for any rotational symmetric distribution.

Shadow-bound $= \Theta(n^2 m^{1/(n-1)})$ for the uniform distribution on the sphere.

## Smoothed analysis [Dadush, Huiberts] '19

Assume that $A = \bar{A} + \sigma G$, where $\bar{A}$ is fixed with rows of $\ell_2$ norm at most $1$, and $G$ has i.i.d. $\mathcal{N}(0,1)$ entries and $\sigma > 0$.

Shadow-bound $= O(n^2 \sqrt{\log m}/\sigma^2)$, when $\sigma \leq 1/\sqrt{n \log m}$.

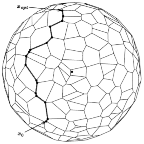
**Theorem.** [B., Dadush, Grupel, Huiberts, Livshitz] '22

Suppose that $n, m \in \mathbb{N}$ satisfy $n \geq 2$ and $m \geq 2^{\Omega(n)}$.
Let $A^\mathsf{T} := (a_1, \ldots, a_M) \in \mathbb{R}^{n \times M}$, where $M \sim \text{Poisson}(m)$, and $a_1, \ldots, a_M$ are sampled independently and uniformly from $\mathbb{S}^{n-1}$.
Set $P(A) := \{x \in \mathbb{R}^n : Ax \leq 1\}$.

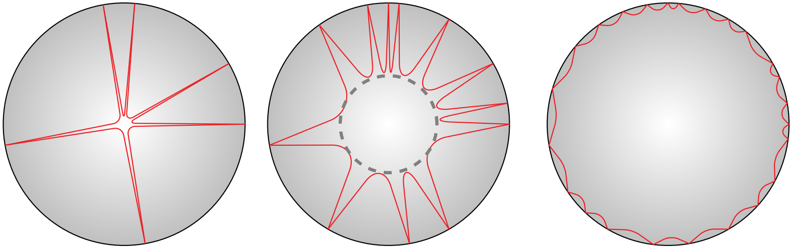
Then, with probability at least $1 - m^{-n}$, we have that

$$\Omega(nm^{\frac{1}{n-1}}) \leq \text{diam}(P(A)) \leq O(n^2 m^{\frac{1}{n-1}} + n^5 4^n).$$

For $m = 2^{\Omega(n^2)}$, $\qquad \Omega(nm^{\frac{1}{n-1}}) \leq \text{diam}(P(A)) \leq O(n^2 m^{\frac{1}{n-1}})$.



For the convex hull of $m$ i.i.d. uniform unit vector in $\mathbb{R}^n$ the following picture holds:



**(a) sub-exponential regime**
$(\ln m)/n \to 0$.
Facets' heights $\simeq O(1/\sqrt{n})$.

**(b) exponential regime**
$(\ln m)/n \to \alpha$.
Facets' heights $\simeq \sqrt{1-e^{-\alpha}}$.

**(c) super-exponential regime**
$(\ln m)/n \to \infty$.
Facets' heights $\simeq 1$.

Let $A \in \mathbb{R}^{m\times n}$ a random matrix with i.i.d. $\beta$-distributed rows,

i.e. with density proportional to $\mathbf{1}(a \in \mathbb{B}^n)(1-\|a\|^2)^\beta$.

Set $P = \{x \in \mathbb{R}^n : Ax \le \mathbf{1}\} \subseteq \mathbb{R}^n$.

[Bogwardt and Huhn '99] provide a lower bound on the Diameter of $P(A)$.

For $\beta \to -1$ that gives:

**Best previous lower bound** [Bogwardt, Huhn] '99

Assume that the rows of $A$ are i.i.d. uniform unit vectors, then

$$\Omega\left(\frac{m^{1/(n-1)}}{(m^{1/(n-1)}n)^{\delta(n)}}\right) \le \mathbb{E}\,\mathrm{diam}(P(A)),$$

where $\delta(n) \to 0$ as $n \to \infty$.

Our results is a triple improvement (when $n \ge 2^{\Omega(n)}$):

1. we remove the denominator $(m^{1/(n-1)}n)^{\delta(n)}$,
2. we add a factor $n$ (Borgwardt and Huhn wrote this would be difficult),
3. we show that it holds with high probability rather then only in expectation.



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### 1-neighbourly random polytope [Bárány, Füredi] '88

Let $Q(A) = \mathrm{conv}(a_1, \ldots, a_m)$ with $a_i$ i.i.d. uniformly distributed in the unit ball. With probability $1 - o(1)$

$$\mathrm{diam}(Q(A)) = 1, \qquad \text{if } m \leq 1.125^n,$$
$$\mathrm{diam}(Q(A)) > 1, \qquad \text{if } m \geq 1.4^n.$$

### Shadows of 3-dimensional convex hull [Glisse et al.] '16

Let $Q(A) = \mathrm{conv}(a_1, \ldots, a_M)$ with $a_i \in \mathbb{S}^2$ i.i.d. uniformly distributed in the unit sphere, and $M$ a Poisson random variable with $\mathbb{E}(M) = m$.

1. With high probability, the maximum number of edges in any 2-dimensional projection is $\Theta(\sqrt{m})$.
2. In particular $\mathrm{diam}(Q(A)) = O(\sqrt{m})$.

$Q(A) = \mathrm{conv}(a_1, \ldots, a_m)\text{“} = \text{"}\{x : \langle x, y\rangle \leq 1, \forall y \in Q(A)\} = P(A)^\circ$

$P(A) = Q(A)^\circ$

The proof of our main theorem (for $P(A)$) leads to similar bounds for $Q(A)$, smaller by a factor $n$.

**Theorem (convex hull)** [B., Dadush, Grupel, Huiberts, Livshitz] '22

Suppose that $n, m \in \mathbb{N}$ satisfy $n \geq 2$ and $m \geq 2^{\Omega(n)}$. Let $M$ be Poisson distributed with $\mathbb{E}[M] = m$, and $a_1, \ldots, a_M$ i.i.d. uniform unit vectors. Then, letting $Q(A) := \mathrm{conv}(a_1, \ldots, a_M)$, with probability at least $1 - m^{-n}$, we have that

$$\Omega(m^{\frac{1}{n-1}}) \leq \mathrm{diam}(Q(A)) \leq O(nm^{\frac{1}{n-1}} + n^5 4^n).$$

**Theorem** [Eisenbrand, Hähnle, Razborov and Rothvoß] '10

Let $G = (V, E)$ be a connected graph, where the vertices $V$ of $G$ are subsets of $\{1, \ldots, k\}$ of cardinality $n$ and the edges $E$ of $G$ are such that for each $u, v \in V$ there exists a path connecting $u$ and $v$ whose intermediate vertices all contain $u \cap v$. Then the following upper bounds on the diameter of $G$ hold:

$$2^{n-1} \cdot k - 1 \text{ (Barnette–Larman)}, \qquad k^{1+\log n} - 1 \text{ (Kalai–Kleitman)}.$$

Let $A = \{a_1, ..., a_m\} \subseteq \mathbb{S}^{n-1}$ be in general position. For a vertex $x \in P(A)$, we denote $A_x = \{a \in A : \langle a, x\rangle = 1\}$. Consider the following sets

$$V = \{A_x : x \text{ is a vertex of } P(A)\}, \qquad E = \{\{A_x, A_y\} : [x, y] \text{ is an edge of } P(A)\}.$$

$$\mathrm{diam}(P(A)) \leq 2^{n-1} m - 1, \qquad \mathrm{diam}(P(A)) \leq m^{1+\log n} - 1.$$

These are almost the bounds presented earlier:

$$\mathrm{diam}(P(A)) \leq 2^{n-3} \cdot m, \qquad \mathrm{diam}(P(A)) \leq m^{\log_2 n+1}.$$

1. Choose a *scale* $\varepsilon = \varepsilon(n, m)$ at which, with high probability,
   - $\{a_1, \ldots, a_M\} \subseteq \mathbb{S}^{n-1}$ is sufficiently dense: $C(v, \varepsilon) \cap A \neq \emptyset$ for any $v \in \mathbb{S}^{n-1}$,
   - $\{a_1, \ldots, a_M\} \subseteq \mathbb{S}^{n-1}$ is not too dense: $\#(C(v, t\varepsilon) \cap A) \leq 45 \log(p) t^{n-1}$ for any $t \geq 1$ and $0 < p < m^{-n}$.
2. Bound the diameter of the shadow on an arbitrary plane $W$
   - Let $w_1, w_2, \ldots, w_{1/\varepsilon} = w_1 \in \mathbb{S}^{n-1} \cap W$.
   - (localization) Show that the shadow path "between" $w_i$ and $w_{i+1}$ is determined (whp) by $A \cap C(w_i, 6\varepsilon)$.
   - Use the abstract diameter bound to bound the length of the local shadow.
   - Sum the local contributions
3. Bound the diameter
   - Let $N \subseteq \mathbb{S}^{n-1}$ be a fixed minimal $\varepsilon$-net.
   - Bound the shadow diameter for any plane $W = \mathrm{span}(e_1, v)$, $v \in N$.
   - Connect the $a_i$ to the closest shadow path.

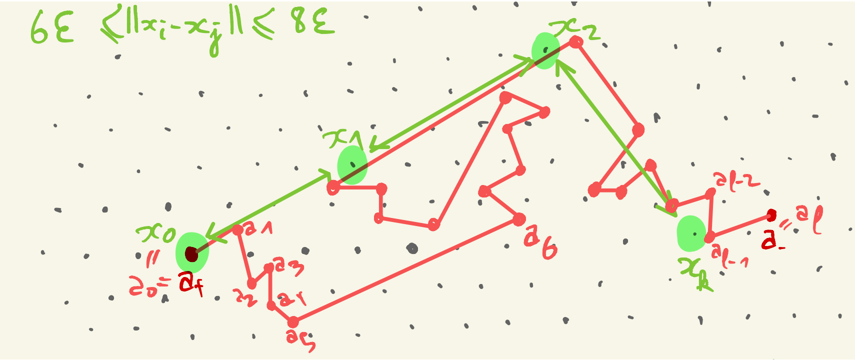
Quantify carefully everything... and you are done!

**Lemma**

*For $n \geq 2$, let $P \subseteq \mathbb{R}^n$ be a simple bounded polytope containing the origin in its interior and let $Q = P^\circ := \{x \in \mathbb{R}^n : \langle x, y \rangle \leq 1, \forall y \in P\}$ denote the polar of $P$. Then,*

$$\operatorname{diam}(P) \geq (n-1)(\operatorname{diam}(Q) - 2).$$

Thus, we only need a lower bound for $\operatorname{diam}(Q(A))$.




- Let $a_+, a_- \in A$ such that $\|a_+ - a_-\| \geq 1$.
- Let $a_+ \in N \subseteq \mathbb{S}^{n-1}$ be a minimal $\varepsilon$-net, $\varepsilon = cm^{-1/(n-1)}$.
- Set $x_0 = a_+$ and for any $k \in \mathbb{N}$,

$$X_k = \{\mathbf{x} \in N^k : x_i \neq x_j \text{ and } 6\varepsilon \leq \|x_i - x_{i+1}\| \leq 8\varepsilon\}$$

1. Setting
   - Let $a_+, a_- \in A$ such that $\|a_+ - a_-\| \geq 1$.
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2. Intermediate lemmas:
   - $\# X_k \leq (17^{n-1})^k$
   - For any path $[a_0, a_1], [a_1, a_2], \ldots, [a_{\ell-1}, a_\ell]$ with $a_0 = a_+$ and $a_\ell = a_-$, there exists $k \geq k_0 = \Omega(1/\varepsilon)$ and $\mathbf{x} \in X_k$ such that, for any $i \in [k]$, there exists $j \in [\ell]$ and $x \in [a_{j-1}, a_j]$ such that $x/\|x\| \in C(x_i, \varepsilon)$
   - It implies

$$\operatorname{diam}(Q(A)) \geq \min_{k \geq k_0} \min_{\mathbf{x} \in X_k} \sum_{0 \leq i \leq k-1} \mathbf{1}(C(x_i, \varepsilon/2) \cap A \neq \emptyset) \mathbf{1}(C(x_{i+1}, \varepsilon/2) \cap A \neq \emptyset).$$

3. Conclude with the union bound.

## THANK YOU FOR YOUR ATTENTION !


*Monotonicity of facet numbers of random convex hulls* — J. Math. Anal. Appl.
**B.**, Grote, Temesvari, Thäle, Turchi, Wespi — vol 455, 1351-1364 ('17)

*Cells with many facets in a hyperplane mosaic* — Adv. in Math.
**B.**, Calka, Reitzner — vol 324, 203-240 ('18)

*Small cells in a Poisson hyperplane tessellation* — Adv. Appl. Math.
**B.** — vol 95, 31-52 ('18)

*Polytopal approximation of elongated convex bodies* — Adv. Geom.
**B.** — vol 1, 105-114 ('18)

*Threshold phenomena for random high-dimensional...* — Commun. Contemp. Math.
**B.**, Chasapis, Grote, Temesvari, Turchi — vol 21, 5 ('18)

*Phase transition for the volume of high dimensional...* — Rand. Struct. Algo
**B.**, Kabluchko, Turchi — vol 58 ('21)

*Facets of high dimensional random polytopes* — Math. Nachr.
**B.**, O'Reilly — accepted ('22)

*Asymptotic bounds on the combinatorial diameter...* — Leibniz Int. Proc. Infor.
**B.**, Dadusch, Grupel, Huiberts, Livschyts — vol 224, SoCG 2022 ('22)