Concentration inequalities for Poisson U-statistics

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joint work with:
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$$F_m = F_m(\eta, h) := \sum_{(x_1,\dots,x_m)\in\eta^m_{\neq}} h(x_1, \dots, x_m),$$

where $\eta^m_{\neq}$ is the collection of m-tuples of distinct points of η and $h : \mathbb{X}^m \mapsto \mathbb{R}$ is some symmetric function (kernel), satisfying $h \in L^1(\Lambda^m)$

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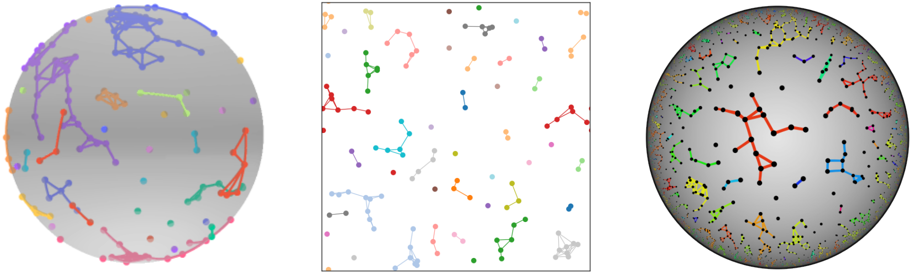
Some interesting functionals of stochastic geometry models and random graphs are Poisson U-statistics!

Examples: random geometric graph



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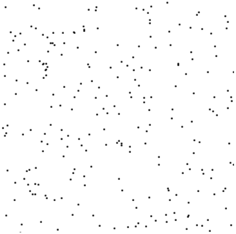

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  Vertices: $V = \eta$
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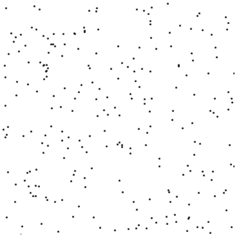

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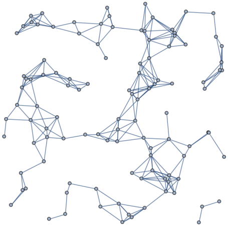

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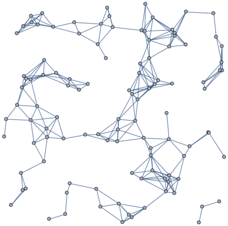
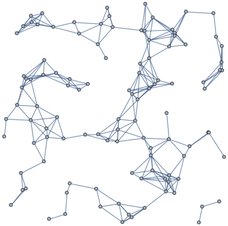


Power edge length:

$$F_2^{(\tau)}(\gamma, \rho) = \sum_{\substack{(x_1, x_2) \text{ - edge of} \\ G(\eta, \rho)}} \mathrm{dist}(x_1, x_2)^\tau$$

where $\tau \geq 0$.



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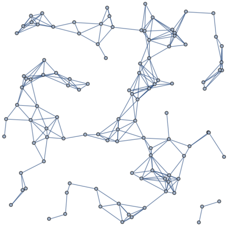

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$$F_2^{(0)}(\gamma, \rho) = \frac{1}{2} \sum_{(x_1, x_2) \in \eta^2_{\neq}} \mathbf{1}\{\mathrm{dist}(x_1, x_2) \leq \rho\} = \text{total number of edges}$$

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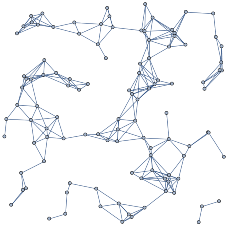

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$$F_2^{(1)}(\gamma, \rho) = \frac{1}{2} \sum_{(x_1, x_2) \in \eta^2_{\neq}} \mathrm{dist}(x_1, x_2) \mathbf{1}\{\mathrm{dist}(x_1, x_2) \leq \rho\} = \text{total edge length}$$

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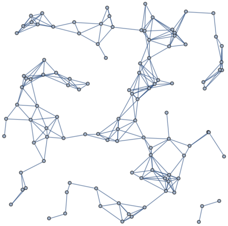

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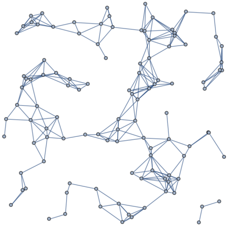

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Subgraph count: Let H_m be some connected graph on m vertices, then

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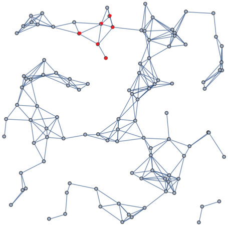

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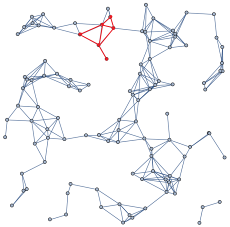

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- $A_{d,d-1}$ - space of hyperplanes in $\mathbb{R}^d$
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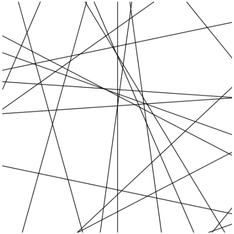

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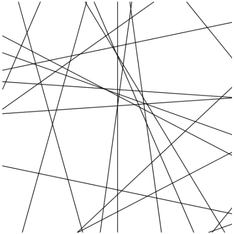

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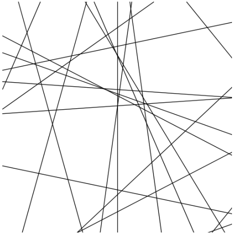
Intersection process:

$$\frac{1}{m!}\sum_{(H_1,\dots,H_m)\in\eta^m_{\neq}}\delta_{H_1\cap\dots\cap H_m}\mathbf{1}\{\dim(H_1\cap\dots\cap H_m)=d-m\}$$

where

- $1\le m\le d$



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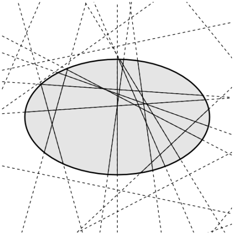
- V_i - i-th intrinsic volume, $0 \leq i \leq d - m$



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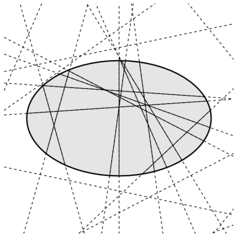

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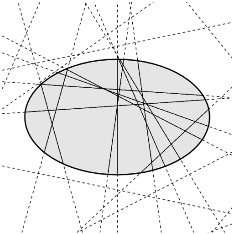

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$F_{d,0}$ is the number of intersection points within W

Variance: For a kernel h and $1 \le k \le m$ define

$$h_k(y_1, \dots, y_k) := \binom{m}{k} \int_{\mathbb{X}^{m-k}} h(y_1, \dots, y_k, z_1, \dots, z_{m-k}) \Lambda(\mathrm{d}z_1) \dots \Lambda(\mathrm{d}z_{m-k}).$$

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Then if $h_k \in L^2(\Lambda^k)$ for all $1 \leq k \leq m$ we have

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CLT and consequences

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Reitzner, Schulte, 2013; Schulte, 2016; Quantitative CLT w.r.t. Kolmogorov distance: Let F_m be a U-statistic, such that kernel h and measure Λ are independent of γ, $\mathbb{E}[F_m]^2 < \infty$ and $\|h_1\|_{L^2(\Lambda)} > 0$. Then

$$\sup_{s \in \mathbb{R}} \left| \mathbb{P}\Big(F_m - \mathbb{E} F_m \leq s\sqrt{\operatorname{var} F_m} \Big) - \Phi(s) \right| \leq C\gamma^{-1/2},$$

where $\Phi(s)$ is cumulative distribution function of standard Gaussian random variable.

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Find optimal rate function $I : \mathbb{R}_+^2 \to \mathbb{R}_+$ satisfying $\lim_{t\to\infty} I(\gamma, t) = \infty$ for any fixed γ, and such that

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- Local U-statistics (Bachmann, Reitzner'17): If $\mathbb{X} = \mathbb{R}^d$, $F_m < \infty$ a.s. and there are $0 < \rho_1 \leq \rho_2$, $0 < M_1 \leq M_2$ such that

$$h(x_1,\dots,x_m)\begin{cases} > 0 \text{ if } \operatorname{diam}(x_1,\dots,x_m) \leq \rho_1, \\ = 0 \text{ if } \operatorname{diam}(x_1,\dots,x_m) > \rho_2, \end{cases} \qquad h \in \{0\} \cup [M_1, M_2],$$

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Main result

Theorem (B., Gusakova, 2024+)

Assume F_m satisfies condition (A1) [shown on next slide] and $\|h_1\|_{L^2(\Lambda)} > 0$. Then there are constants $c, C \in (0,\infty)$, s.t. for all $t \geq 0$ and $\gamma \geq C^{-1}$ we have

$$\mathbb{P}(|F_m - \mathbb{E}F_m| \geq t) \leq 2\exp\left(-ct^{\frac{1}{m}} \min\left(\left[\frac{t}{\gamma^m}\right]^{2-\frac{1}{m}}, 1 + \log_+\left[\frac{t}{\gamma^m}\right]\right)\right),$$

where $\log_+(\cdot) = \mathbf{1}(\log(\cdot) \geq 0)\log(\cdot)$ denotes the positive part of the logarithm.

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What is this condition (A1)?

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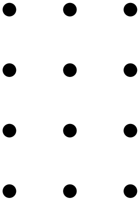

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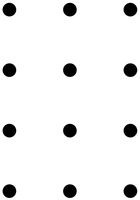

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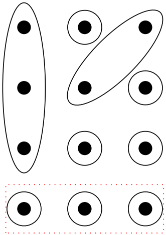

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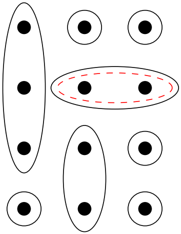

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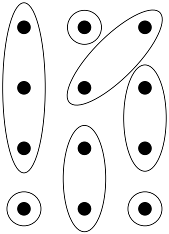

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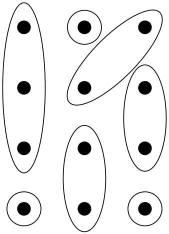

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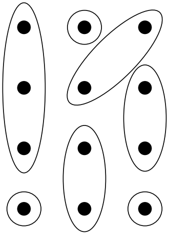

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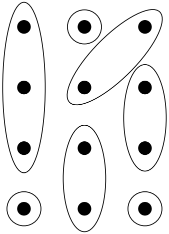

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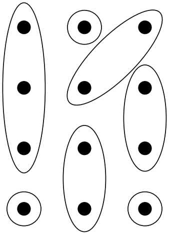

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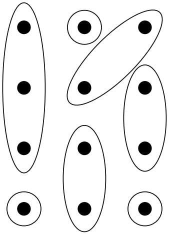

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Main result (revised)

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Assume F_m satisfies (A1). Then for all $\gamma \geq 8m\beta_0\beta_1^{-1}$ and $t \geq 0$ we have

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Idea of the proof

Use

$$\mathbb{P}(|F_m - \mathbb{E}F_m| \geq t) \leq \min_{\ell-\text{even}} \frac{\mathbb{E}(F_m - \mathbb{E}F_m)^\ell}{t^\ell}. \tag{1}$$

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Combing this inequalities and minimizing (1) with respect to ℓ gives concentration bounds for $t \geq C'\beta_2(\gamma\beta_1)^{m-\frac{1}{2}} \asymp \sqrt{\mathrm{var} F_m}$

Idea of the proof

Use

$$\mathbb{P}(|F_m - \mathbb{E}F_m| \geq t) \leq \min_{\ell-\text{even}} \frac{\mathbb{E}(F_m - \mathbb{E}F_m)^\ell}{t^\ell}. \tag{1}$$

Due to (A1) and careful analysis of the set of partitions $\Pi(m, \ell)$ we get

$$\mathbb{E}[(F_m - \mathbb{E}F_m)^\ell] \leq \beta_0 \left(\frac{(m\ell)^m \beta_2}{\log^m \left(1 + \frac{m\ell}{\gamma\beta_1}\right)} \right)^\ell \quad \ell \geq 2, \gamma\beta_1 > 0,$$

$$\mathbb{E}[(F_m - \mathbb{E}F_m)^\ell] \leq \beta_0 \Big(2^{2m+1} m\ell\beta_2^2 (\gamma\beta_1)^{2m-1} \Big)^{\ell/2}, \ell \geq 2, \gamma\beta_1 \geq 2m\ell.$$

Combing this inequalities and minimizing (1) with respect to ℓ gives concentration bounds for $t \geq C'\beta_2(\gamma\beta_1)^{m-\frac{1}{2}} \asymp \sqrt{\mathrm{var} F_m}$

For $0 \leq t \leq \sqrt{\mathrm{var} F_m}$ we use Chebyshev-Cantelli inequality

$$\mathbb{P}(F_m - \mathbb{E}F_m \geq t) \leq \exp\Big(-\log\Big(1 + \frac{t^2}{\mathrm{var} F_m}\Big)\Big).$$

Simplified conditions: (A2) and (A3)

Condition (A2): there are $\alpha_1, \alpha_2 > 0$ such that $\Lambda(\mathbb{X}) = \alpha_1$ and $|h| \leq \alpha_2$.

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Example of *U*-statistic: Intrinsic volumes of intersection process of Poisson hyperplanes.

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Condition (A3): $\mathbb{X}$ is equipped with a metric dist, $h \in L^1(\Lambda^m)$ and there are $\rho > 0$ and $M > 0$ such that

$$0 \leq h(x_1, \dots, x_m) \leq M \cdot \mathbf{1}\{\mathrm{diam}(x_1, \dots, x_m) \leq \rho\}, \text{ (local } U\text{-statistic)}$$
$$C(\rho, \Lambda) := \sup_{x \in \mathbb{X}} \Lambda(B(x, \rho)) < \infty,$$

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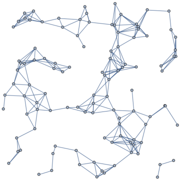
Lemma (B., Gusakova, 2024+)

1. *(A2) $\Rightarrow$ (A1) with* $\beta_0 = 1$, $\beta_1 = \alpha_1$, $\beta_2 = \alpha_2$.
2. *(A3) $\Rightarrow$ (A1) with* $\beta_0 = 1$, $\beta_1 = C(\rho, \Lambda)$, $\beta_2 = M \max\left(1, \frac{\|h\|_{L^1(\Lambda^m)}}{MC(\rho,\Lambda)^m}\right)^{\frac{1}{2}}$.



- $\mathbb{X}$ is a metric space with metric dist
- Λ - σ-finite non-atomic measure
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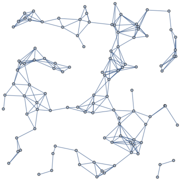
- $G(\eta, \rho)$, $\rho > 0$ - random geometric graph

Poisson U-statistics:

$$F_2^{(\tau)}(\gamma, \rho) = \frac{1}{2} \sum_{(x_1, x_2) \in \eta^2_{\neq}} \mathrm{dist}(x_1, x_2)^{\tau} \mathbf{1}\{\mathrm{dist}(x_1, x_2) \le \rho\}, \qquad \tau \ge 0.$$



- $\mathbb{X} = \mathbb{R}^d$ with usual Euclidean distance
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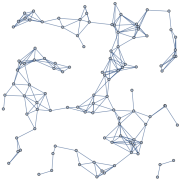

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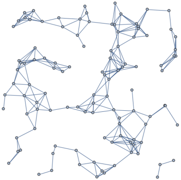
Remark: $F_2^{(\tau)}$ satisfies (A3).



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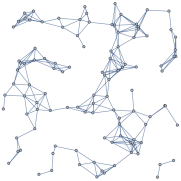
For any $\rho \leq r$ and $\gamma\rho^d > c''$ we have

$$\mathbb{P}(|F_2^{(\tau)} - \mathbb{E}F_2^{(\tau)}| \geq t) \leq 2\exp\Big(- c\Big(\frac{t}{\rho^\tau (r/\rho)^{\frac{d}{2}}}\Big)^{\frac{1}{2}} \times \min\Big(\Big[\frac{t}{\rho^\tau (r/\rho)^{d/2} (\rho^d\gamma)^2}\Big]^{3/2}, 1 + \log_+ \Big[\frac{t}{\rho^\tau (r/\rho)^{d/2} (\rho^d\gamma)^2}\Big]\Big)\Big).$$

Applications: random geometric graphs



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Remark: Reitzner, Schulte, Thäle, 2017 showed inequalities with rate $t^{\frac{1}{3}}$ as $t \to \infty$.

- $A_{d,d-1}$ - space of hyperplanes in $\mathbb{R}^d$
- μ_{d-1} - rigid motion invariant measure on $A_{d,d-1}$
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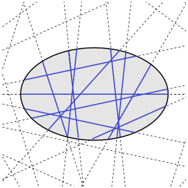

$$F_{m,i}(r,\gamma) = \frac{1}{m!} \sum_{(H_1,\ldots,H_m)\in\eta^m_{\neq}} V_i(H_1 \cap \ldots \cap H_m \cap B(o,r))\mathbf{1}\{\dim(H_1 \cap \ldots \cap H_m) = d-m\}.$$

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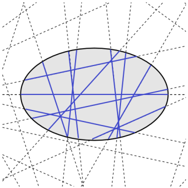

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- $F_{m,i}(r,\gamma)$ only depends on restriction of η to $A_{d,d-1}(r)$

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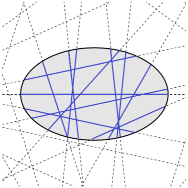

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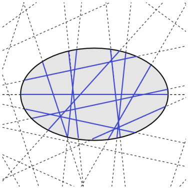

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$\Rightarrow$ (A2) applies.

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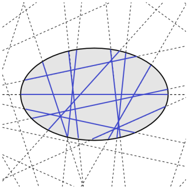

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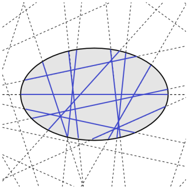

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Which regime to consider $\gamma \to \infty$ or $r \to \infty$?

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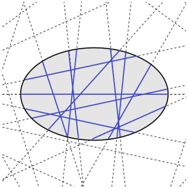

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Which regime to consider $\gamma \to \infty$ or $r \to \infty$?
Since $F_{m,i}(r,\gamma) \stackrel{d}{=} r^i F_{m,i}(1, r\gamma)$ it doesn't matter; in particular $F_{m,i}$ satisfies CLT in both regimes.

Applications: Poisson hyperplane process (hyperbolic)

- $\mathbb{H}^d$ - d-dimensional hyperbolic space
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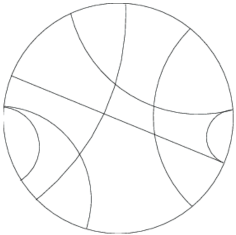

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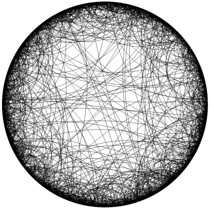

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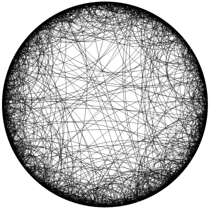

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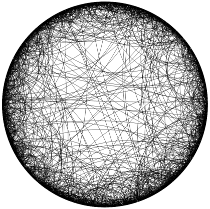

Intersection process:

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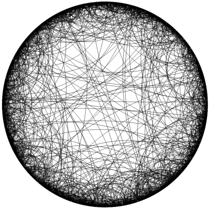

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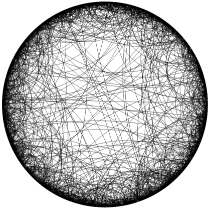

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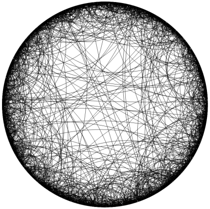

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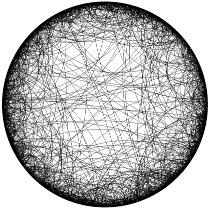

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What do we know about CLT?

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$c(r) \to 0$ as $r \to \infty$, so condition (A2) is not good enough in this case and we need to work with (A1).

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Theorem (B., Gusakova, 2024+)

For $\gamma > 1$ and sufficiently big r we get

$$\mathbb{P}(|F(r,\gamma) - \mathbb{E}F(r,\gamma)| \geq \sqrt{\mathrm{var} F(r,\gamma)}s) \leq \exp\left(-c \cdot s^2\right),$$

which holds for

$$0 < s \leq C(\gamma e^r)^{1/2} \quad \textit{if} \quad d = 2,$$
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For any $d \geq 4$, $r > 0$ and $\gamma > 0$ we have

$$\mathbb{P}\Big(\frac{F(r,\gamma) - \mathbb{E}(r,\gamma)}{c_d e^{(d-2)r}} \geq s\Big) \leq \exp\left(-s\log(s) + cs\right), \qquad s > c'\gamma.$$

Thank you for attention!