Model Reduction of Networks

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Key features

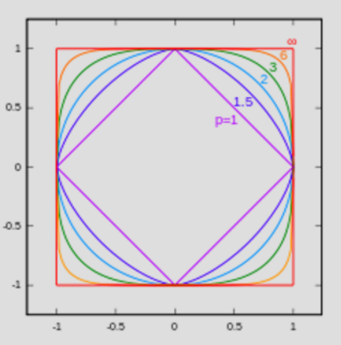
- Reducing the complexity
- Approximating the original behavior
- Preserving system/network properties
- Preserving structural properties

Outline

1. System norms and Lyapunov equalities
2. Model reduction of networks: Introduction
3. Model reduction of networks: Reducing the dynamics
4. Model reduction of networks: Reducing the interconnections
5. Conclusions and research directions

- $p-$norms

$$\|x\|_p = (\Sigma_i |x_i|^p)^{\frac{1}{p}}, \qquad p \geqslant 1$$



Induced norm:

$$\|A\|_p = \sup_{x \neq 0} \frac{\|Ax\|_p}{\|x\|_p}$$

- $\|A\|_1$: maximum column sums
- $\|A\|_\infty$: maximum row sums
- $\|A\|_2$: maximum singular value

L_2-norm of a scalar-valued signal $v(t)$

$$\|v\|_2^2 = \int_0^\infty v(t)^2 dt$$

H_2-norm: SISO

$$\|\Sigma\|_{H_2}^2 = \int_0^\infty h(t)^2 dt$$

H_2-norm: MIMO

$$\|\Sigma\|_{H_2}^2 = \int_0^\infty \text{trace}\, h(t)^T h(t) dt$$

H_2-norm: frequency domain

$$\|\Sigma\|_{H_2}^2 = \frac{1}{2\pi} \int_{-\infty}^\infty \text{trace}\, H^*(j\omega) H(j\omega) d\omega$$

Lyapunov equalities

$$A^T Q + QA + C^T C = 0$$
$$AP + PA + BB^T = 0$$

H_2-norm: state-space formulae

$$\|\Sigma\|_{H_2}^2 = \text{trace}\, B^T Q B = \text{trace}\, C P C^T$$

H_∞-norm

$$\|H\|_\infty = \sup_\omega \overline{\sigma}(H(j\omega))$$

Bounded real

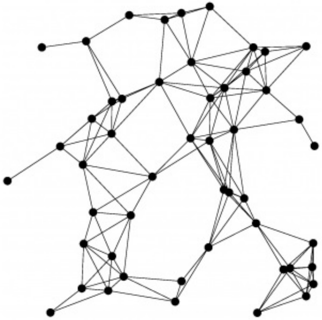
$\|H\|_\infty \leqslant \gamma$ if and only if there exists $X \geqslant 0$ s.t.

$$\begin{bmatrix} A^T X + XA + C^T C & XB + C^T D \\ B^T X + D^T C & -\gamma^2 I + D^T D \end{bmatrix} \leqslant 0.$$

Outline

Reducing the complexity

- Targeting agents dynamics
- Targeting the interconnection



Balanced Truncation

1. **Balancing:** Simultaneous diagonalization of (P, Q).
2. **Truncating:** Discarding the least important state components

Type	Equations
Lyapunov	$AP + PA^{\mathrm{T}} + BB^{\mathrm{T}} = 0$ $A^{\mathrm{T}}Q + QA + C^{\mathrm{T}}C = 0$
Bounded Real	$AP + PA^{\mathrm{T}} + BB^{\mathrm{T}} + (PC^{\mathrm{T}} + BD^{\mathrm{T}})(I - DD^{\mathrm{T}})^{-1}(PC^{\mathrm{T}} + BD^{\mathrm{T}})^{\mathrm{T}} = 0$ $A^{\mathrm{T}}Q + QA + C^{\mathrm{T}}C + (QB + C^{\mathrm{T}}D)(I - D^{\mathrm{T}}D)^{-1}(QB + C^{\mathrm{T}}D)^{\mathrm{T}} = 0$
Positive Real	$AP + PA^{\mathrm{T}} + (PC^{\mathrm{T}} - B)(D + D^{\mathrm{T}})^{-1}(PC^{\mathrm{T}} - B)^{\mathrm{T}} = 0$ $A^{\mathrm{T}}Q + QA + (QB - C^{\mathrm{T}})(D^{\mathrm{T}} + D)^{-1}(QB - C^{\mathrm{T}})^{\mathrm{T}} = 0$

Key notions

- Vertex set, Edge set, Edge weights
- Adjacency matrix
- Incidence matrix
- Laplacian matrix

Example

On the board

Outline

Node dynamics

$$\dot{x}_i(t) = Ax_i(t) + Bu_i(t)$$
$$y_i(t) = Cx_i(t)$$

Coupling rule

$$u_i(t) = -\sum_{j \in \mathcal{N}_i} y_i(t) - y_j(t)$$

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Network representation

$$\dot{x}(t) = (I_p \otimes A - L \otimes BC)x(t)$$

Network model

$$\dot{x}(t) = (I_p \otimes A - L \otimes BC)x(t)$$

Network reduced model

$$\dot{x}(t) = (I_p \otimes A_r - L \otimes B_r C_r)x(t)$$

Desired features

- Preserving nice properties (synchronization)
- Comparable behavior

Sunchronization

$$\lim_{t \to \infty} y_i(t) - y_j(t) = 0, \qquad \forall i, j \in V$$

Synchronization iff $A - \lambda_i BC$ is Hurwitz for each $i = 2, 3, \ldots, p$.

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Truncation loses synchrony

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Truncation loses synchrony

Synchronization if

$$(\lambda_p - \lambda_2)\|\overline{G}\|_\infty < 1, \qquad \overline{G}(s) = C(sI - A + \lambda_2 BC)^{-1}B.$$

Bounded real $\equiv \quad \|G\|_\infty < 1$

Bounded real iff there exists $Q = Q^T$ satisfying

$$A^T Q + QA + C^T C + QBB^T Q = 0.$$

BR balancing

Balancing (Q_M^{-1}, Q_m), $\quad 0 < Q_m \leqslant Q \leqslant Q_M$

Synchronization if

$$(\lambda_p - \lambda_2)\|\overline{G}\|_\infty < 1, \qquad \overline{G}(s) = C(sI - A + \lambda_2 BC)^{-1}B.$$

Network reduction

- $(\lambda_p - \lambda_2)\|\overline{G}\|_\infty < \gamma < 1$
- $(A - \lambda_2 BC)^T Q + Q(A - \lambda_2 BC) + C^T C + (\frac{\lambda_p - \lambda_2}{\gamma})^2 QBB^T Q = 0$
- Balance (Q_M^{-1}, Q_m)

Synchronizied reduced network

$$\dot{x} = (I \otimes A - L \otimes BC)x \quad \longrightarrow \quad \dot{x}_r = (I \otimes A_r - L \otimes B_r C_r)x_r$$

$$(\lambda_p - \lambda_2)\|\overline{G}\|_\infty < 1 \quad \longrightarrow \quad (\lambda_p - \lambda_2)\|\overline{G}_r\|_\infty < 1$$

- Nodal behavior: $(A, B, C) \to (A_r, B_r, C_r)$
- Network behavior $(\mathcal{A}, \mathcal{B}, \mathcal{C}) \to (\mathcal{A}_r, \mathcal{B}_r, \mathcal{C}_r)$,

$$\mathcal{A} = I \otimes A - L \otimes BC, \; \mathcal{A}_r = I \otimes A_r - L \otimes B_r C_r$$

External ports

$$\dot{x} = (I \otimes A - L \otimes BC)x$$

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External ports

$$\dot{x} = (I \otimes A - L \otimes BC)x + (I \otimes B)d$$

$$z = (R^T \otimes I)y = (R^T \otimes C)x$$

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Error bound?

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Error bound

$$\|T - T_r\|_\infty = \frac{2\gamma\sqrt{\lambda_p}}{(\lambda_p - \lambda_2)(1 - \gamma^2)} \sum_{i=r+1}^{N} \sigma_i$$