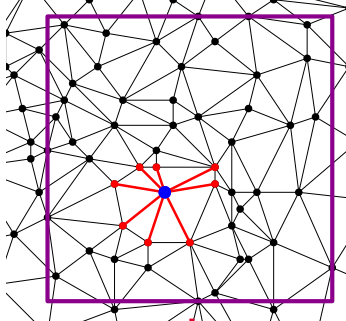




university of
groningen

faculty of science
and engineering



Maximal degree Beta (prime)-Delaunay graph

Gilles Bonnet

joint work with my PhD student [Joseph Gordon](#)

follow up of a project with [Nicolas Chenavier](#) (Université du Littoral Côte d'Opale)

October 10 2024, SPP 2265 Workshop: Geometric extreme value theory, Bochum

1 Introduction: Maximal degree in random graphs

2 Delaunay graph

3 Beta-(prime)-Delaunay graph

4 Typical degree

5 Proof ingredients

Random combinatorial graphs (non exhaustive list!)

- ▷ Erdős-Rényi graph $G_{n,p}$ with $p = o(\frac{\log n}{n})$ [Bollobás '85]
- ▷ Uniformly distributed among a class of graphs with n vertices
 - Labelled tree [Carr, Goh, Schmutz '94]
 - Planar [Drmotá, Giménez, Panagiotou, Steger '14]
 - Triangulation of an n -gon [Gao, Wormald '00]
 - Excluded minor [Giménez, Mitsche, Noy '16]

Random geometric graphs

- ▷ Gilbert graph [Penrose '03]

$\Delta_n :=$ maximal degree over

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all vertices in $[0, n^{1/d}]^d$	(geometric graphs)

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Δ_n concentrates on two consecutive values.

(Erdős-Rényi, Labelled tree, Gilbert graph)

Let $X, X_1, X_2, \dots \in \mathbb{N}$ be i.i.d. random variables and set

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Theorem [Anderson, Journal of Applied Probability, 1970]

- $\frac{\mathbb{P}(X > k)}{\mathbb{P}(X > k + 1)} \xrightarrow{k \rightarrow \infty} \infty$ (concentration)
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Idea: Show **concentration on one value** for a sequence $(k, n_k)_k$.

For this, pick n_k such that $\mathbb{P}(X > k) \gg n_k^{-1} \gg \mathbb{P}(X > k + 1)$.

$$\Rightarrow \mathbb{P}(\Delta_{n_k} \leq k') = \mathbb{P}(X \leq k')^{n_k} \cong \exp(-n_k \mathbb{P}(X > k')) \cong \begin{cases} 1 & \text{if } k' \leq k, \\ 0 & \text{if } k' > k. \end{cases}$$

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$$2. \quad \frac{\mathbb{P}(X > k)}{\mathbb{P}(X > k + 1)} \xrightarrow{k \rightarrow \infty} e^\alpha \text{ for some } \alpha > 0 \quad (\text{intermediate regime})$$

$$\Leftrightarrow \text{There exists } (k_n)_n \text{ such that for all } x \in \mathbb{R} \quad (\approx \text{Gumbel})$$

$$e^{-e^{-\alpha(x-1)}} \leq \liminf_{n \rightarrow \infty} \mathbb{P}(\Delta_n \leq k_n + x) \leq \limsup_{n \rightarrow \infty} \mathbb{P}(\Delta_n \leq k_n + x) \leq e^{-e^{-\alpha x}}$$

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$$3. \quad \frac{\mathbb{P}(X > k)}{\mathbb{P}(X > k + 1)} \xrightarrow{k \rightarrow \infty} 1 \quad (\text{spread})$$

$$\Leftrightarrow \mathbb{P}(|\Delta_n - k_n| < y) \xrightarrow{n \rightarrow \infty} 0 \text{ for any } (k_n)_n \text{ and } y > 0.$$

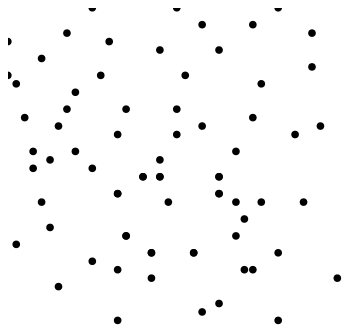
1 Introduction: Maximal degree in random graphs

2 Delaunay graph

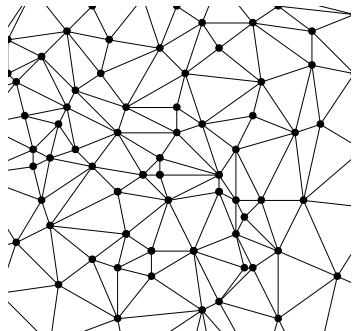
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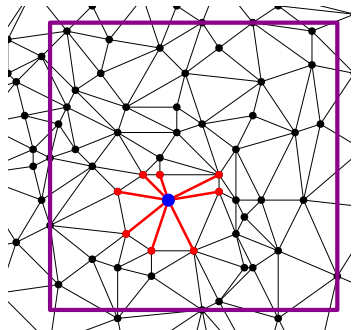
5 Proof ingredients



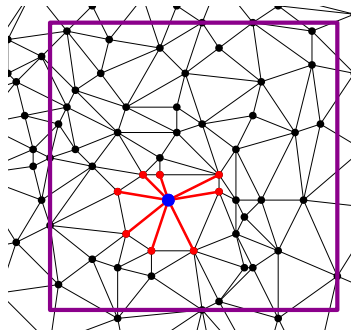
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 $\Delta_n = \max_{x \in \eta \cap W_n} \deg_\eta(x)$.



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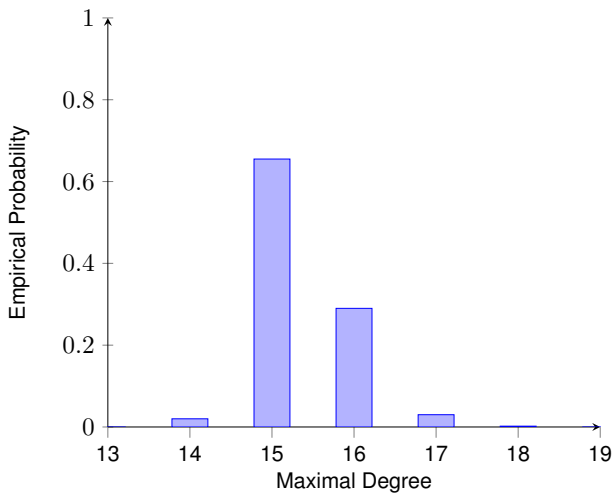
Theorem [Bern, Eppstein, Yao '91]

$$\mathbb{E}\Delta_n = \Theta\left(\frac{\log n}{\log \log n}\right).$$

Theorem [Broutin, Devillers, Hemsley '14]

In dimension $d = 2$. For any $\xi > 0$,

$$\mathbb{P}(\Delta_n \leq (\log n)^{2+\xi}) \rightarrow 1, \text{ as } n \rightarrow \infty.$$



- ▷ 75000 simulations
- ▷ window $\mathbf{W}_{10^6} = 10^3[0, 1]^2$

Δ_n = maximal degree over all vertices of a Poisson-Delaunay graph in $[0, n^{\frac{1}{d}}]^d$.

Theorem 1 [B., Chenavier, 2020]

Assume $d = 2$. There exists a map $n \mapsto I_n$ such that

1. $\mathbb{P}(\Delta_n \in \{I_n, I_n + 1\}) \rightarrow 1$;
2. $I_n \sim \frac{\log n}{2 \log \log n}$.

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Theorem 2 [B., Chenavier, 2020]

For any $d \geq 2$. There exists a map $n \mapsto J_n$, such that

1. $\mathbb{P}(\Delta_n \in \{J_n, J_n + 1, \dots, J_n + \ell_d\}) \rightarrow 1$, where $\ell_d = \lfloor \frac{d+3}{2} \rfloor$;
2. $J_n \sim \frac{\log n}{\frac{2}{d-1} \log \log n}$.

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Corollaries [B., Chenavier, 2020]

1. For $d = 2$, there exists a sequence n_i such that $\mathbb{P}(\Delta_{n_i} = I_{n_i}) \rightarrow 1$.
2. For any $d \geq 2$, $\mathbb{E} \Delta_n \sim \frac{\log n}{\frac{2}{d-1} \log \log n}$.

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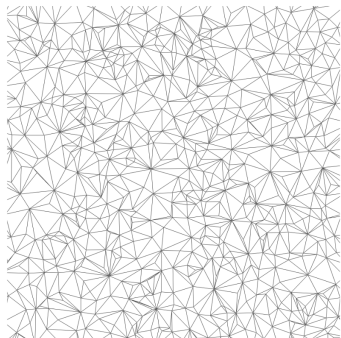


Figure: β -Delaunay tessellation, $\beta = 15$

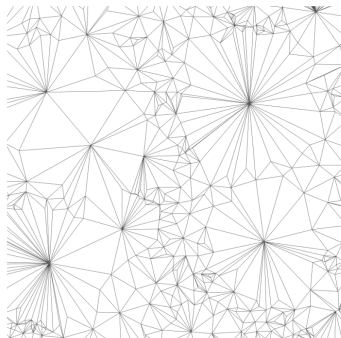


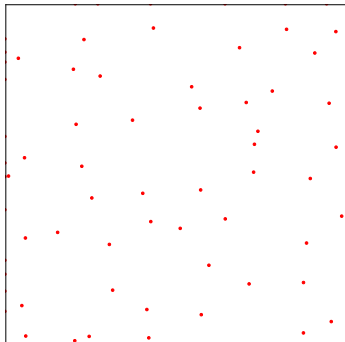
Figure: β' -Delaunay tessellation, $\beta = 2.5$

New parametric models introduced by [\[Gusakova, Kabluchko, Thäle\]](#):

- ▷ The β -Delaunay tessellation I: . . . , Advances in Applied Probability (2022)
- ▷ The β -Delaunay tessellation II: . . . , Electronic Journal of Probability (2022)
- ▷ The β -Delaunay tessellation III: . . . , ALEA (2022)
- ▷ The β -Delaunay tessellation IV: . . . , Stochastics and Dynamics (2023+)
- ▷ . . .

Poisson-Voronoi tessellation:

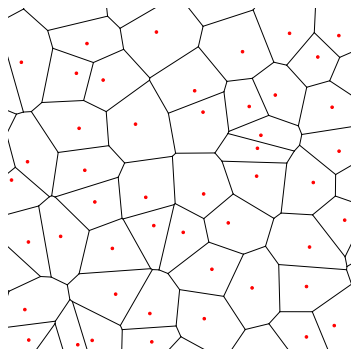
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Poisson-Voronoi tessellation:

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2. Every point $v_0 \in \eta$ has an associated Voronoi cell

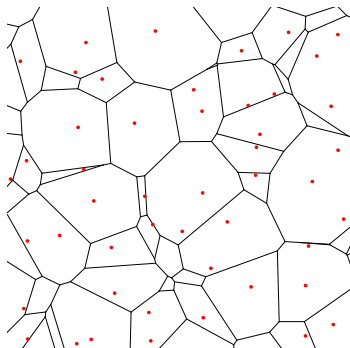
$$Z_\eta(v_0) = \{u : \|u - v_0\|^2 = \min_{v \in \eta} \|u - v\|^2\}.$$



Beta-Voronoi tessellation:

1. Poisson point process η_β in $\mathbb{R}^d \times \mathbb{R}_+$
... intensity measure of density $d\mu = h^\beta dv dh$, $\beta > -1$
2. Every point $(v_0, h_0) \in \eta_\beta$ has an associated Voronoi cell

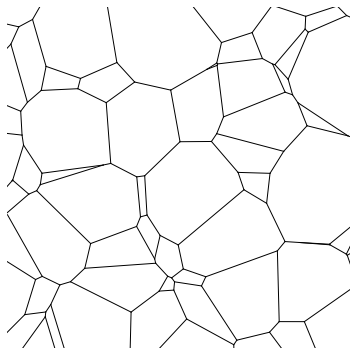
$$Z_{\eta_\beta}(v_0, h_0) = \{u : \|u - v_0\|^2 + h_0 = \min_{(v,h) \in \eta_\beta} \|u - v\|^2 + h\}.$$



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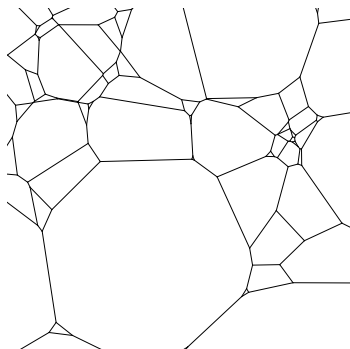
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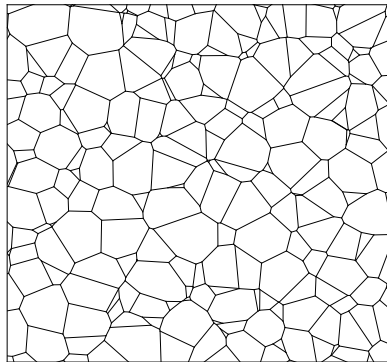
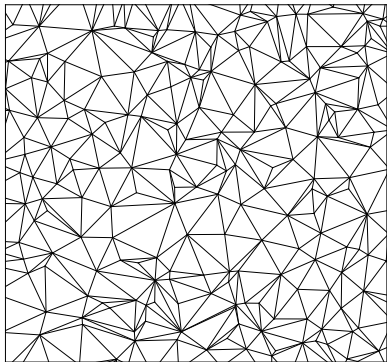


Beta-Prime-Voronoi tessellation:

1. Poisson point process η_{β}' in $\mathbb{R}^d \times \mathbb{R}_-$
 ... intensity measure of density $d\mu = (-h)^{-\beta} dv dh$, $\beta > \frac{d}{2} + 1$
2. Every point $(v_0, h_0) \in \eta_{\beta}'$ has an associated Voronoi cell

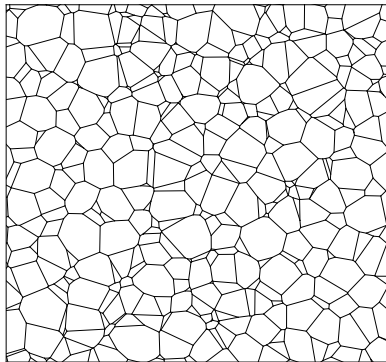
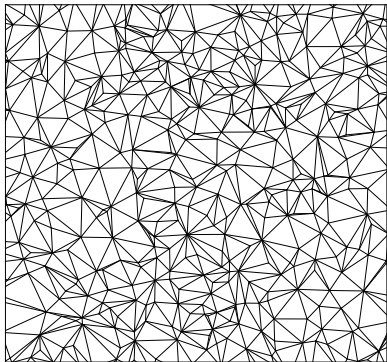
$$Z_{\eta_{\beta}'}(v_0, h_0) = \{u : \|u - v_0\|^2 + h_0 = \min_{(v, h) \in \eta_{\beta}'} \|u - v\|^2 + h\}.$$



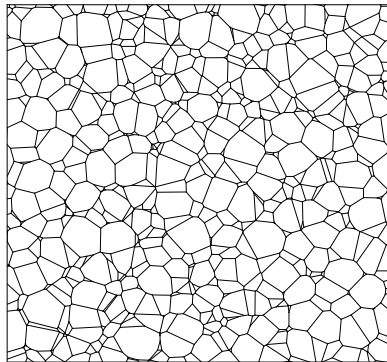
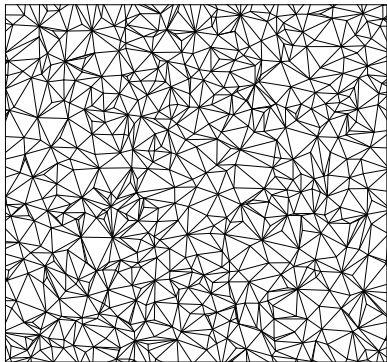


$$\gamma = 1, \beta = 0$$

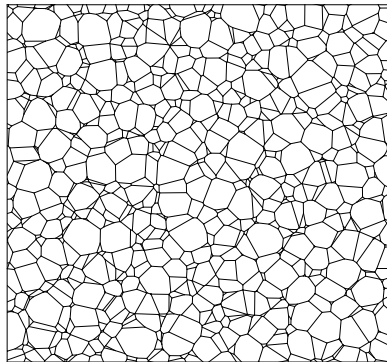
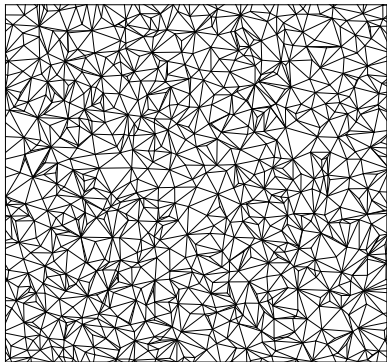
$\beta \rightarrow -1$: Classical Delaunay/Voronoi limit (ongoing work: Mathias Wolde-Lübke)



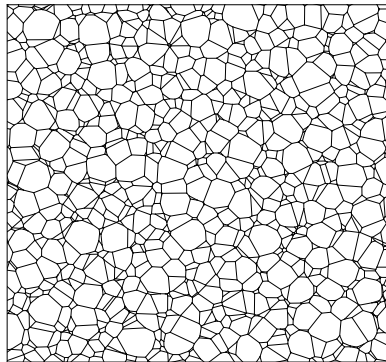
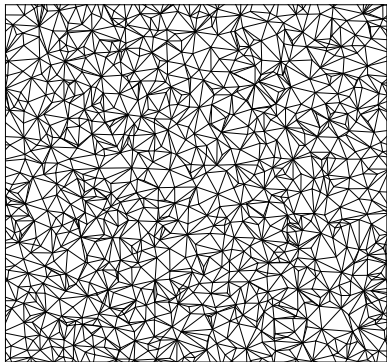
$$\gamma = 1, \beta = 1$$



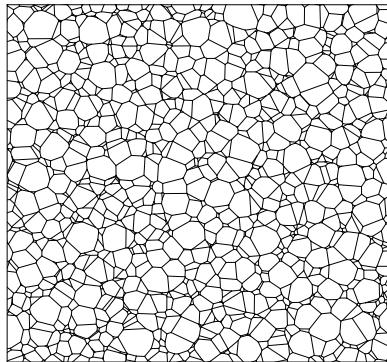
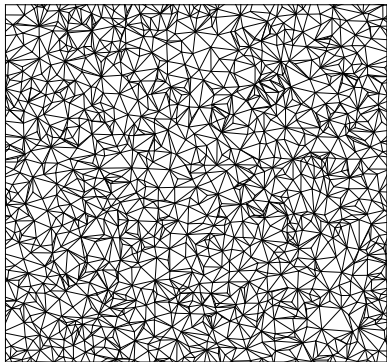
$$\gamma = 1, \beta = 2$$



$$\gamma = 1, \beta = 3$$

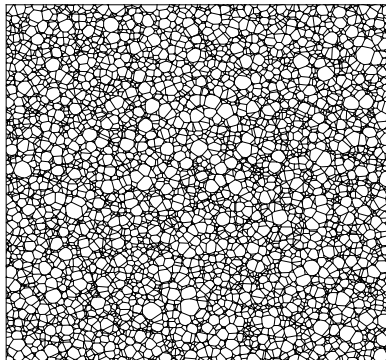
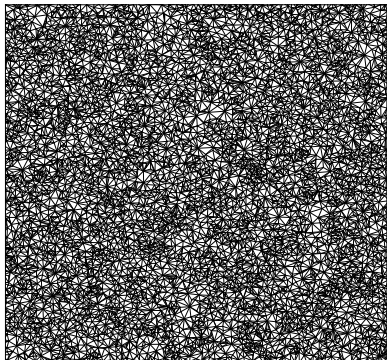


$$\gamma = 1, \beta = 4$$



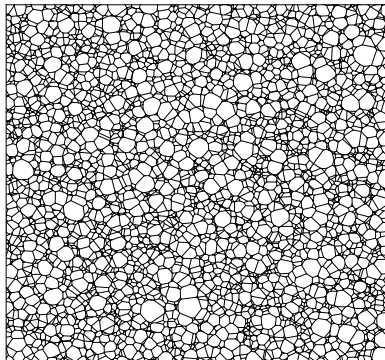
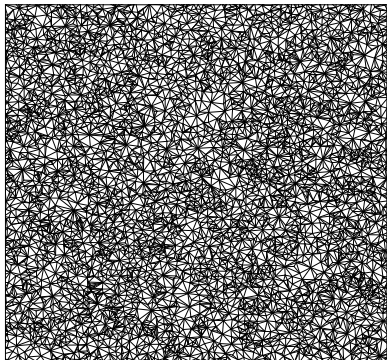
$$\gamma = 1, \beta = 5$$

$\beta \rightarrow \infty$: Gaussian Delaunay/Voronoi limit

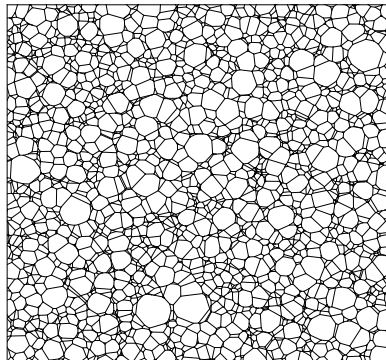
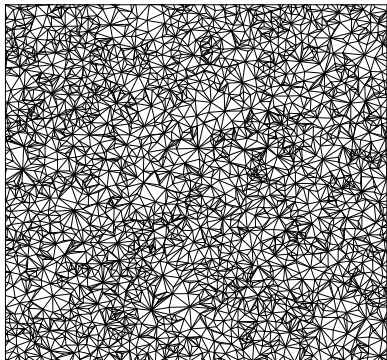


$$\gamma = 1, \beta = 30$$

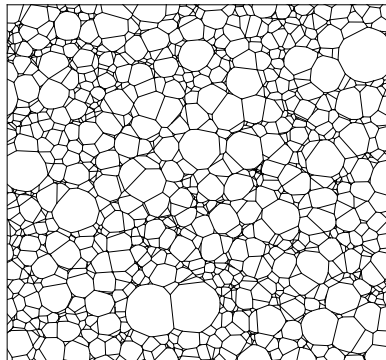
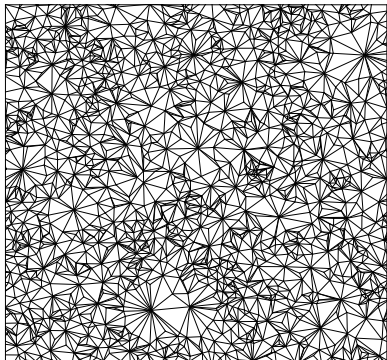
$\beta \rightarrow \infty$: Gaussian Delaunay/Voronoi limit



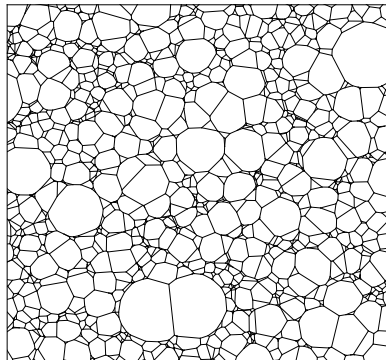
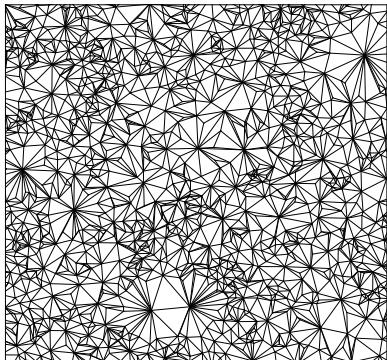
$$\gamma = 1, \beta = 20$$



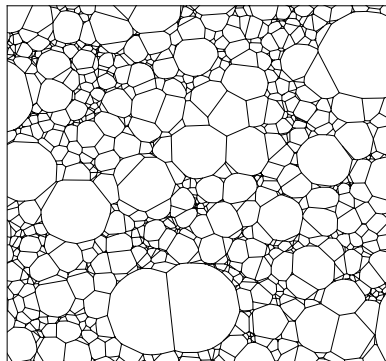
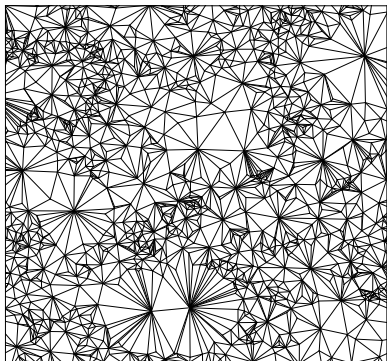
$$\gamma = 1, \beta = 10$$



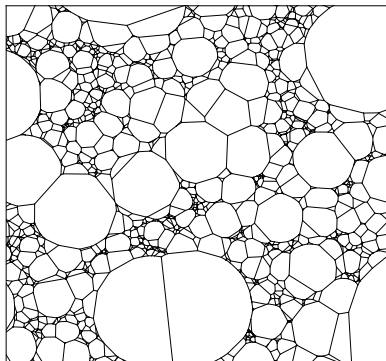
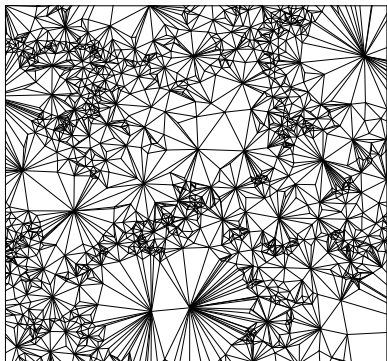
$$\gamma = 1, \beta = 5$$



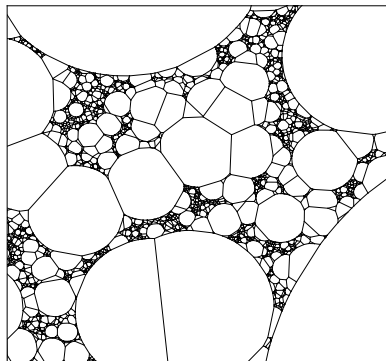
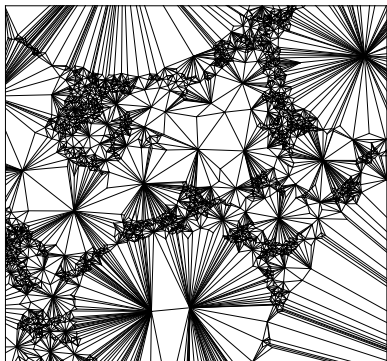
$$\gamma = 1, \beta = 4$$



$$\gamma = 1, \beta = 3$$



$$\gamma = 1, \beta = 2.5$$



$$\gamma = 1, \beta = 2.1$$

$$\beta \rightarrow 2 : ???$$

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For a stationary random graph in \mathbb{R}^d :

$$\mathbb{P}(\text{deg}_{\text{typ}} = k) = \frac{\mathbb{E} \left[\sum_{v \in \text{Vertices}} \mathbf{1}(v \in [0, 1]^d) \mathbf{1}(\text{deg}(x) = k) \right]}{\mathbb{E} \left[\sum_{v \in \text{Vertices}} \mathbf{1}(v \in [0, 1]^d) \right]}.$$

Theorem : [Hilhorst '05]

For the **Delaunay** graph in the plane ($d = 2$),

$$\mathbb{P}(\text{deg}_{\text{typ}} = k) \sim k^{-2k} k^{-1/2} c,$$

as $k \rightarrow \infty$, where $c = \frac{e^2}{4\sqrt{\pi}} \prod_{q=1}^{\infty} \left(1 - \frac{1}{q} + \frac{4}{q^4}\right)^{-1}$

See also [Hilhorst, Calka, 08].

Theorem [B. '16]

There exist constants $c_1, c_2, c_3 > 0$, depending on d , such that,

$$c_1^k k^{\frac{-2}{d-1}k} \leq \mathbb{P}(\deg_{\text{typ}} = k) \leq c_2 k^{\frac{-2}{d-1}} \mathbb{P}(\deg_{\text{typ}} = k-1) \leq \dots \leq c_3^k k^{\frac{-2}{d-1}k}$$

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... + involved/technical work to handle dependencies

\Rightarrow **concentration** (on 2 values for $d = 2$, finitely many for $d > 2$) [B., Chenavier 2020]

Ongoing work with Joseph Gordon.

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Same as for Classical Delaunay:

Theorem (Beta Delaunay)

There exist constants $c_1, c_2, c_3 > 0$, depending on d , and β , such that,

$$c_1^k k^{\frac{-2}{d-1}k} \leq \mathbb{P}(\text{deg}_{\text{typ}} = k) \leq c_2 k^{\frac{-2}{d-1}} \mathbb{P}(\text{deg}_{\text{typ}} = k - 1) \leq \dots \leq c_3^k k^{\frac{-2}{d-1}k}$$

In particular $\frac{\mathbb{P}(\text{deg}_{\text{typ}} \geq k)}{\mathbb{P}(\text{deg}_{\text{typ}} \geq k + 1)} \rightarrow \infty$.

... + work to handle dependencies (simplified approach, more robust)

\Rightarrow **concentration** (on two values for $d = 2$, finitely many for $d > 2$)

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Ongoing work with Joseph Gordon. **Different behaviour:**

Theorem (Beta-Prime Delaunay)

$$c^k \leq \mathbb{P}(\text{deg}_{\text{typ}} = k) \leq ??? \quad \text{for some } c = c_{d,\beta} \in (0, 1).$$

In particular $\liminf_{k \rightarrow \infty} \frac{\mathbb{P}(\text{deg}_{\text{typ}} \geq k)}{\mathbb{P}(\text{deg}_{\text{typ}} \geq k + 1)} \leq \frac{1}{c}$.

\Rightarrow There does **not** exist a sequence $(n_k)_{k \in \mathbb{N}}$ for which $\mathbb{P}(\Delta_{n_k} = k) \xrightarrow{k \rightarrow \infty} 1$.

\Rightarrow **anti-concentration result.** (extend the above argument to finitely many values)

Ongoing work with Joseph Gordon. **Different behaviour:**

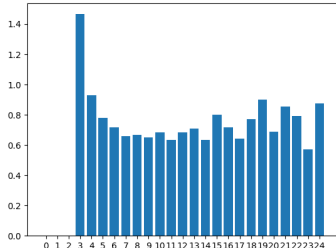
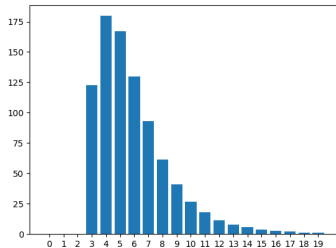
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1 Introduction: Maximal degree in random graphs

2 Delaunay graph

3 Beta-(prime)-Delaunay graph

4 Typical degree

5 Proof ingredients

$$\begin{aligned}
 \mathbb{P}(\Delta_n > k) &= \mathbb{P}\left(\sum_{x \in \eta \cap W_n} \mathbb{1}(\deg_\eta(x) > k) \geq 1\right) \\
 &\leq \mathbb{E}\left[\sum_{x \in \eta \cap W_n} \mathbb{1}(\deg_\eta(x) > k)\right] \\
 &= n \mathbb{P}(\mathcal{D}^0 > k) \\
 &\leq cn \mathbb{P}(\mathcal{D}^0 \geq k) k^{-\frac{2}{d-1}}.
 \end{aligned}$$



For $R > 0$, we construct $\{x_1, \dots, x_N\} \subset \eta \cap W_n$ such that:

- ▷ x_i is picked uniformly at random in $\eta \cap W_n \setminus \cup_{j=1}^{i-1} B(x_j, 2R)$,
- ▷ N is maximal, i.e. $\eta \cap W_n \setminus \cup_{j=1}^N B(x_j, 2R) = \emptyset$.

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▷ $|W_n \setminus \bigcup_{j=1}^N B(x_j, 2R)| \leq n - cNR^d \Rightarrow \mathbb{P}(N \leq n) \leq e^{n - cnR^d} \Rightarrow N \asymp \frac{n}{R^d}$

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- ▷ conditioned on $\{x_1, \dots, x_N\}$, the events

$$E_i = \{\text{Flower}_\eta(x_i) \subset B(x_i, R) \text{ and } \deg_{\text{typ}} \geq k\}, \quad 1 \leq i \leq N,$$

are independent, and have probability

$$\begin{aligned} p_{k,R} = \mathbb{P}(E_i) &\geq \mathbb{P}(|\text{Flower}_\eta(o)| \leq cR^d \text{ and } \deg x_i = k) \\ &= \mathbb{P}(\Gamma_k \leq cR^d) \mathbb{P}(\deg x_i = k) \quad [\text{complem. thm.}] \end{aligned}$$

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- ▷ $\mathbb{P}(\Delta_n < k) \leq \dots \leq \mathbb{E}[(1 - p_{k,R})^N]$.

7-9 April 2025, Groningen, Netherlands

WORDS GRONINGEN

Workshop On Randomness and Discrete
Structures

Thank you!