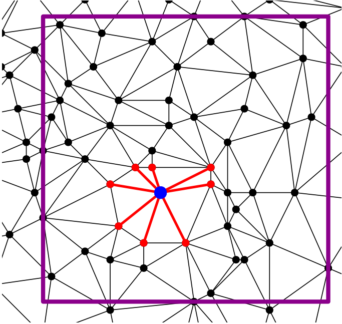
university of groningen / faculty of science and engineering



Maximal degree in random graphs

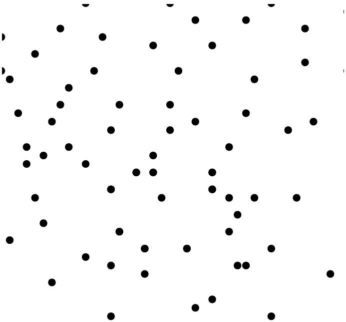
Gilles Bonnet

joint work with Nicolas Chenavier (Calais):

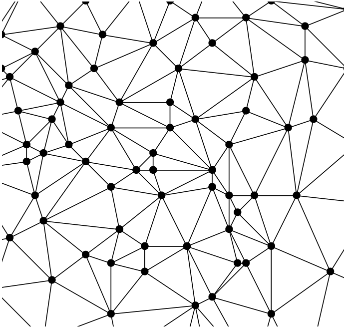
The maximal degree in a Poisson-Delaunay graph, Bernoulli, vol 26, no 2 (2020)

+ current work with my PhD student Joseph Gordon and BSc student Szymon Urban

June 30, EVA 2023, Milan



1. Poisson point process η of intensity 1



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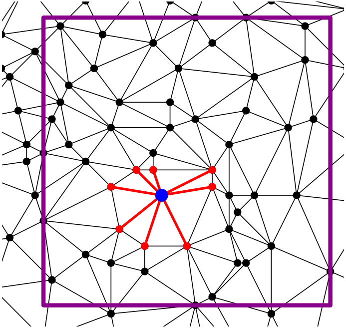
   Draw triangle if the circumscribed circle is empty



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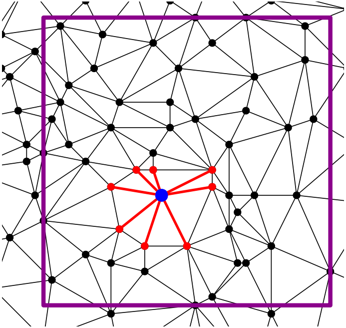
3. Let $\rho > 0$.
   $\Delta_\rho = \max_{x \in \eta \cap [0, \rho^{1/2}]^2} \deg_\eta(x)$.



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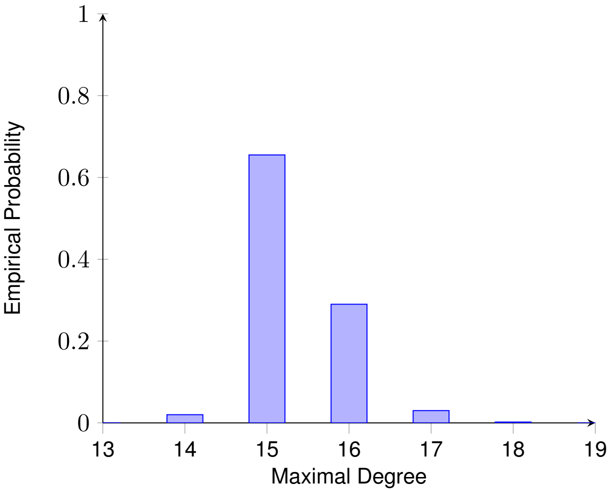
Theorem [Bern, Eppstein, Yao '91]

$$\mathbb{E}\Delta_\rho = \Theta\left(\frac{\log \rho}{\log \log \rho}\right).$$

Theorem [Broutin, Devillers, Hemsley '14]

In dimension $d = 2$. For any $\xi > 0$,

$$\mathbb{P}\left(\Delta_\rho \leq (\log \rho)^{2+\xi}\right) \to 1, \text{ as } \rho \to \infty.$$



- 75000 simulations
- window $\mathbf{W}_{10^6} = 10^3[0,1]^2$

Random combinatorial graphs (non exhaustive list!)

- ▷ Erdös-Rényi graph $G_{n,p}$ with $p = o(\frac{\log n}{n})$ [Bollobás '85]
- ▷ Uniformly distributed among a class of graphs with n vertices
  - Labelled tree [Carr, Goh, Schmutz '94]
  - Planar [Drmota, Giménez, Panagiotou, Steger '14]
  - Triangulation of a n-gon [Gao, Wormald '00]
  - Excluded minor [Giménez, Mitsche, Noy '16]

Random geometric graphs

- ▷ Gilbert graph [Penrose '03]

$$\Delta_n := \text{maximal degree over} \begin{cases} \text{all vertices} & \text{(combinatorial graphs)} \\ \text{all vertices in } [0, n^{1/d}]^d & \text{(geometric graphs)} \end{cases}$$

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$$\Delta_n \simeq \frac{\log n}{f(n)}$$ where $1 \leq f(n) \leq \log \log n$ depends on the model.

For the **Erdös-Rényi** graphs, **labelled tree** model and **Gilbert** graphs,

Δ_n concentrates on two consecutive values.

$\Delta_\rho =$ maximal degree over all vertices of a Poisson-Delaunay graph in $[0, \rho^{\frac{1}{d}}]^d$.

Theorem 1 [B., Chenavier, 2020]

Assume $d = 2$. There exists a map $\rho \mapsto I_\rho$ such that

1. $\mathbb{P}\left(\Delta_\rho \in \{I_\rho, I_\rho + 1\} \right) \to 1;$
2. $I_\rho \sim \frac{\log \rho}{2 \log \log \rho}$.

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Theorem 2 [B., Chenavier, 2020]

For any $d \geq 2$. There exists a map $\rho \mapsto J_\rho$, such that

1. $\mathbb{P}\left(\Delta_\rho \in \{J_\rho, J_\rho + 1, \dots, J_\rho + \ell_d\} \right) \to 1$, where $\ell_d = \left\lfloor \frac{d+3}{2} \right\rfloor$;
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Corollaries [B., Chenavier, 2020]

1. For $d = 2$, there exists a sequence ρ_i such that $\mathbb{P}\left(\Delta_{\rho_i} = I_{\rho_i} \right) \to 1$.
2. For any $d \geq 2$, $\mathbb{E}\, \Delta_\rho \sim \frac{\log \rho}{\frac{2}{d-1} \log \log \rho}$.

$\eta \ldots$ stationary Poisson point process

$\mathcal{D}^0 = \deg_{\eta \cup \{0\}}(0) =$ typical degree

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Theorem [B. '16]

There exist constants $c_1, c_2, c_3 > 0$, depending on d, such that,

$$c_1^k k^{\frac{-2}{d-1}k} \leq \mathbb{P}\left(\mathcal{D}^0 = k\right) \leq c_2 k^{\frac{-2}{d-1}} \mathbb{P}\left(\mathcal{D}^0 = k-1\right) \leq \ldots \leq c_3^k k^{\frac{-2}{d-1}k}$$

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Fast decay of $k \mapsto \mathbb{P}\left(\mathcal{D}^0 = k\right) \implies \mathbb{P}\left(\mathcal{D}^0 \geq k\right) \sim \mathbb{P}\left(\mathcal{D}^0 = k\right)$.

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Lemma

There exists a map $\rho \in \mathbf{R}_+ \mapsto I_\rho \in \mathbf{N}$ with the following properties:

1. $\rho\,\mathbb{P}\left(\mathcal{D}^0 \geq I_\rho + 2\right) \to 0$;
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2bis. $\left(\frac{\log\log\rho}{\log\rho}\right)^{\frac{2\ell}{d-1}} \rho\, \mathbb{P}\left(\mathcal{D}^0 \geq I_\rho - \ell\right) \to \infty$, for any $\ell \in \mathbf{N}$;

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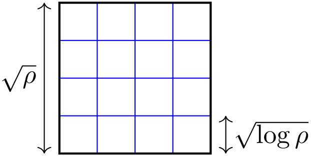
3. $I_\rho \sim \frac{\log\rho}{\frac{2}{d-1}\log\log\rho}$.

$$
\begin{aligned}
\mathbb{P}\left(\Delta_\rho \geq I_\rho + 2\right) &= \mathbb{P}\left(\sum_{x \in \eta \cap [0,\rho^{1/d}]^d} \mathbb{1}(d_\eta(x) \geq I_\rho + 2) \geq 1\right) \\
&\leq \mathbb{E}\left[\sum_{x \in \eta \cap [0,\rho^{1/d}]^d} \mathbb{1}(d_\eta(x) \geq I_\rho + 2)\right] \\
&= \rho\, \mathbb{P}\left(\mathcal{D}^0 \geq I_\rho + 2\right) \\
&\to 0. \quad \square
\end{aligned}
$$

1. Divide $[0, \rho^{1/2}]^2$ into squares of area $\simeq \log\rho$.

$$\mathbb{P}\left(\Delta_\rho < I_\rho\right) \leq \mathbb{P}\left(\Delta_{\log\rho} < I_\rho\right)^{c\frac{\rho}{\log\rho}}$$

$$\sim \exp\left(-c\frac{\rho}{\log\rho}\,\mathbb{P}\left(\Delta_{\log\rho} \geq I_\rho\right)\right).$$



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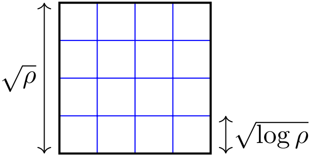

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2. In each of these squares there is at most 4 vertices with degree $\geq I_\rho$.

Lemma [B., Chenavier] **No large cluster**

Let S be a set of 5 vertices in a **planar graph**. Then there exist 2 vertices in S which have at most 23 neighbors in common.

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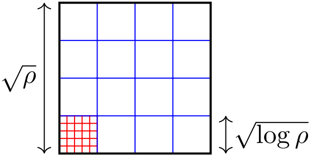

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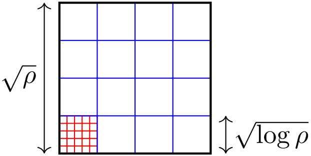

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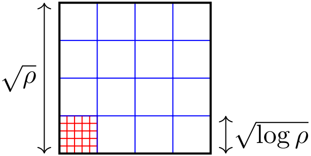
4. Put everything together. □

Proof II ($d = 2$): $\mathbb{P}\left(\Delta_\rho < I_\rho\right) \to 0$.



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2. In each of these squares there is at most 4 vertices with degree $\geq I_\rho$.

Lemma [~~B., Chenavier~~ Rémi de Verclos] **No large cluster**

Let S be a set of 5 vertices in a **planar graph**. Then there exist 2 vertices in S which have at most ~~23~~ $\mathbf{4}$ neighbors in common.

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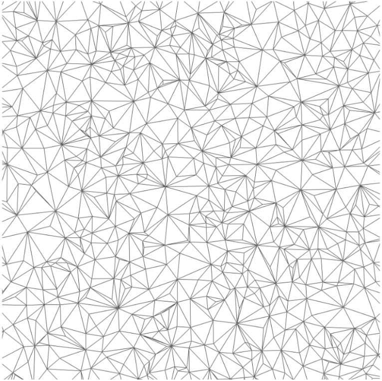


Figure: β-Delaunay tessellation, $\beta = 15$

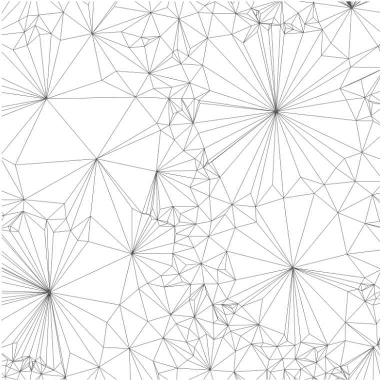


Figure: β'-Delaunay tessellation, $\beta = 2.5$

New parametric models introduced by [Gusakova, Kabluchko, Thäle]:

- The β-Delaunay tessellation I: . . . , Advances in Applied Probability (2022)
- The β-Delaunay tessellation II: . . . , Electronic Journal of Probability (2022)
- The β-Delaunay tessellation III: . . . , ALEA (2022)
- The β-Delaunay tessellation IV: . . . , Stochastics and Dynamics (2023+)
- . . .

university of groningen

$\mathcal{D}^0$ = typical degree in Poisson Delaunay

Theorem [B. '16]

$$c_1^k k^{\frac{-2}{d-1}k} \leq \mathbb{P}\left(\mathcal{D}^0 = k\right) \leq c_2^k k^{\frac{-2}{d-1}k}$$

$\mathcal{D}^0_\beta$ = typical degree in Poisson β-Delaunay

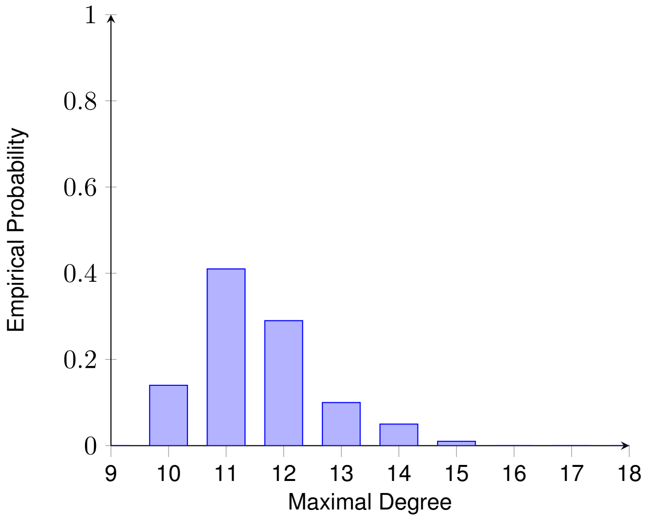
Work in progress [Joseph Gordon]

$$c_1^k k^{\frac{-2}{d-1}k} \overset{\checkmark}{\leq} \mathbb{P}\left(\mathcal{D}^0_\beta = k\right) \overset{\checkmark ?}{\leq} c_2^k k^{\frac{-2}{d-1}k}$$

$\mathcal{D}^0_{\beta'}$ = typical degree in Poisson β'-Delaunay

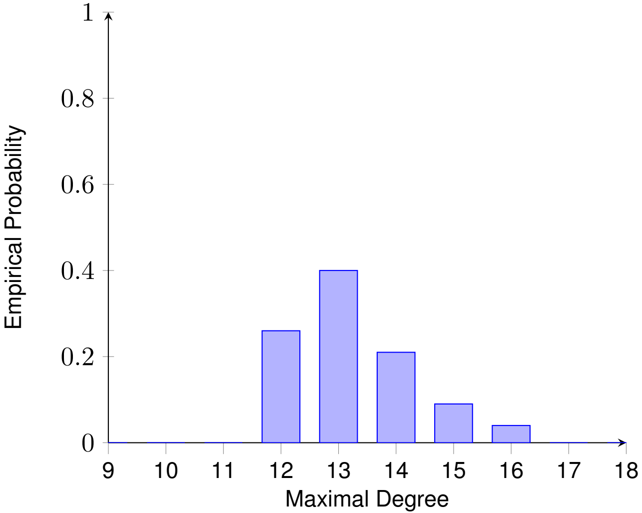
Work in progress [Joseph Gordon]

$$c_1^k \overset{\checkmark}{\leq} \mathbb{P}\left(\mathcal{D}^0_{\beta'} = k\right) \leq ?$$



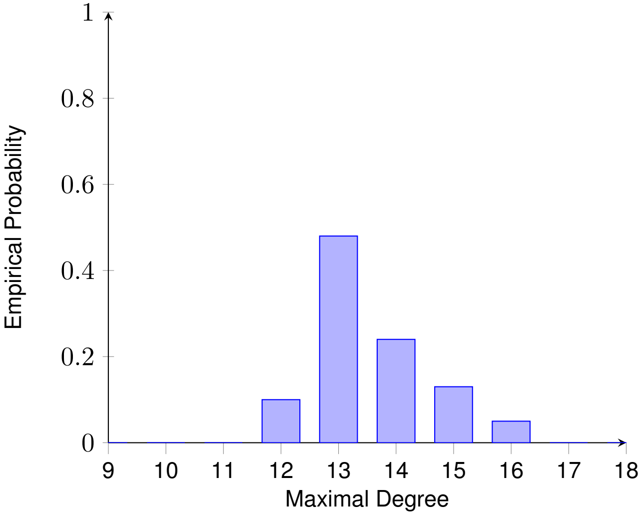
- $d = 2$
- $\beta = 1$
- $\gamma = 1$
- $\rho = 15^2$

Credits: Szymon Urban



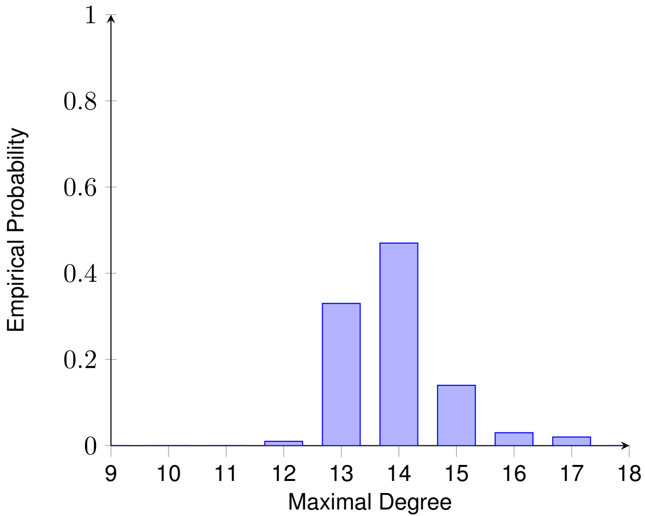
- $d = 2$
- $\beta = 1$
- $\gamma = 1$
- $\rho = 30^2$

Credits: Szymon Urban



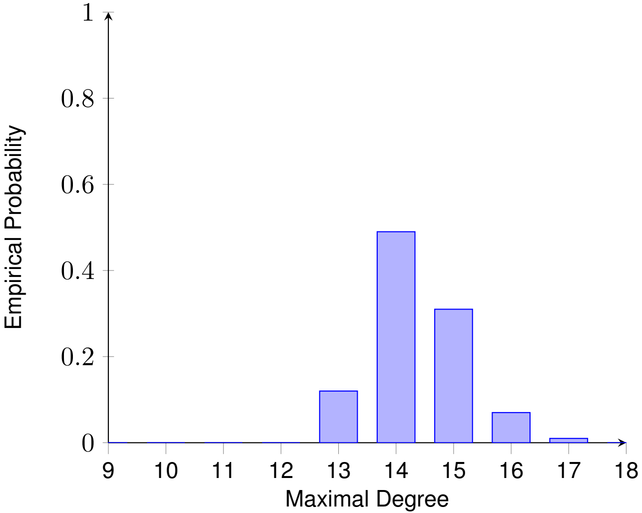
- $d = 2$
- $\beta = 1$
- $\gamma = 1$
- $\rho = 40^2$

Credits: Szymon Urban



- $d = 2$
- $\beta = 1$
- $\gamma = 1$
- $\rho = 50^2$

Credits: Szymon Urban



- $d = 2$
- $\beta = 1$
- $\gamma = 1$
- $\rho = 70^2$

Credits: Szymon Urban

Thank you!