university of groningen

Model reduction for linear systems by balancing

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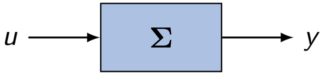
Outline

Main topics

- Problem formulation
- Observability Gramian
- Controllability Gramian
- Balanced truncation
- Balanced singular perturbation
- Example
- Conclusions

More advanced topics

- Hankel operator
- Fundamental lower bound
- Positive real balancing
- Conclusions



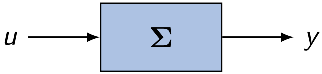
State-space representation

$$\Sigma : \begin{cases} \dot{x} = Ax + Bu \\ y = Cx + Du \end{cases}$$

with

$$x \in \mathbb{R}^n, \quad u \in \mathbb{R}^m, \quad y \in \mathbb{R}^p, \qquad m, p \ll n$$

Linear systems



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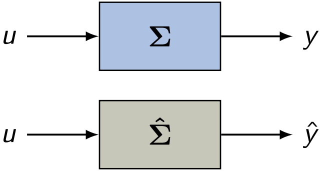
Transfer function (input-output representation)

$$G(s) = C(sI - A)^{-1}B + D$$

Problem statement

Problem. Given a dynamical system Σ, find a reduced-order system $\hat{\Sigma}$ that approximates its input-output behavior



$$\Sigma : \begin{cases} \dot{x} = Ax + Bu \\ y = Cx + Du \end{cases}$$

with $x \in \mathbb{R}^n$, n large

$$G(s) = C(sI - A)^{-1}B + D$$

$$\hat{\Sigma}_k : \begin{cases} \dot{\xi} = \hat{A}\xi + \hat{B}u \\ \hat{y} = \hat{C}\xi + \hat{D}u \end{cases}$$

with $\xi \in \mathbb{R}^k$, $k < n$ small

$$\hat{G}_k(s) = \hat{C}(sI - \hat{A})^{-1}\hat{B} + \hat{D}$$

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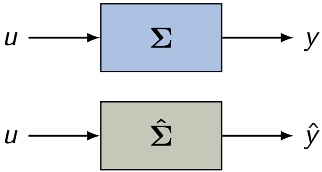

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with $x \in \mathbb{R}^n$, n large — with $\xi \in \mathbb{R}^k$, $k < n$ small

Objectives

1. Preservation of stability

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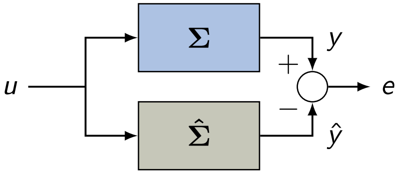

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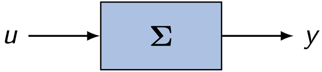
with $x \in \mathbb{R}^n$, n large — with $\xi \in \mathbb{R}^k$, $k < n$ small

Objectives

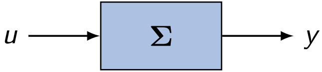
1. Preservation of stability
2. Error $e = y - \hat{y}$ "small"



Signal norm: $\mathcal{L}_2$ signal norm for $x : [0, \infty) \to \mathbb{R}^n$

$$\|x\|_{\mathcal{L}_2} = \sqrt{\int_0^\infty \|x(t)\|^2 \, dt}, \quad \text{with} \quad \|x(t)\|^2 = x^\mathsf{T}(t)x(t)$$

Signal and system norms



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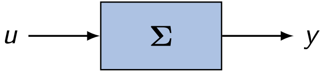
System norm: $\mathcal{L}_2$ gain

$$\|\boldsymbol{\Sigma}\|_{\mathcal{L}_2} = \sup_{u\in\mathcal{L}_2,\, u\neq 0} \frac{\|y\|_{\mathcal{L}_2}}{\|u\|_{\mathcal{L}_2}} \qquad (\text{for } x_0 = 0)$$



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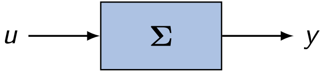

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Properties

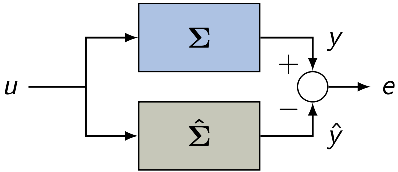
- Let $\gamma \geq \|\boldsymbol{\Sigma}\|_{\mathcal{L}_2}$. For any input function u, it holds that

$$\|y\|_{\mathcal{L}_2} \leq \gamma \|u\|_{\mathcal{L}_2}$$

- For stable $\boldsymbol{\Sigma}$, the $\mathcal{L}_2$ gain equals the $\mathcal{H}_\infty$ norm of G, i.e.,

$$\|\boldsymbol{\Sigma}\|_{\mathcal{L}_2} = \|G\|_{\mathcal{H}_\infty} = \sup_{\omega \in \mathbb{R}} \sigma_{\max}\big(G(j\omega)\big)$$

Objectives



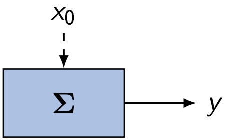
Objectives

1. Preservation of stability
2. Bound ε on the $\mathcal{L}_2$ gain of the reduction error $e = y - \hat{y}$, i.e.,

$$\|y - \hat{y}\|_{\mathcal{L}_2} \leq \varepsilon \|u\|_{\mathcal{L}_2}$$

   Equivalently,

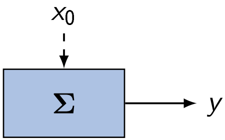
$$\|G - \hat{G}\|_{\mathcal{H}_\infty} \leq \varepsilon$$



Observability energy function

$$L_o(x_0) = \int_0^\infty \|y(t)\|^2 \, dt, \qquad x(0) = x_0,\ u(t) = 0 \ \ \forall t \geq 0$$

The observability energy function gives the energy associated by observing the output of Σ for initial condition x_0



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Theorem. If A is Hurwitz, then

$$L_o(x_0) = x_0^\top Q x_0, \qquad Q = \int_0^\infty e^{A^\top t} C^\top C e^{At} \, dt$$

Observability Gramian

$$Q = \int_0^\infty e^{A^\mathsf{T} t} C^\mathsf{T} C e^{At} \, \mathrm{d}t$$

Properties

- If A is Hurwitz, Q exists and is the unique solution of

$$A^\mathsf{T} Q + QA + C^\mathsf{T} C = 0$$

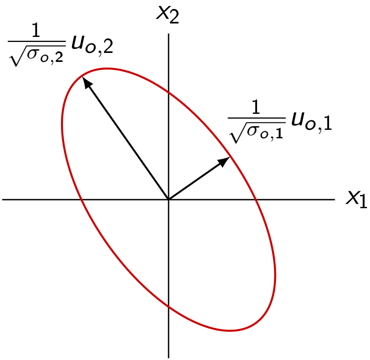
- Q is symmetric and positive (semi-)definite, i.e., $Q = Q^\mathsf{T} \succcurlyeq 0$
- If (A, C) is observable, then $Q \succ 0$

Observability Gramian: interpretation

Singular value decomposition

$$Q = U_o \Sigma_o U_o^\mathsf{T} = \sum_{i=1}^{n} \sigma_{o,i} u_{o,i} u_{o,i}^\mathsf{T}$$

with $\sigma_{o,i} \geq \sigma_{o,i+1}$



The ellipse gives all initial states x_0 for which $x_0^\mathsf{T} Q x_0 = c$, $c > 0$, i.e., all initial states that give the same output energy

- $u_{o,1}$: easily observable direction ($\sigma_{o,1}$ large)
- $u_{o,2}$: poorly observable direction ($\sigma_{o,2}$ small)

Singular value decomposition

$$Q = U_o \Sigma_o U_o^\top = \sum_{i=1}^{n} \sigma_{o,i} u_{o,i} u_{o,i}^\top$$

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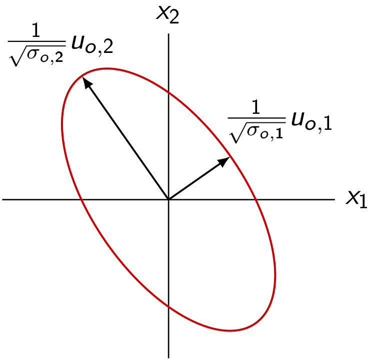
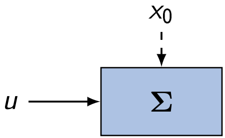


Unobservable subspace. If $\sigma_{o,k+1} = \ldots = \sigma_{o,n} = 0$ and $\sigma_{o,k} > 0$,

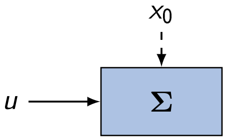
$$\mathcal{N} = \text{span}\{u_{o,k+1}, \ldots, u_{o,n}\} = \ker \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}$$



Controllability energy function

$$L_c(x_0) = \min \left\{ \int_{-\infty}^{0} \|u(t)\|^2 \,\mathrm{d}t \;\middle|\; u \in \mathcal{L}_2,\ x(-\infty) = 0,\ x(0) = x_0 \right\}$$

The controllability energy function gives the least input energy needed to steer the system state from 0 to x_0 in infinite time



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Theorem. If A is Hurwitz and (A, B) is controllable, then

$$L_c(x_0) = x_0^\top P^{-1} x_0, \qquad P = \int_0^\infty e^{At} B B^\top e^{A^\top t} \,\mathrm{d}t$$

Controllability Gramian

$$P = \int_0^\infty e^{At} B B^\mathsf{T} e^{A^\mathsf{T} t} \, \mathrm{d}t$$

Properties

- If A is Hurwitz, P exists and is the unique solution of

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Controllability Gramian: interpretation

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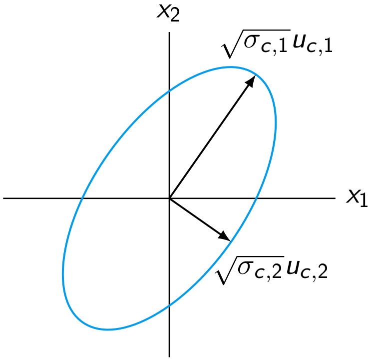
$$P = U_c \Sigma_c U_c^\mathsf{T} = \sum_{i=1}^{n} \sigma_{c,i} u_{c,i} u_{c,i}^\mathsf{T}$$

with $\sigma_{c,i} \geq \sigma_{c,i+1}$, i.e.,

$$P^{-1} = \sum_{i=1}^{n} \sigma_{c,i}^{-1} u_{c,i} u_{c,i}^\mathsf{T}$$



The ellipse gives all states x_0 for which $x_0^\mathsf{T} P^{-1} x_0 = c$, $c > 0$, i.e., all initial states that can be reached with bounded energy c

- $u_{c,1}$: easily reachable direction ($\sigma_{c,1}$ large)
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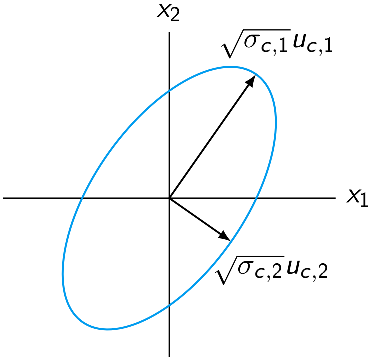
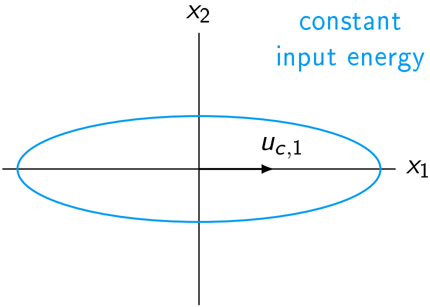


Reachability subspace. If $\sigma_{c,k+1} = \ldots = \sigma_{c,n} = 0$ and $\sigma_{c,k} > 0$,

$$\mathcal{W} = \operatorname{span}\{u_{c,1}, \ldots, u_{c,k}\} = \operatorname{im} \begin{bmatrix} B & AB & \ldots & A^{n-1}B \end{bmatrix}$$

Example

$$\dot{x} = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} x + \begin{bmatrix} 100 \\ 1 \end{bmatrix} u, \qquad y = \begin{bmatrix} 1 & 200 \end{bmatrix} x$$



From a controllability perspective, x_1 is important

Example

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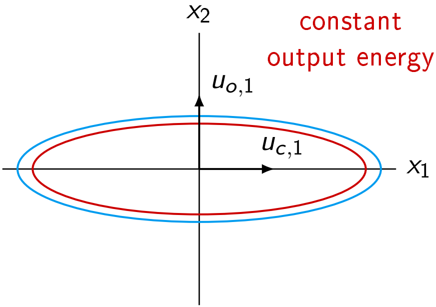

From an observability perspective, x_2 is important

Example

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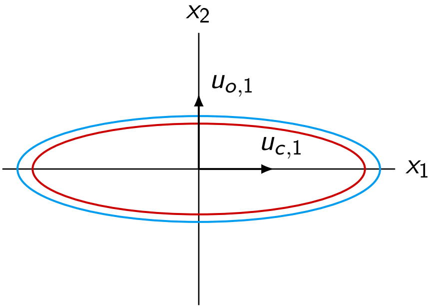

States that are easy to control ($\sigma_{c,1}$ large) are not necessarily easy to observe ($\sigma_{o,1}$ large) or vice versa

Coordinate transformations

$$\mathbf{\Sigma} : \begin{cases} \dot{x} = Ax + Bu \\ y = Cx + Du \end{cases}$$

Coordinate transformation T ($T \in \mathbb{R}^{n \times n}$ nonsingular)

$$\bar{x} = Tx$$

In new coordinates,

$$\mathbf{\Sigma} : \begin{cases} \dot{\bar{x}} = \bar{A}\bar{x} + \bar{B}u \\ y = \bar{C}\bar{x} + \bar{D}u \end{cases}$$

with

$$\bar{A} = TAT^{-1}, \quad \bar{B} = TB, \quad \bar{C} = CT^{-1}, \quad \bar{D} = D$$

Properties

- Stability properties and input-output behavior are unchanged

Hankel singular values

Gramians in new coordinates $\bar{x} = Tx$

$$\bar{P} = TPT^{\mathsf{T}}, \qquad \bar{Q} = T^{-\mathsf{T}} Q T^{-1}$$

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Transformation of PQ

$$\bar{P}\bar{Q} = TPQT^{-1}$$

- The eigenvalues of PQ are invariant under transformation
- The eigenvalues of PQ equal the **Hankel singular values** σ_i

$$\sigma_i = \sqrt{\lambda_i(PQ)}, \quad i \in \{1, \dots, n\}, \ \sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_n \geq 0$$

Hankel singular values

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$$\sigma_i = \sqrt{\lambda_i(PQ)}, \quad i \in \{1, \ldots, n\}, \ \sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_n \geq 0$$

Theorem. There exists a transformation T such that

$$\bar{P} = \bar{Q} = \Sigma := \operatorname{diag}\{\sigma_1, \sigma_2, \ldots, \sigma_n\}$$

A realization for which $\bar{P} = \bar{Q} = \Sigma$ is called balanced. Then,

$$\sigma_i = \sigma_{c,i} = \sigma_{o,i}$$

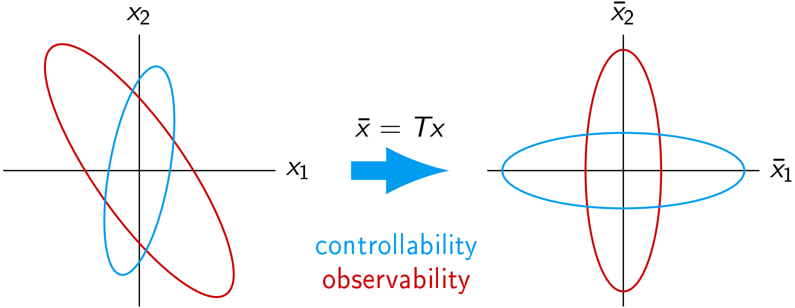
$$\bar{P} = \bar{Q} = \Sigma = \begin{bmatrix} \sigma_1 & 0 & \cdots & 0 \\ 0 & \sigma_2 & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & \sigma_n \end{bmatrix}$$

Interpretation. $\bar{P}$ and $\bar{Q}$ are

- Equal. States that are easy to control are also easy to observe and vice versa ($\sigma_{c,i} = \sigma_{o,i} = \sigma_i$ and $u_{c,i} = u_{o,i}$)
- Diagonal. Controllability and observability can be interpreted per state component ($u_{c,i} = u_{o,i} = e_i$)
- Ordered. The first state is the easiest to control and easiest to observe ($\sigma_i \geq \sigma_{i+1}$)

Balanced realization: interpretation (1)

$$\bar{P} = \bar{Q} = \Sigma = \begin{bmatrix} \sigma_1 & 0 & \cdots & 0 \\ 0 & \sigma_2 & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & \sigma_n \end{bmatrix}$$



Balanced realization: interpretation (2)

Observability and controllability energy functions

$$L_o(\bar{x}_0) = \bar{x}_0^\mathsf{T} \Sigma \bar{x}_0, \qquad L_c(\bar{x}_0) = \bar{x}_0^\mathsf{T} \Sigma^{-1} \bar{x}_0$$

Consider $\bar{x}_0 = e_1 = [\,1\ 0\ \ldots\ 0\,]^\mathsf{T}$

- $L_o(e_1) = \sigma_1$, Starting at e_1 gives large output energy
- $L_c(e_1) = \sigma_1^{-1}$, Small input energy needed to reach e_1

Balanced realization: interpretation (2)

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Balanced realization: interpretation (2)

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Consider $\bar{x}_0 = e_n = [\,0\ \ldots\ 0\ 1\,]^{\mathsf{T}}$

- $L_o(e_n) = \sigma_n$, Starting at e_n gives small output energy
- $L_c(e_n) = \sigma_n^{-1}$, Large input energy needed to reach e_n

In a balanced realization, states are ordered according to their contribution to the input-output behavior

Balancing algorithm

1. Obtain P and Q as solutions of the Lyapunov equations
$$AP + PA^{\mathsf{T}} + BB^{\mathsf{T}} = 0$$
$$A^{\mathsf{T}}Q + QA + C^{\mathsf{T}}C = 0$$
2. Determine U through a Cholesky factorization of P as
$$P = UU^{\mathsf{T}}$$
3. Perform an eigenvalue decomposition of $U^{\mathsf{T}}QU$ as
$$U^{\mathsf{T}}QU = K\Sigma^2K^{\mathsf{T}}$$
to obtain K and Σ
4. The transformation matrix T and its inverse are given as
$$T = \Sigma^{\frac{1}{2}}K^{\mathsf{T}}U^{-1}, \qquad T^{-1} = UK\Sigma^{-\frac{1}{2}}$$

In Matlab, use `lyap()`, `chol()`, and `eig()`

Balanced realization: partitioning

Consider a balanced realization

$$\Sigma : \begin{cases} \dot{x} = Ax + Bu \\ y = Cx + Du \end{cases}$$

Partitioning of the energy functions and Gramian

$$\begin{aligned} L_o(x_0) &= x_0^T \Sigma x_0 &= x_{1,0}^T \Sigma_1 x_{1,0} + x_{2,0}^T \Sigma_2 x_{2,0} \\ L_c(x_0) &= x_0^T \Sigma^{-1} x_0 &= x_{1,0}^T \Sigma_1^{-1} x_{1,0} + x_{2,0}^T \Sigma_2^{-1} x_{2,0} \end{aligned}$$

with

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \qquad \Sigma = \begin{bmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{bmatrix}$$

- $\Sigma_1 \in \mathbb{R}^{k \times k}$ contains large Hankel singular values
- $\Sigma_2 \in \mathbb{R}^{(n-k)\times(n-k)}$ contains small Hankel singular values

Balanced realization: partitioning

Consider a balanced realization

$$\boldsymbol{\Sigma} : \begin{cases} \dot{x} = Ax + Bu \\ y = Cx + Du \end{cases}$$

Partitioning of the energy functions and Gramian

$$L_o(x_0) = x_0^T \Sigma x_0 \quad = x_{1,0}^T \Sigma_1 x_{1,0} + x_{2,0}^T \Sigma_2 x_{2,0}$$
$$L_c(x_0) = x_0^T \Sigma^{-1} x_0 = x_{1,0}^T \Sigma_1^{-1} x_{1,0} + x_{2,0}^T \Sigma_2^{-1} x_{2,0}$$

with

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \qquad \Sigma = \begin{bmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{bmatrix}$$

Partitioning of system matrices

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \qquad B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \qquad C = \begin{bmatrix} C_1 & C_2 \end{bmatrix}$$

Model reduction: truncation

Reduced-order model

$$\hat{\boldsymbol{\Sigma}}_t : \begin{cases} \dot{\xi} = A_{11}\xi + B_1 u \\ \hat{y}_t = C_1\xi + Du \end{cases}$$

Properties

- Obtained by setting $x_2 = 0$
- Approximation is exact for $s \to \infty$, i.e., $G(\infty) = \hat{G}_t(\infty)$

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Theorem. Let $\hat{\Sigma}_t$ be obtained by balanced truncation. Then,

- $\hat{\Sigma}_t$ is in balanced realization with controllability Gramian Σ_1 and observability Gramian Σ_1
- If $\sigma_k > \sigma_{k+1}$, then A_{11} is Hurwitz, i.e., $\hat{\Sigma}_t$ is asympt. stable

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Theorem. The following error bound holds:

$$\left\| G(s) - \hat{G}_t(s) \right\|_{\mathcal{H}_\infty} \leq 2 \sum_{i=k+1}^{n} \sigma_i$$

Note: multiplicities of σ_i do not have to be counted

Model reduction: singular perturbation

Reduced-order model

$$\hat{\Sigma}_s : \begin{cases} \dot{\xi} = \left(A_{11} - A_{12}A_{22}^{-1}A_{21}\right)\xi + \left(B_1 - A_{12}A_{22}^{-1}B_2\right)u \\ \hat{y}_s = \left(C_1 - C_2A_{22}^{-1}A_{21}\right)\xi + \left(D - C_2A_{22}^{-1}B_2\right)u \end{cases}$$

Properties

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Model reduction: singular perturbation

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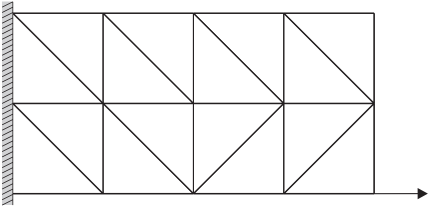
Truss frame example



Dynamics

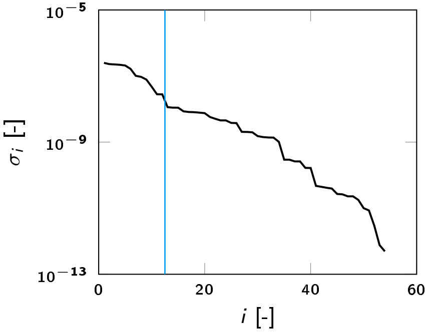
$$M\ddot{q} + D\dot{q} + Kq = Su$$

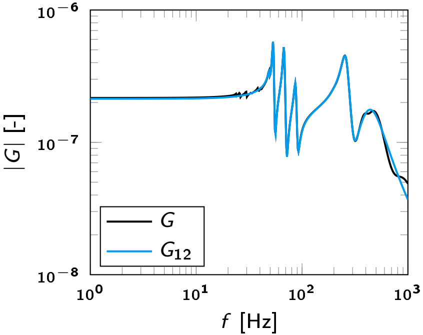
State-space form with $x = [\, q^{\mathsf{T}} \;\; \dot{q}^{\mathsf{T}} \,]^{\mathsf{T}} \in \mathbb{R}^{72}$

$$\dot{x} = \begin{bmatrix} 0 & I \\ -M^{-1}K & -M^{-1}D \end{bmatrix} x + \begin{bmatrix} 0 \\ M^{-1}S \end{bmatrix} u, \qquad y = Cx$$

Truss frame example: results

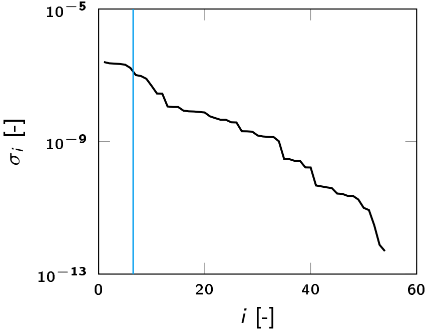
Reduction to $k = 12$ using balanced truncation

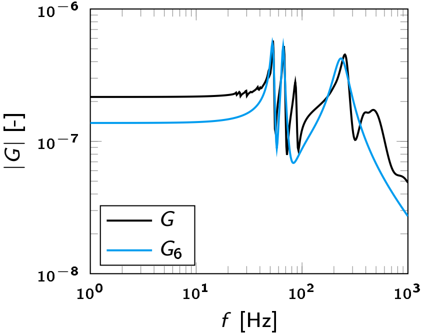




Truss frame example: results

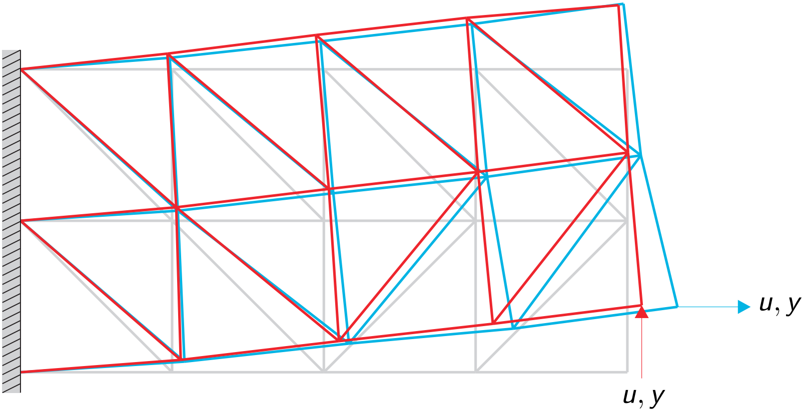
Reduction to $k = 6$ using balanced truncation





Truss frame example: modes

The most important "mode" depends on the input and output



Conclusions (so far...)

- Gramians as measure of controllability and observability
- Balancing to identify most important states
- Two methods for reduction:
  1. Truncation
  2. Singular perturbation
- Preservation of stability
- Error bound in terms of the truncated Hankel singular values

Conclusions (so far...)

- Gramians as measure of controllability and observability
- Balancing to identify most important states
- Two methods for reduction:
  1. Truncation
  2. Singular perturbation
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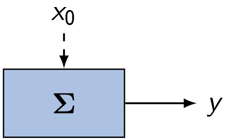
- Error bound in terms of the truncated Hankel singular values

Questions

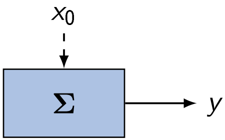
- What do the parameters σ_i represent?
- What is the most accurate reduced-order model possible?
- Can we preserve additional properties (for example, passivity or a bounded $\mathcal{L}_2$ gain)?

Observability and controllability operator



Observability operator $\Psi_o : \mathbb{R}^n \to \mathcal{L}_2^p([0, \infty))$

$$\Psi_o x_0 = Ce^{At} x_0, \quad t \geq 0$$



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$$\Psi_o x_0 = Ce^{At}x_0, \quad t \geq 0$$

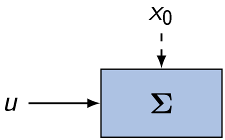
Properties

- Ψ_o is bounded for A Hurwitz
- Relation to the observability energy function for $y = \Psi_o x_0$

$$L_o(x_0) = \|y\|_{\mathcal{L}_2}^2 = \langle \Psi_o x_0, \Psi_o x_0 \rangle_{\mathcal{L}_2}$$
$$= \langle x_0, \Psi_o^* \Psi_o x_0 \rangle_{\mathbb{R}^n} = x_0^\top Q x_0$$

with Ψ_o^* the adjoint of Ψ_o

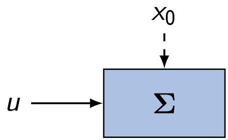
Observability and controllability operator



Controllability operator $\Psi_c : \mathcal{L}_2^m((-\infty, 0]) \to \mathbb{R}^n$

$$\Psi_c u = \int_{-\infty}^{0} e^{-At} B u(t) \, \mathrm{d}t$$

Observability and controllability operator



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$$\Psi_c u = \int_{-\infty}^{0} e^{-At} Bu(t)\,\mathrm{d}t$$

Properties

- If (A, B) is controllable, then $\mathrm{im}(\Psi_c) = \mathbb{R}^n$
- Relation to the controllability energy function

$$\begin{aligned} L_c(x_0) &= \min\left\{ \|u\|_{\mathcal{L}_2}^2 \mid \Psi_c u = x_0 \right\} \\ &= \|u_{\text{opt}}\|_{\mathcal{L}_2}^2 = \left\langle P^{-1}x_0, \Psi_c \Psi_c^* P^{-1} x_0 \right\rangle_{\mathbb{R}^n} = x_0^{\mathrm{T}} P^{-1} x_0 \end{aligned}$$

with $u_{\text{opt}} = \Psi_c^* P^{-1} x_0$

Hankel operator

Hankel operator $\mathcal{H} : \mathcal{L}_2^m((-\infty, 0]) \to \mathcal{L}_2^p([0, \infty))$

$$\mathcal{H}u = \Psi_o \Psi_c u = Ce^{At} \int_{-\infty}^{0} e^{-A\tau} Bu(\tau)\, \mathrm{d}\tau, \quad t \geq 0$$

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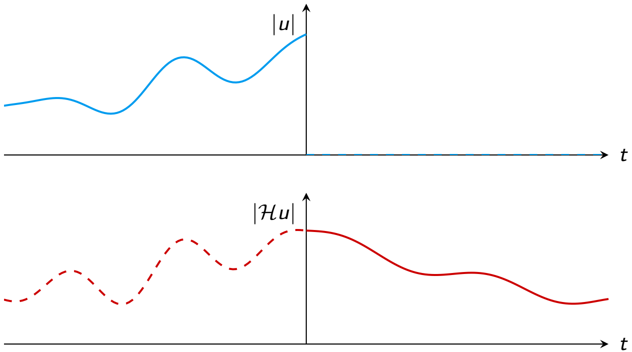

The Hankel operator maps past inputs to future outputs

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Properties

- $\mathcal{H}$ is a bounded operator for A Hurwitz
- $\mathcal{H}$ is a finite-dimensional operator with $\dim \operatorname{im}(\mathcal{H}) \leq n$

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Singular value decomposition of the Hankel operator

$$\mathcal{H}u = \sum_{i=1}^{n} \sigma_i \langle u, \bar{u}_i \rangle_{\mathcal{L}_2} \bar{y}_i$$

with $\bar{u}_i \in \mathcal{L}_2^m((-\infty, 0])$ and $\bar{y}_i \in \mathcal{L}_2^p([0, \infty))$ orthonormal, i.e.,

$$\langle \bar{u}_i, \bar{u}_j \rangle_{\mathcal{L}_2} = \langle \bar{y}_i, \bar{y}_j \rangle_{\mathcal{L}_2} = \begin{cases} 1 \ , & i = j \\ 0 \ , & i \neq j \end{cases}$$

Hankel singular values

Singular value decomposition of the Hankel operator

$$\mathcal{H}u = \sum_{i=1}^{n} \sigma_i \langle u, \bar{u}_i \rangle_{\mathcal{L}_2} \bar{y}_i$$

Theorem. The Hankel singular values σ_i satisfy

$$PQ\bar{x}_i = \sigma_i^2 \bar{x}_i, \text{ i.e., } \sigma_i^2 = \lambda_i(PQ)$$

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Hankel norm. Let $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_n \geq 0$ and $y = \mathcal{H}u$. Then,

$$\|\boldsymbol{\Sigma}\|_H = \sup\left\{ \frac{\|y\|_{\mathcal{L}_2^p(\mathbb{R}_+)}}{\|u\|_{\mathcal{L}_2^m(\mathbb{R}_-)}} \;\middle|\; u \in \mathcal{L}_2^m((-\infty, 0]),\ u \neq 0 \right\} = \sigma_1$$

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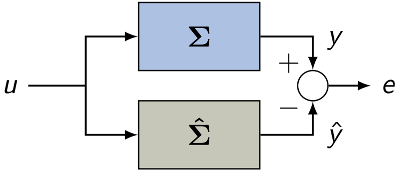
Properties

- The Hankel norm is a lower bound on the $\mathcal{L}_2$ norm, i.e.,

$$\|\mathbf{\Sigma}\|_H \leq \|\mathbf{\Sigma}\|_{\mathcal{L}_2} = \|G\|_{\mathcal{H}_\infty}$$

A fundamental lower bound



Theorem. Let $\mathbf{\Sigma}$ be a minimal realization with A Hurwitz. For any asymptotically stable $\hat{\mathbf{\Sigma}}_k$ of dimension $k < n$,

$$\left\| \mathbf{\Sigma} - \hat{\mathbf{\Sigma}}_k \right\|_{\mathcal{L}_2} = \left\| G - \hat{G}_k \right\|_{\mathcal{H}_\infty} \geq \sigma_{k+1}$$

with $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_n > 0$ the Hankel singular values of $\mathbf{\Sigma}$

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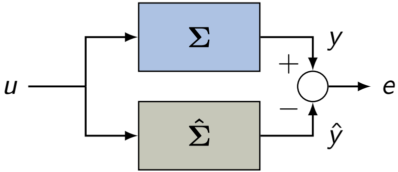

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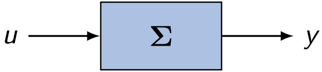
Properties

- Lower bound is independent of the reduction procedure
- For balanced truncation and balanced singular perturbation,

$$\sigma_{k+1} \leq \left\| \boldsymbol{\Sigma} - \hat{\boldsymbol{\Sigma}}_k \right\|_{\mathcal{L}_2} \leq 2\left(\sigma_{k+1} + \sigma_{k+2} + \ldots + \sigma_n\right)$$

Passive systems

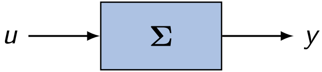


Passivity. $\mathbf{\Sigma}$ is passive if there exists $V : \mathbb{R}^n \to [0, \infty)$ such that

$$V(x(t_1)) \leq V(x(t_0)) + \int_{t_0}^{t_1} u^\mathsf{T}(t) y(t) \, \mathrm{d}t$$

for all $t_0 \leq t_1$ and for all trajectories of $\mathbf{\Sigma}$

Passive systems



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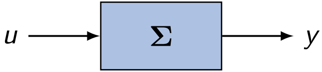
Properties

- Examples: physical systems, e.g., mechanical or electrical
- Special case of dissipativity theory, with storage function V and supply rate $s(u, y) = u^{\mathsf{T}} y$

Passive systems



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Properties

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- Special case of dissipativity theory, with storage function V and supply rate $s(u, y) = u^\mathsf{T} y$

Question. Can we find a reduced-order model that preserves passivity?

Postive real lemma

Theorem. Let $\mathbf{\Sigma}$ be minimal. Then, the following are equivalent

1. $\mathbf{\Sigma}$ is passive
2. The transfer function G is positive real, i.e.,

$$G(s) + G^*(s) \succcurlyeq 0, \qquad \mathrm{Re}(s) > 0$$

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3. $\boldsymbol{\Sigma}$ is passive with $V(x) = \frac{1}{2}x^\mathsf{T}Kx$ for some $K = K^\mathsf{T} \succ 0$

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3. $\boldsymbol{\Sigma}$ is passive with $V(x) = \frac{1}{2}x^\top Kx$ for some $K = K^\top \succ 0$
4. There exist $K = K^\top$, L, and W such that

$$A^\top K + KA = -LL^\top$$
$$PB - C^\top = -LW^\top$$
$$D + D^\top = WW^\top$$

where $0 \prec K_- \preccurlyeq K \preccurlyeq K_+$

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Note. The equivalence 2. $\Leftrightarrow$ 4. is known as the positive real lemma or Kalman-Yakubovich-Popov (KYP) lemma

Extremal storage functions

Available storage function $V_a : \mathbb{R}^n \to [0, \infty)$

$$V_a(x_0) = \tfrac{1}{2} x_0^\mathsf{T} K_- x_0$$
$$= \sup \left\{ -\int_0^\infty u^\mathsf{T}(t) y(t)\, \mathrm{d}t \;\middle|\; x(0) = x_0,\ x(\infty) = 0,\ u \in \mathcal{L}_2 \right\}$$

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Required supply function $V_r : \mathbb{R}^n \to [0, \infty)$

$$V_r(x_0) = \tfrac{1}{2} x_0^\mathsf{T} K_+ x_0$$
$$= \inf \left\{ \int_{-\infty}^0 u^\mathsf{T}(t) y(t)\, \mathrm{d}t \;\middle|\; x(-\infty) = x_0,\ x(0) = x_0,\ u \in \mathcal{L}_2 \right\}$$

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Interpretation

- $V_a(x_0)$ gives the maximum energy one can extract from x_0
- $V_r(x_0)$ gives the minimal energy needed to reach x_0

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Interpretation

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- $V_r(x_0)$ gives the minimal energy needed to reach x_0

Idea: perform balancing using K_- and K_+^{-1}

Positive real balancing

Theorem. The eigenvalues of $K_- K_+^{-1}$ denoted as

$$\pi_i = \sqrt{\lambda_i(K_- K_+^{-1})}$$

are system invariants and satisfy $0 < \pi_i \leq 1$

Positive real balancing

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Notes

- π_i are called positive real singular values
- A realization with $K_- = K_+^{-1} = \Pi$ is positively real balanced

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In positively real balanced coordinates, partition

$$\Pi = \begin{bmatrix} \Pi_1 & 0 \\ 0 & \Pi_2 \end{bmatrix}, \quad A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \quad C = \begin{bmatrix} C_1 & C_2 \end{bmatrix}$$

Positive real balanced truncation

Reduced-order model obtained by truncation with $\xi \in \mathbb{R}^k$, $k < n$

$$\hat{\boldsymbol{\Sigma}}_t : \begin{cases} \dot{\xi} = A_{11}\xi + B_1 u \\ \hat{y}_t = C_1\xi + Du \end{cases}$$

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Properties

- There exists an extension to strictly positive real systems that preserves, in addition, minimality and asymptotic stability

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- If $D + D^\mathsf{T} \succ 0$, then K_- and K_+ are also extremal solutions of

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- A similar method known as bounded real balancing preserves an $\mathcal{L}_2$ gain bounded by 1, i.e., dissipativity with respect to the supply rate $s(u, y) = \|u\|_2^2 - \|y\|_2^2$

Conclusions

Summary

- Gramians as measure of controllability and observability
- Balanced truncation or singular perturbation for reduction
  - Preservation of asymptotic stability and minimality
  - Error bound in terms of Hankel singular values
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Extensions of balancing

- Optimal Hankel norm approximation
- Frequency-weighted balancing
- Balancing for controllers, structured systems, networks, nonlinear systems, ...

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