Model Reduction of Networks

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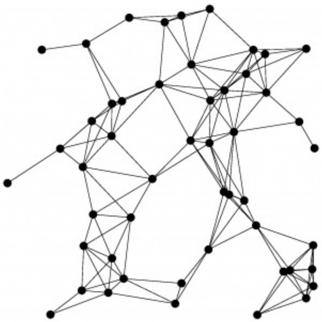
Key features

- Reducing the complexity
- Approximating the original behavior
- Preserving system/network properties
- Preserving structural properties



Network model defined on $\mathcal{G}$

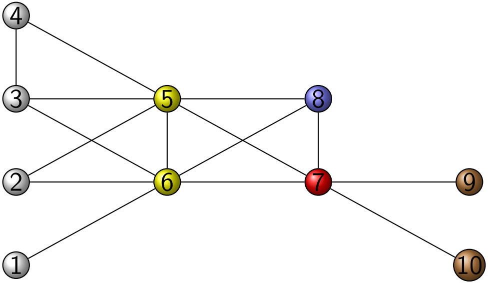
$$M\dot{x} = -D\Gamma D^T x + Eu$$

- Mass-damper systems
- Chemical reaction networks
- Consensus type dynamics

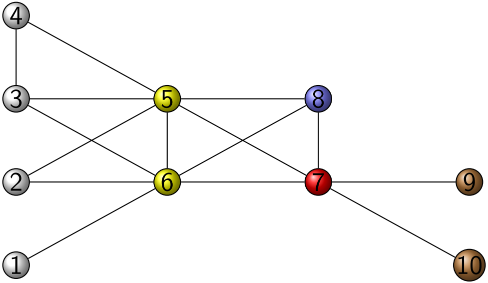
Reduced network defined on $\hat{\mathcal{G}}$

$$\hat{M}\dot{\hat{x}} = -\hat{D}\hat{\Gamma}\hat{D}^T x + \hat{E}u$$

- $|\mathcal{V}(\hat{\mathcal{G}})| < |\mathcal{V}(\mathcal{G})|$



- Vertex set: $V = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$.
- Cell: any nonempty subset of V
- Partition: $\pi = \{\{1, 2, 3, 4\}, \{5, 6\}, \{7\}, \{8\}, \{9, 10\}\}$
- Characteristic matrix:



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- Cell: any nonempty subset of V
- Partition: $\pi = \{\{1, 2, 3, 4\}, \{5, 6\}, \{7\}, \{8\}, \{9, 10\}\}$
- Characteristic matrix: $P(\pi) = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}^\top .$

Projected Model

- Network model: $M\dot{x} = -D\Gamma D^T x + Eu$
- Let π be a graph partition, $P := P(\pi)$

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$$\begin{bmatrix} P^T \\ S^T \end{bmatrix} M \begin{bmatrix} P & S \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = -\begin{bmatrix} P^T \\ S^T \end{bmatrix} D\Gamma D^T \begin{bmatrix} P & S \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} P^T \\ S^T \end{bmatrix} Eu$$

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- Choose S s.t. $P^T MS = 0$ and $\begin{bmatrix} P & S \end{bmatrix}$ is invertible.

$$\begin{bmatrix} P^T MP & 0 \\ 0 & S^T MS \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = - \begin{bmatrix} P^T D\Gamma D^T P & P^T D\Gamma D^T S \\ S^T D\Gamma D^T P & S^T D\Gamma D^T S \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} P^T E \\ S^T E \end{bmatrix} u$$

- Truncation

$$P^T MP\dot{\hat{x}} = -P^T D\Gamma D^T P\hat{x} + P^T Eu$$
$$\hat{M}\dot{\hat{x}} = -\hat{D}\hat{\Gamma}\hat{D}^T\hat{x} + \hat{E}u$$

- Structure is preserved

- An alternative approach: singular perturbation

- Original dynamics:

$$\begin{bmatrix} (P^T MP)\dot{x}_1 \\ (S^T MS)\dot{x}_2 \end{bmatrix} = - \begin{bmatrix} P^T D\Gamma D^T P & P^T D\Gamma D^T S \\ S^T D\Gamma D^T P & S^T D\Gamma D^T S \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} P^T E \\ S^T E \end{bmatrix} u$$

- Reduced dynamics: $(P^T MP)\dot{\hat{x}} = -(P^T D\Gamma D^T P)\hat{x} + P^T E$
- $x = Px_1 + Sx_2$, $P^T MS = 0$

Question

Under what condition $\hat{x} \equiv x_1$?

- Unweighted vertices and unweighted edges: $DD^T \operatorname{im} P \subseteq \operatorname{im} P$
- Unweighted vertices and weighted edges: $D\Gamma D^T \operatorname{im} P \subseteq \operatorname{im} P$
- Weighted vertices and weighted edges: $D\Gamma D^T \operatorname{im} P \subseteq M \operatorname{im} P$

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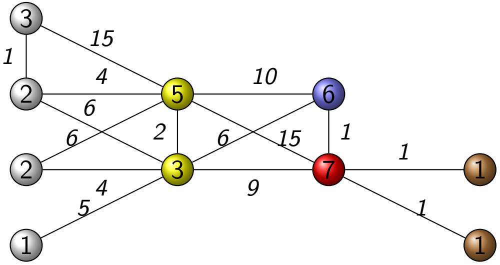
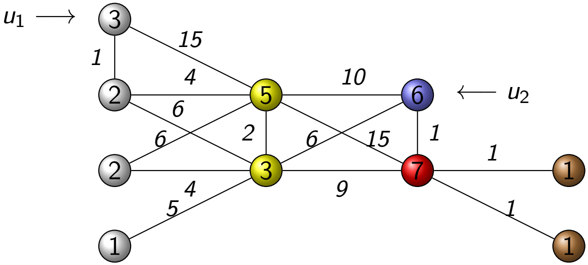


Proposition

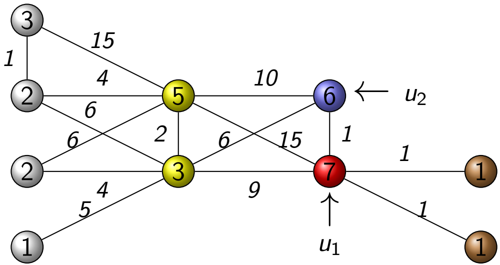
$$\hat{x} \equiv x_1 \iff \pi \text{ is an AEP w.r.t } (\Gamma, M)$$



Theorem

$$\|G - G_r\|_2^2 = \frac{1}{2}\sum_{i \in V_b}\left(\frac{1}{m_i} - \frac{1}{\sigma^i_M}\right)$$

- $\|G - G_r\|_2^2 = \left(\frac{1}{3} - \frac{1}{3+2+2+1}\right) + \left(\frac{1}{6} - \frac{1}{6}\right) = \frac{5}{24}$



Theorem

$$\|G - G_r\|_2^2 = \frac{1}{2} \sum_{i \in V_b} \left(\frac{1}{m_i} - \frac{1}{\sigma_M^i} \right)$$

- $\|G - G_r\|_2^2 = (\frac{1}{7} - \frac{1}{7}) + (\frac{1}{6} - \frac{1}{6}) = 0$ (Removing uncontrollable modes)

Further Reading

- N. Monshizadeh, H. L. Trentelman, and M.K. Camlibel, "Stability and synchrnization preserving model reduction of multi-agent systems", SCL '13
- N. Monshizadeh, H. L. Trentelman, and M.K. Camlibel, "Projection based model reduction of multi-agent systems using graph partitions", TCON '14
- N. Monshizadeh, A.J. van der Schaft, "Structure-preserving model reduction of physical network systems by clustering", CDC 14

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- X. Cheng, Y. Kawano, J.M.A. Scherpen, "Graph structure-preserving model reduction of linear network systems", ECC '16
- X. Cheng, Y. Kawano, J.M.A. Scherpen, "Reduction of second-order network systems with structure preservation", TAC (accepted)
- B. Besselink, H. Sandberg, K.H. Johansson, "Clustering-based model reduction of networked passive systems, TAC 2016.
- H. J. Jongsma, H. L. Trentelman, M. K. Camlibel, "Model reduction of consensus networks by graph simplification", CDC '15
- H. J. Jongsma, P. Mlinaric, S. Grundel, P. Benner, H.L. Trentelman, "Model reduction of linear multi-agent systems by clustering and associated H_2 and H_∞ error bounds", MCSS (under review).