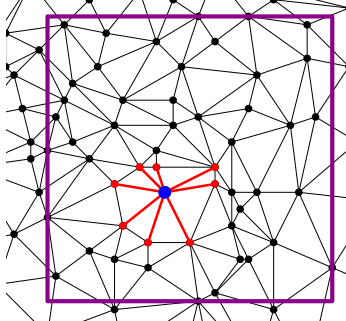




university of
groningen

faculty of science
and engineering



Maximal degree in random graphs

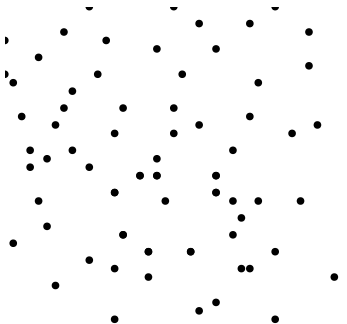
Gilles Bonnet

joint work with [Nicolas Chenavier](#) (Calais):

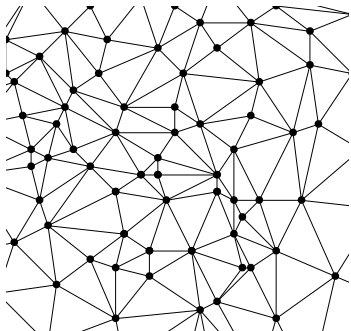
[The maximal degree in a Poisson-Delaunay graph](#), Bernoulli, vol 26, no 2 (2020)

+ current work with my PhD student [Joseph Gordon](#) and BSc student [Szymon Urban](#)

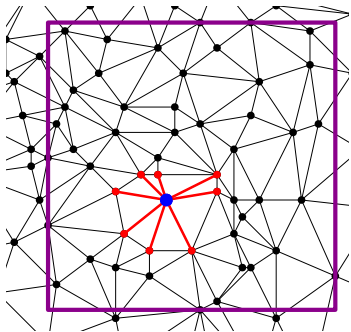
July 19 2023, Oberseminar Stokastic, Münster



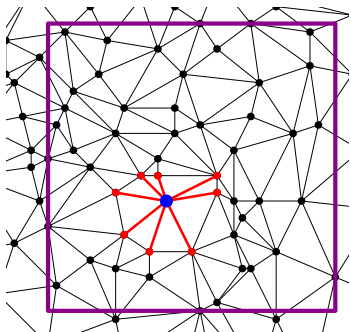
1. Poisson point process η of intensity 1



1. Poisson point process η of intensity 1
2. For any triple of points ($d = 2$):
Draw triangle if the circumscribed circle is empty



1. Poisson point process η of intensity 1
2. For any triple of points ($d = 2$):
Draw triangle if the circumscribed circle is empty
3. Let $\rho > 0$.
$$\Delta_\rho = \max_{x \in \eta \cap [0, \rho^{1/2}]^2} \deg_\eta(x).$$



1. Poisson point process η of intensity 1
2. For any triple of points ($d = 2$):
Draw triangle if the circumscribed circle is empty
3. Let $\rho > 0$.
$$\Delta_\rho = \max_{x \in \eta \cap [0, \rho^{1/2}]^2} \deg_\eta(x).$$

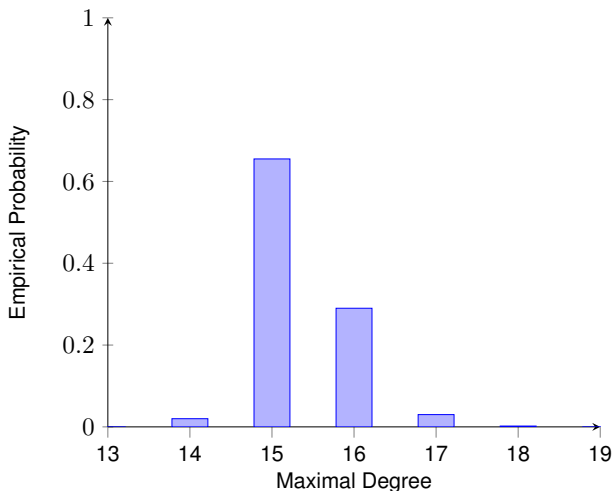
Theorem [Bern, Eppstein, Yao '91]

$$\mathbb{E}\Delta_\rho = \Theta\left(\frac{\log \rho}{\log \log \rho}\right).$$

Theorem [Broutin, Devillers, Hemsley '14]

In dimension $d = 2$. For any $\xi > 0$,

$$\mathbb{P}(\Delta_\rho \leq (\log \rho)^{2+\xi}) \rightarrow 1, \text{ as } \rho \rightarrow \infty.$$



- ▷ 75000 simulations
- ▷ window $\mathbf{W}_{10^6} = 10^3[0, 1]^2$

Random combinatorial graphs (non exhaustive list!)

- ▷ Erdős-Rényi graph $G_{n,p}$ with $p = o(\frac{\log n}{n})$ [Bollobás '85]
- ▷ Uniformly distributed among a class of graphs with n vertices
 - Labelled tree [Carr, Goh, Schmutz '94]
 - Planar [Drmota, Giménez, Panagiotou, Steger '14]
 - Triangulation of a n -gon [Gao, Wormald '00]
 - Excluded minor [Giménez, Mitsche, Noy '16]

Random geometric graphs

- ▷ Gilbert graph [Penrose '03]

$\Delta_n :=$ maximal degree over

all vertices	(combinatorial graphs)
all vertices in $[0, n^{1/d}]^d$	(geometric graphs)

Random combinatorial graphs (non exhaustive list!)

- ▷ Erdős-Rényi graph $G_{n,p}$ with $p = o(\frac{\log n}{n})$ [Bollobás '85]
- ▷ Uniformly distributed among a class of graphs with n vertices
 - Labelled tree [Carr, Goh, Schmutz '94]
 - Planar [Drmota, Giménez, Panagiotou, Steger '14]
 - Triangulation of a n -gon [Gao, Wormald '00]
 - Excluded minor [Giménez, Mitsche, Noy '16]

Random geometric graphs

- ▷ Gilbert graph [Penrose '03]

$\Delta_n :=$ maximal degree over

all vertices	(combinatorial graphs)
all vertices in $[0, n^{1/d}]^d$	(geometric graphs)

$$\Delta_n \simeq \frac{\log n}{f(n)}$$

where $1 \leq f(n) \leq \log \log n$ depends on the model.

For the **Erdős-Rényi** graphs, **labelled tree** model and **Gilbert** graphs,

Δ_n concentrates on two consecutive values.

$\Delta_\rho =$ maximal degree over all vertices of a Poisson-Delaunay graph in $[0, \rho^{\frac{1}{d}}]^d$.

Theorem 1 [B., Chenavier, 2020]

Assume $d = 2$. There exists a map $\rho \mapsto I_\rho$ such that

1. $\mathbb{P}(\Delta_\rho \in \{I_\rho, I_\rho + 1\}) \rightarrow 1$;
2. $I_\rho \sim \frac{\log \rho}{2 \log \log \rho}$.

Δ_ρ = maximal degree over all vertices of a Poisson-Delaunay graph in $[0, \rho^{\frac{1}{d}}]^d$.

Theorem 1 [B., Chenavier, 2020]

Assume $d = 2$. There exists a map $\rho \mapsto I_\rho$ such that

1. $\mathbb{P}(\Delta_\rho \in \{I_\rho, I_\rho + 1\}) \rightarrow 1$;
2. $I_\rho \sim \frac{\log \rho}{2 \log \log \rho}$.

Theorem 2 [B., Chenavier, 2020]

For any $d \geq 2$. There exists a map $\rho \mapsto J_\rho$, such that

1. $\mathbb{P}(\Delta_\rho \in \{J_\rho, J_\rho + 1, \dots, J_\rho + \ell_d\}) \rightarrow 1$, where $\ell_d = \lfloor \frac{d+3}{2} \rfloor$;
2. $J_\rho \sim \frac{\log \rho}{\frac{2}{d-1} \log \log \rho}$.

Δ_ρ = maximal degree over all vertices of a Poisson-Delaunay graph in $[0, \rho^{\frac{1}{d}}]^d$.

Theorem 1 [B., Chenavier, 2020]

Assume $d = 2$. There exists a map $\rho \mapsto I_\rho$ such that

1. $\mathbb{P}(\Delta_\rho \in \{I_\rho, I_\rho + 1\}) \rightarrow 1$;
2. $I_\rho \sim \frac{\log \rho}{2 \log \log \rho}$.

Theorem 2 [B., Chenavier, 2020]

For any $d \geq 2$. There exists a map $\rho \mapsto J_\rho$, such that

1. $\mathbb{P}(\Delta_\rho \in \{J_\rho, J_\rho + 1, \dots, J_\rho + \ell_d\}) \rightarrow 1$, where $\ell_d = \lfloor \frac{d+3}{2} \rfloor$;
2. $J_\rho \sim \frac{\log \rho}{\frac{2}{d-1} \log \log \rho}$.

Corollaries [B., Chenavier, 2020]

1. For $d = 2$, there exists a sequence ρ_i such that $\mathbb{P}(\Delta_{\rho_i} = I_{\rho_i}) \rightarrow 1$.
2. For any $d \geq 2$, $\mathbb{E} \Delta_\rho \sim \frac{\log \rho}{\frac{2}{d-1} \log \log \rho}$.

Let $X_1, X_2, \dots \in \mathbb{N}$ be i.i.d. random variables and set $M_n := \max(X_1, \dots, X_n)$.

Let $\bar{F}(x) := \mathbb{P}(X_1 > x)$.

Assume that $\mathbb{P}(X_1 = n) > 0$ for all n sufficiently large.

Let $X_1, X_2, \dots \in \mathbb{N}$ be i.i.d. random variables and set $M_n := \max(X_1, \dots, X_n)$.

Let $\bar{F}(x) := \mathbb{P}(X_1 > x)$.

Assume that $\mathbb{P}(X_1 = n) > 0$ for all n sufficiently large.

Theorem [Anderson, Journal of Applied Probability, 1970]

As $n \rightarrow \infty$,

$$\triangleright \frac{\bar{F}(n)}{\bar{F}(n+1)} \rightarrow \infty$$

\Leftrightarrow There exists $(I_n)_n$ such that $\mathbb{P}(M_n \in \{I_n, I_{n+1}\}) \rightarrow 1$.

Let $X_1, X_2, \dots \in \mathbb{N}$ be i.i.d. random variables and set $M_n := \max(X_1, \dots, X_n)$.

Let $\bar{F}(x) := \mathbb{P}(X_1 > x)$.

Assume that $\mathbb{P}(X_1 = n) > 0$ for all n sufficiently large.

Theorem [Anderson, Journal of Applied Probability, 1970]

As $n \rightarrow \infty$,

$$\triangleright \frac{\bar{F}(n)}{\bar{F}(n+1)} \rightarrow \infty$$

\Leftrightarrow There exists $(I_n)_n$ such that $\mathbb{P}(M_n \in \{I_n, I_{n+1}\}) \rightarrow 1$.

$$\triangleright \frac{\bar{F}(n)}{\bar{F}(n+1)} \rightarrow e^\alpha \text{ for some } \alpha > 0$$

\Leftrightarrow There exists $(I_n)_n$ such that for all $x \in \mathbb{R}$

$$e^{-e^{-\alpha(x-1)}} \leq \liminf \mathbb{P}(M_n \leq I_n + x) \leq \limsup \mathbb{P}(M_n \leq I_n + x) \leq e^{-e^{-\alpha x}}$$

Let $X_1, X_2, \dots \in \mathbb{N}$ be i.i.d. random variables and set $M_n := \max(X_1, \dots, X_n)$.

Let $\bar{F}(x) := \mathbb{P}(X_1 > x)$.

Assume that $\mathbb{P}(X_1 = n) > 0$ for all n sufficiently large.

Theorem [Anderson, Journal of Applied Probability, 1970]

As $n \rightarrow \infty$,

$$\triangleright \frac{\bar{F}(n)}{\bar{F}(n+1)} \rightarrow \infty$$

\Leftrightarrow There exists $(I_n)_n$ such that $\mathbb{P}(M_n \in \{I_n, I_{n+1}\}) \rightarrow 1$.

$$\triangleright \frac{\bar{F}(n)}{\bar{F}(n+1)} \rightarrow e^\alpha \text{ for some } \alpha > 0$$

\Leftrightarrow There exists $(I_n)_n$ such that for all $x \in \mathbb{R}$

$$e^{-e^{-\alpha(x-1)}} \leq \liminf \mathbb{P}(M_n \leq I_n + x) \leq \limsup \mathbb{P}(M_n \leq I_n + x) \leq e^{-e^{-\alpha x}}$$

$$\triangleright \frac{\bar{F}(n)}{\bar{F}(n+1)} \rightarrow 1$$

$\Leftrightarrow \mathbb{P}(|M_n - I_n| < y) \rightarrow 0$ for any $(I_n)_n$ and $y > 0$.

η . . . stationary Poisson point process

$$\mathcal{D}^0 = \deg_{\eta \cup \{0\}}(0) = \text{typical degree}$$

η . . . stationary Poisson point process

$\mathcal{D}^0 = \deg_{\eta \cup \{0\}}(0) = \text{typical degree}$

Theorem [B. '16]

There exist constants $c_1, c_2, c_3 > 0$, depending on d , such that,

$$c_1^k k^{\frac{-2}{d-1}k} \leq \mathbb{P}(\mathcal{D}^0 = k) \leq c_2 k^{\frac{-2}{d-1}} \mathbb{P}(\mathcal{D}^0 = k-1) \leq \dots \leq c_3^k k^{\frac{-2}{d-1}k}$$

η . . . stationary Poisson point process

$\mathcal{D}^0 = \deg_{\eta \cup \{0\}}(0) = \text{typical degree}$

Theorem [B. '16]

There exist constants $c_1, c_2, c_3 > 0$, depending on d , such that,

$$c_1^k k^{\frac{-2}{d-1}k} \leq \mathbb{P}(\mathcal{D}^0 = k) \leq c_2 k^{\frac{-2}{d-1}} \mathbb{P}(\mathcal{D}^0 = k-1) \leq \dots \leq c_3^k k^{\frac{-2}{d-1}k}$$

In particular $\frac{\mathbb{P}(\mathcal{D}^0 \geq k)}{\mathbb{P}(\mathcal{D}^0 \geq k+1)} \rightarrow \infty$.

η . . . stationary Poisson point process

$\mathcal{D}^0 = \deg_{\eta \cup \{0\}}(0) = \text{typical degree}$

Theorem [B. '16]

There exist constants $c_1, c_2, c_3 > 0$, depending on d , such that,

$$c_1^k k^{\frac{-2}{d-1}k} \leq \mathbb{P}(\mathcal{D}^0 = k) \leq c_2 k^{\frac{-2}{d-1}} \mathbb{P}(\mathcal{D}^0 = k-1) \leq \dots \leq c_3^k k^{\frac{-2}{d-1}k}$$

In particular $\frac{\mathbb{P}(\mathcal{D}^0 \geq k)}{\mathbb{P}(\mathcal{D}^0 \geq k+1)} \rightarrow \infty$.

Lemma

There exists a map $\rho \in \mathbf{R}_+ \mapsto I_\rho \in \mathbf{N}$ with the following properties:

1. $\rho \mathbb{P}(\mathcal{D}^0 \geq I_\rho + 2) \rightarrow 0$;
2. $\rho \mathbb{P}(\mathcal{D}^0 \geq I_\rho) \rightarrow \infty$;

η . . . stationary Poisson point process

$\mathcal{D}^0 = \deg_{\eta \cup \{0\}}(0) = \text{typical degree}$

Theorem [B. '16]

There exist constants $c_1, c_2, c_3 > 0$, depending on d , such that,

$$c_1^k k^{\frac{-2}{d-1}k} \leq \mathbb{P}(\mathcal{D}^0 = k) \leq c_2 k^{\frac{-2}{d-1}} \mathbb{P}(\mathcal{D}^0 = k-1) \leq \dots \leq c_3^k k^{\frac{-2}{d-1}k}$$

In particular $\frac{\mathbb{P}(\mathcal{D}^0 \geq k)}{\mathbb{P}(\mathcal{D}^0 \geq k+1)} \rightarrow \infty$.

Lemma

There exists a map $\rho \in \mathbf{R}_+ \mapsto I_\rho \in \mathbf{N}$ with the following properties:

- $\rho \mathbb{P}(\mathcal{D}^0 \geq I_\rho + 2) \rightarrow 0$;
- $\rho \mathbb{P}(\mathcal{D}^0 \geq I_\rho) \rightarrow \infty$;

2bis. $\left(\frac{\log \log \rho}{\log \rho}\right)^{\frac{2\ell}{d-1}} \rho \mathbb{P}(\mathcal{D}^0 \geq I_\rho - \ell) \rightarrow \infty$, for any $\ell \in \mathbf{N}$;

η . . . stationary Poisson point process

$\mathcal{D}^0 = \deg_{\eta \cup \{0\}}(0) = \text{typical degree}$

Theorem [B. '16]

There exist constants $c_1, c_2, c_3 > 0$, depending on d , such that,

$$c_1^k k^{\frac{-2}{d-1}k} \leq \mathbb{P}(\mathcal{D}^0 = k) \leq c_2 k^{\frac{-2}{d-1}} \mathbb{P}(\mathcal{D}^0 = k-1) \leq \dots \leq c_3^k k^{\frac{-2}{d-1}k}$$

In particular $\frac{\mathbb{P}(\mathcal{D}^0 \geq k)}{\mathbb{P}(\mathcal{D}^0 \geq k+1)} \rightarrow \infty$.

Lemma

There exists a map $\rho \in \mathbf{R}_+ \mapsto I_\rho \in \mathbf{N}$ with the following properties:

- $\rho \mathbb{P}(\mathcal{D}^0 \geq I_\rho + 2) \rightarrow 0$;
- $\rho \mathbb{P}(\mathcal{D}^0 \geq I_\rho) \rightarrow \infty$;

2bis. $\left(\frac{\log \log \rho}{\log \rho}\right)^{\frac{2\ell}{d-1}} \rho \mathbb{P}(\mathcal{D}^0 \geq I_\rho - \ell) \rightarrow \infty$, for any $\ell \in \mathbf{N}$;

- $I_\rho \sim \frac{\log \rho}{\frac{2}{d-1} \log \log \rho}$.

$$\begin{aligned}\mathbb{P}(\Delta_\rho \geq I_\rho + 2) &= \mathbb{P}\left(\sum_{x \in \eta \cap [0, \rho^{1/d}]^d} \mathbb{1}(d_\eta(x) \geq I_\rho + 2) \geq 1\right) \\ &\leq \mathbb{E}\left[\sum_{x \in \eta \cap [0, \rho^{1/d}]^d} \mathbb{1}(d_\eta(x) \geq I_\rho + 2)\right] \\ &= \rho \mathbb{P}(\mathcal{D}^0 \geq I_\rho + 2) \\ &\rightarrow 0. \quad \square\end{aligned}$$

1. Divide $[0, \rho^{1/2}]^2$ into squares of area $\simeq \log \rho$.

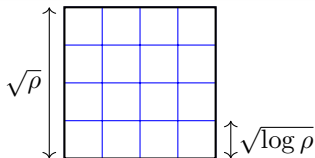
$$\begin{aligned}\mathbb{P}(\Delta_\rho < I_\rho) &\leq \mathbb{P}(\Delta_{\log \rho} < I_\rho)^{c \frac{\rho}{\log \rho}} \\ &\sim \exp\left(-c \frac{\rho}{\log \rho} \mathbb{P}(\Delta_{\log \rho} \geq I_\rho)\right).\end{aligned}$$



1. Divide $[0, \rho^{1/2}]^2$ into squares of area $\simeq \log \rho$.

$$\mathbb{P}(\Delta_\rho < I_\rho) \leq \mathbb{P}(\Delta_{\log \rho} < I_\rho)^{c \frac{\rho}{\log \rho}}$$

$$\sim \exp\left(-c \frac{\rho}{\log \rho} \mathbb{P}(\Delta_{\log \rho} \geq I_\rho)\right).$$



2. In each of these squares there is at most 4 vertices with degree $\geq I_\rho$.

Lemma [B., Chenavier]

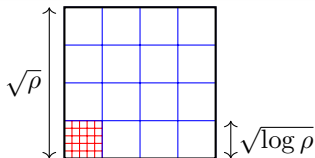
No large cluster

Let S be a set of 5 vertices in a planar graph. Then there exist 2 vertices in S which have at most 23 neighbors in common.

1. Divide $[0, \rho^{1/2}]^2$ into squares of area $\simeq \log \rho$.

$$\mathbb{P}(\Delta_\rho < I_\rho) \leq \mathbb{P}(\Delta_{\log \rho} < I_\rho)^{c \frac{\rho}{\log \rho}}$$

$$\sim \exp\left(-c \frac{\rho}{\log \rho} \mathbb{P}(\Delta_{\log \rho} \geq I_\rho)\right).$$



2. In each of these squares there is at most 4 vertices with degree $\geq I_\rho$.

Lemma [B., Chenavier]

No large cluster

Let S be a set of 5 vertices in a planar graph. Then there exist 2 vertices in S which have at most 23 neighbors in common.

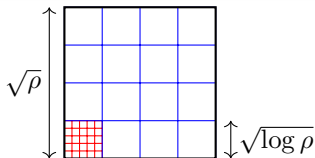
3. Subdivide $[0, \sqrt{\log \rho}]^2$ cubes into unit squares $S_1, \dots, S_{\log \rho}$.

$$\mathbb{P}(\Delta_{\log \rho} \geq I_\rho) = \mathbb{P}\left(\bigcup_{i \leq \log \rho} \{\Delta_{S_i} \geq I_\rho\}\right) \geq \frac{1}{4} \sum_{i \leq \log \rho} \mathbb{P}(\Delta_{S_i} \geq I_\rho) - [\dots]$$

$$\geq \frac{\log \rho}{20} \mathbb{P}(\mathcal{D}^0 \geq I_\rho) - [\dots].$$

1. Divide $[0, \rho^{1/2}]^2$ into squares of area $\simeq \log \rho$.

$$\begin{aligned} \mathbb{P}(\Delta_\rho < I_\rho) &\leq \mathbb{P}(\Delta_{\log \rho} < I_\rho)^{c \frac{\rho}{\log \rho}} \\ &\sim \exp\left(-c \frac{\rho}{\log \rho} \mathbb{P}(\Delta_{\log \rho} \geq I_\rho)\right). \end{aligned}$$



2. In each of these squares there is at most 4 vertices with degree $\geq I_\rho$.

Lemma [B., Chenavier]

No large cluster

Let S be a set of 5 vertices in a planar graph. Then there exist 2 vertices in S which have at most 23 neighbors in common.

3. Subdivide $[0, \sqrt{\log \rho}]^2$ cubes into unit squares $S_1, \dots, S_{\log \rho}$.

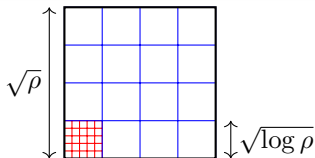
$$\begin{aligned} \mathbb{P}(\Delta_{\log \rho} \geq I_\rho) &= \mathbb{P}\left(\bigcup_{i \leq \log \rho} \{\Delta_{S_i} \geq I_\rho\}\right) \geq \frac{1}{4} \sum_{i \leq \log \rho} \mathbb{P}(\Delta_{S_i} \geq I_\rho) - [\dots] \\ &\geq \frac{\log \rho}{20} \mathbb{P}(\mathcal{D}^0 \geq I_\rho) - [\dots]. \end{aligned}$$

4. Put everything together. \square

1. Divide $[0, \rho^{1/2}]^2$ into squares of area $\simeq \log \rho$.

$$\mathbb{P}(\Delta_\rho < I_\rho) \leq \mathbb{P}(\Delta_{\log \rho} < I_\rho)^{c \frac{\rho}{\log \rho}}$$

$$\sim \exp\left(-c \frac{\rho}{\log \rho} \mathbb{P}(\Delta_{\log \rho} \geq I_\rho)\right).$$



2. In each of these squares there is at most 4 vertices with degree $\geq I_\rho$.

Lemma [B., Chenavier Rémi de Verclos]

No large cluster

Let S be a set of 5 vertices in a planar graph. Then there exist 2 vertices in S which have at most 2 neighbors in common.

3. Subdivide $[0, \sqrt{\log \rho}]^2$ cubes into unit squares $S_1, \dots, S_{\log \rho}$.

$$\mathbb{P}(\Delta_{\log \rho} \geq I_\rho) = \mathbb{P}\left(\bigcup_{i \leq \log \rho} \{\Delta_{S_i} \geq I_\rho\}\right) \geq \frac{1}{4} \sum_{i \leq \log \rho} \mathbb{P}(\Delta_{S_i} \geq I_\rho) - [\dots]$$

$$\geq \frac{\log \rho}{20} \mathbb{P}(\mathcal{D}^0 \geq I_\rho) - [\dots].$$

4. Put everything together. \square

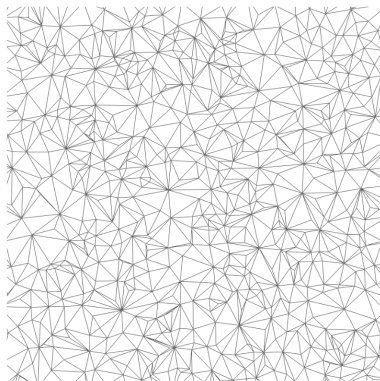


Figure: β -Delaunay tessellation, $\beta = 15$

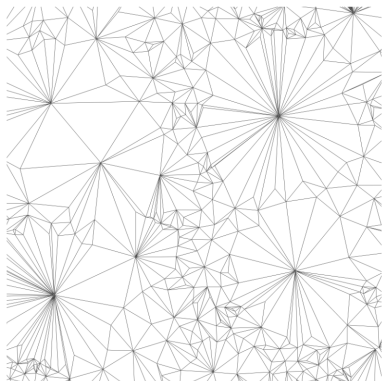


Figure: β' -Delaunay tessellation, $\beta = 2.5$

New parametric models introduced by [\[Gusakova, Kabluchko, Thäle\]](#):

- ▷ The β -Delaunay tessellation I: . . . , Advances in Applied Probability (2022)
- ▷ The β -Delaunay tessellation II: . . . , Electronic Journal of Probability (2022)
- ▷ The β -Delaunay tessellation III: . . . , ALEA (2022)
- ▷ The β -Delaunay tessellation IV: . . . , Stochastics and Dynamics (2023+)
- ▷ . . .

\mathcal{D}^0 = typical degree in Poisson Delaunay

Theorem [B. '16]

$$c_1^k k^{\frac{-2}{d-1}k} \leq \mathbb{P}(\mathcal{D}^0 = k) \leq c_2^k k^{\frac{-2}{d-1}k}$$

\mathcal{D}_β^0 = typical degree in Poisson β -Delaunay

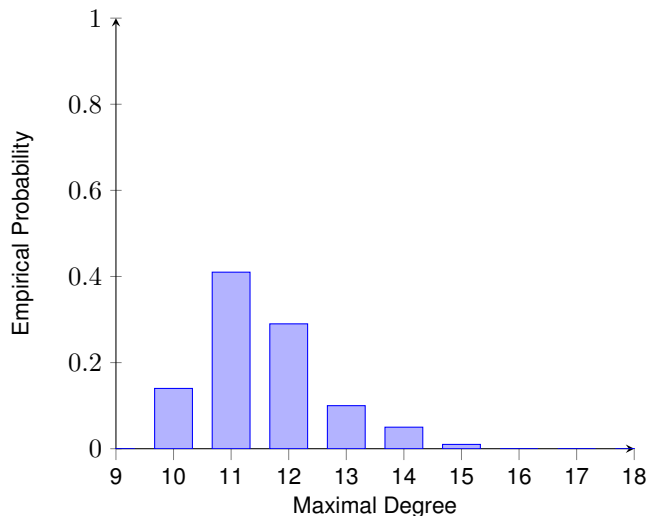
Work in progress [Joseph Gordon]

$$c_1^k k^{\frac{-2}{d-1}k} \overset{\checkmark}{\leq} \mathbb{P}(\mathcal{D}_\beta^0 = k) \overset{\checkmark?}{\leq} c_2^k k^{\frac{-2}{d-1}k}$$

$\mathcal{D}_{\beta'}^0$ = typical degree in Poisson β' -Delaunay

Work in progress [Joseph Gordon]

$$c_1^k \overset{\checkmark}{\leq} \mathbb{P}(\mathcal{D}_{\beta'}^0 = k) \leq ?$$



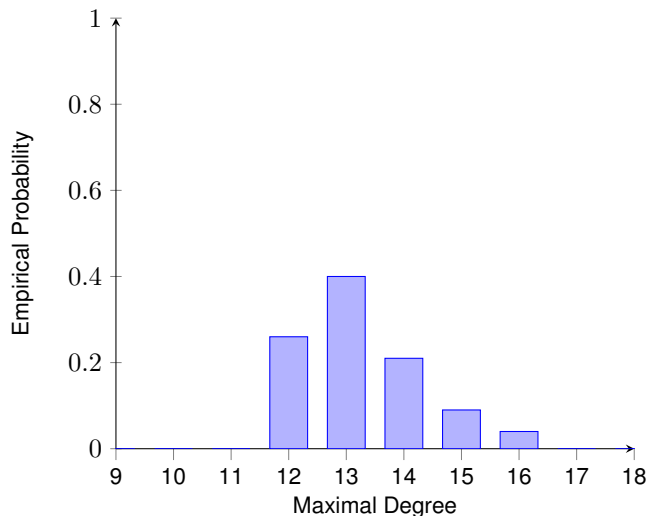
▷ $d = 2$

▷ $\beta = 1$

▷ $\gamma = 1$

▷ $\rho = 15^2$

Credits: [Szymon Urban](#)



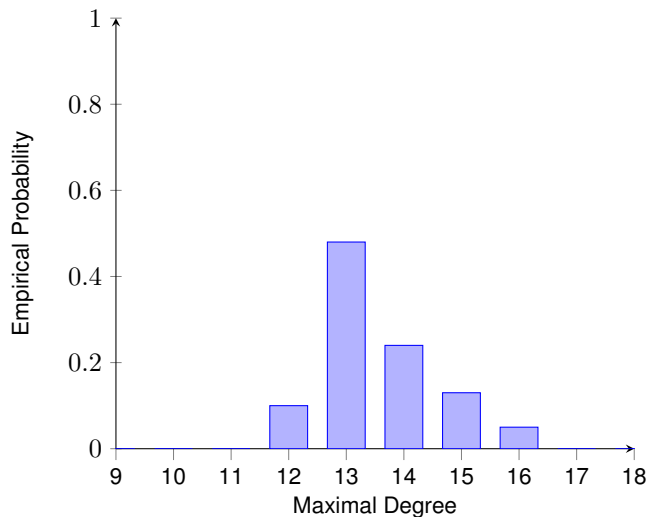
▷ $d = 2$

▷ $\beta = 1$

▷ $\gamma = 1$

▷ $\rho = 30^2$

Credits: [Szymon Urban](#)



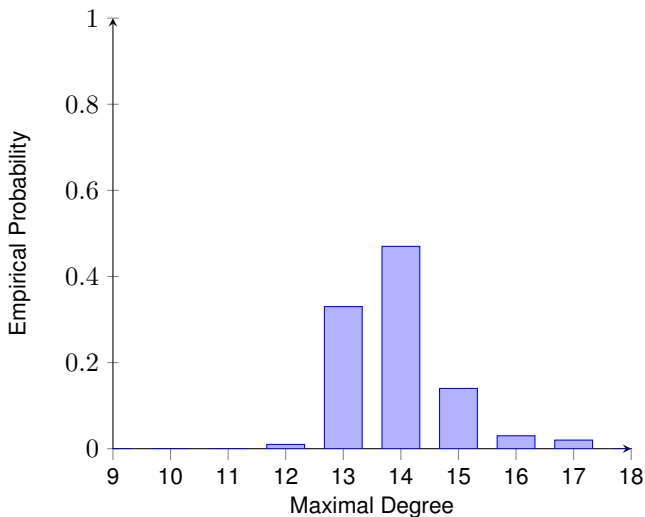
▷ $d = 2$

▷ $\beta = 1$

▷ $\gamma = 1$

▷ $\rho = 40^2$

Credits: [Szymon Urban](#)



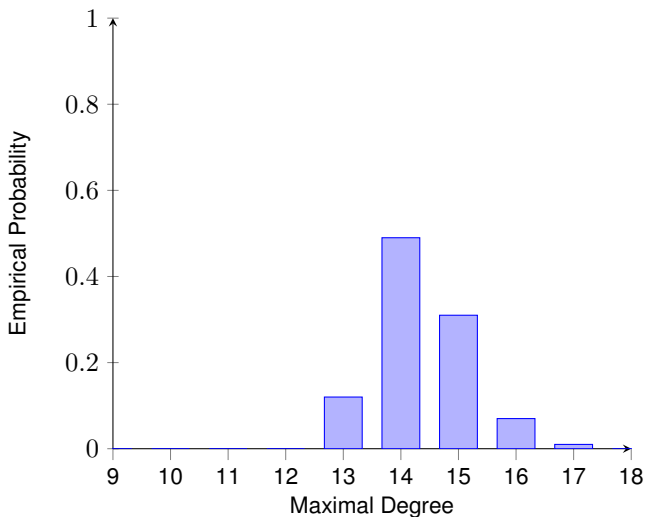
▷ $d = 2$

▷ $\beta = 1$

▷ $\gamma = 1$

▷ $\rho = 50^2$

Credits: [Szymon Urban](#)



▷ $d = 2$

▷ $\beta = 1$

▷ $\gamma = 1$

▷ $\rho = 70^2$

Credits: [Szymon Urban](#)

Thank you!

