Model Reduction for LPV Systems

Siep Weiland

Department of Electrical Engineering
Eindhoven University of Technology

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Outline

1. Introduction and motivation
2. Reduction of parametric systems
3. Reduction LPV systems with time-varying parameters
4. Conclusions

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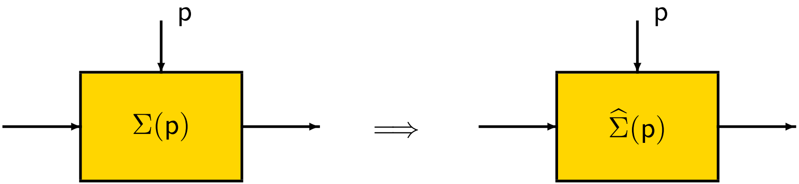
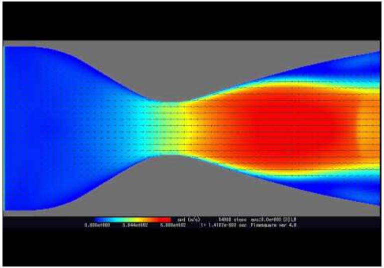
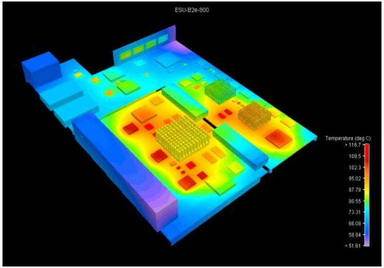


Here

- $\Sigma(\mathsf{p})$ given parameter dependent model
- $\widehat{\Sigma}(\mathsf{p})$ to-be-designed substitute model
- unknown/uncertain parameter $\mathsf{p} \in \mathbb{P}$ or $\mathsf{p} : \mathbb{R} \to \mathbb{P}$.

Motivation




ESU-B2e-800
Temperature (deg C)
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109.5
102.3
95.02
87.78
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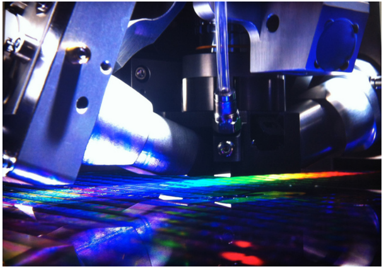
66.08
58.84
< 51.61



Motivation

- Calibration
  Allow adjustments of model or simulation parameters after measurements are compared and adjusted to calibration standards.
- Design optimization
  Physical design parameters on materials or geometry for optimal design based on pROM for fast prototyping.
- Control design
  Model based controller synthesis on basis of parametrized reduced order model.
- Symbolic modeling
  Construction of symbolic models to infer (approximate) simulation relations.
- Uncertainty modeling
  Sampling uncertainty in parametrized models is computationally demanding, benefits from pMOR techniques.

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Setting

Consider MIMO system in state space form

$$\Sigma(\mathsf{p}) : \quad \begin{cases} E(\mathsf{p})\dot{x} = A(\mathsf{p})x + B(\mathsf{p})u \\ y = C(\mathsf{p})x \end{cases} \tag{LPV}$$

where $\dim(u(t)) = \mathsf{u}$, $\dim(y(t)) = \mathsf{y}$, $\dim(x(t)) = \mathsf{n}$ and with parametric dependency of matrices satisfying

$$\begin{aligned} E(\mathsf{p}) &= E_0 + e_1(\mathsf{p})E_1 + \cdots + e_\ell(\mathsf{p})E_\ell \\ A(\mathsf{p}) &= A_0 + a_1(\mathsf{p})A_1 + \cdots + a_\ell(\mathsf{p})A_\ell \\ B(\mathsf{p}) &= B_0 + b_1(\mathsf{p})B_1 + \cdots + b_\ell(\mathsf{p})B_\ell \\ C(\mathsf{p}) &= C_0 + c_1(\mathsf{p})C_1 + \cdots + c_\ell(\mathsf{p})C_\ell \end{aligned}$$

for suitable (linear or nonlinear) functions e_j, a_j, b_j, c_j and $\mathsf{p} \in \mathbb{P} \subset \mathbb{R}^{\mathsf{d}}$

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Assumptions:

- (LPV) is stable for all $\mathsf{p} \in \mathbb{P}$
- complexity indicators (ℓ, n) satisfy $\ell < n$.

General projection framework

Full LPV model

- Model

$$\begin{cases} E(\mathsf{p})\dot{x} = A(\mathsf{p})x + B(\mathsf{p})u \\ y \qquad = C(\mathsf{p})x \end{cases}$$

- Matrices

$$V, W \in \mathbb{R}^{n \times r}$$

- Variable projection

$$x \approx V x_r$$

- Residual projection

$$\operatorname{im} W \perp [E(\mathsf{p})\dot{x} - A(\mathsf{p})x - B(p)u]$$

Reduced LPV model

- Combined:

$$x \approx V x_r$$

$$W^\top [E(\mathsf{p})\dot{x} - A(\mathsf{p})x - Bu] = 0$$

- gives reduced order LPV model

$$\begin{cases} E_r(\mathsf{p})\dot{x}_r = A_r(\mathsf{p})x_r + B_r(\mathsf{p})u \\ y \qquad = C_r(\mathsf{p})x_r \end{cases}$$

where
$E_r(\mathsf{p}) = W^\top E(\mathsf{p})V$, $A_r(\mathsf{p}) = W^\top A(\mathsf{p})V$,
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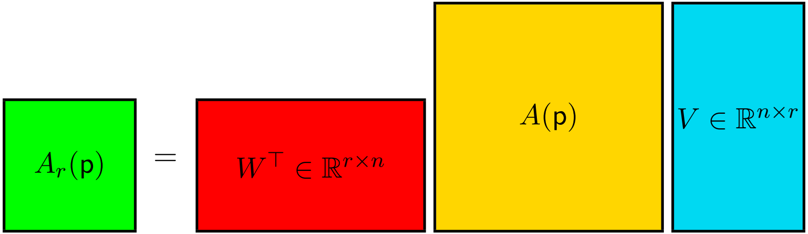

Matrices V and W project state and residual on $\mathcal{V} = \operatorname{im} V$ and $\mathcal{W} = \operatorname{im} W$, respectively.
Assume:

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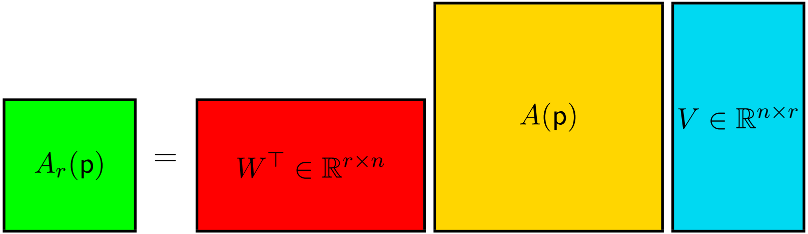
Reduction methods differ in selection of V and W.



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Some observations:

- Structure preserving:

$$E_r(\mathsf{p}) = W^\top E_0 V + e_1(\mathsf{p}) W^\top E_1 V + \cdots + e_m(\mathsf{p}) W^\top E_m V$$

$$\vdots \qquad \vdots$$

$$C_r(\mathsf{p}) = W^\top C_0 V + c_1(\mathsf{p}) W^\top C_1 V + \cdots + c_m(\mathsf{p}) W^\top C_m V$$

  and pre-computable! (No evaluations of $A(\mathsf{p})$, $E(\mathsf{p})$, etc.).
- Generalization to parameter dependent projections $V(\mathsf{p})$ and $W(\mathsf{p})$.
- Defines, for $\mathsf{p} \in \mathbb{P}$, parametrized transfer functions

$$H(\mathsf{s},\mathsf{p}) = C(\mathsf{p})\big(\mathsf{s}E(\mathsf{p}) - A(\mathsf{p})\big)^{-1} B(\mathsf{p})$$

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Error quantification

Parametrized transfer functions $H(\mathsf{s},\mathsf{p})$ and $H_r(\mathsf{s},\mathsf{p})$ define error

$$Y(\mathsf{s},\mathsf{p}) - Y_r(\mathsf{s},\mathsf{p}) = \big(H(\mathsf{s},\mathsf{p}) - H_r(\mathsf{s},\mathsf{p})\big)U(s), \quad (\mathsf{s},\mathsf{p}) \in \mathbb{C} \times \mathbb{P}$$

- Generalized H_∞ sense:

$$\|H - H_r\|_\infty := \sup_{\mathsf{p}\in\mathbb{P}} \sup_{\omega\in\mathbb{R}} \sigma_{\max}\big(H(i\omega,\mathsf{p}) - H_r(i\omega,\mathsf{p})\big)$$

- Generalized H_2 sense:

$$\|H - H_r\|_2^2 := \frac{1}{2\pi} \int_{\mathsf{p}\in\mathbb{P}} \int_{-\infty}^{\infty} \|H(i\omega,\mathsf{p}) - H_r(i\omega,\mathsf{p})]\|_{\mathsf{Frob}}^2 \, \mathsf{d}\omega \; \mathsf{dp}$$

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Sampling techniques

Simple and common reduction technique amounts to take K samples

$$\mathsf{p}_1, \ldots, \mathsf{p}_K \quad \in \quad \mathbb{P}$$

and device *local* projection matrices

$$V_k \in \mathbb{R}^{n \times r}, \qquad W_k \in \mathbb{R}^{n \times r}$$

for $k = 1, \ldots, K$ and *local* approximate models $\Sigma_r(\mathsf{p}_k)$ with transfer

$$H_r(\mathsf{s}, \mathsf{p}_k), \quad k = 1, \ldots, K.$$

then obtain LPV approximate model by interpolation

Sampling techniques

Interpolation techniques: (tomorrow's lectures!)

- define global projection matrices

$$V = \begin{pmatrix} V_1 & \cdots & V_K \end{pmatrix}, \quad W = \begin{pmatrix} W_1 & \cdots & W_K \end{pmatrix}$$

  (or their optimal rank r approximations through an SVD of V, W).

- interpolate, for $\mathsf{p} \in \mathbb{P}$,

$$V(\mathsf{p}) := \mathrm{interpol}(V_1, \ldots, V_K), \quad W(\mathsf{p}) := \mathrm{interpol}(W_1, \ldots, W_K)$$

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Moments of a system

Definition

The mth moment of a transfer function $H(\mathsf{s})$ at $\mathsf{s}_0 \in \mathbb{C}$ is

$$\eta_m(\mathsf{s}_0) = \frac{\mathsf{d}^m}{\mathsf{d}\mathsf{s}^m} H(\mathsf{s})\Big|_{\mathsf{s}=\mathsf{s}_0}$$

- Numbers, vectors or matrices depending on dimensions u and y.
- Called Padé coefficients if $\mathsf{s}_0 = 0$
- Called Markov parameters if $\mathsf{s}_0 = \infty$.

Moments and Taylor/Laurent series

Moments at $s_0 \in \mathbb{C}$ are coefficients of Taylor expansion of H at s_0:

$$H(s) = H(s_0) + \frac{d}{ds}H(s)\Big|_{s=s_0}\frac{(s-s_0)}{1!} + \ldots + \frac{d^m}{ds^m}H(s)\Big|_{s=s_0}\frac{(s-s_0)^m}{m!} + \ldots$$

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$$H(s) = H(s_0) + \frac{d}{ds}H(s)\Big|_{s=s_0} \frac{(s - s_0)}{1!} + \ldots + \frac{d^m}{ds^m}H(s)\Big|_{s=s_0} \frac{(s - s_0)^m}{m!} + \ldots$$

$$= \sum_{m=0}^{\infty} \eta_m(s_0) \frac{(s - s_0)^m}{m!}$$

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Moments at infinity $s_0 = \infty$ are coefficients of Laurent expansion of H:

$$H(s) = \eta_0(\infty) + \eta_1(\infty)s^{-1} + \ldots + \eta_m(\infty)s^{-m} + \ldots$$

$$= \sum_{m=0}^{\infty} \eta_m(\infty)s^{-m}$$

The moment matching problem

Problem

Given LTI system Σ of state dimension n. Let $s_0 \in \mathbb{C} \cup \infty$ and $r < n$.
Find reduced order system Σ_r of state dimension $r < n$ such that

$$\eta_m(s_0) = \eta_{r,m}(s_0) \quad \text{for } m = 0, \ldots, M$$

with M maximal.
Here, $\eta_{r,m}(s_0)$ is mth moment at s_0 of Σ_r.

Fit maximal number of coefficients in Taylor expansion of H in reduced order model H_r.

Of course, maximum M will depend on reduction order r.

Elegant solution for $s_0 = \infty$

Let $H(s) = c(Is - A)^{-1}b$ and store $2r$ Markov parameters in Hankel matrix

$$\mathcal{H} = \underbrace{\begin{pmatrix} c \\ cA \\ \vdots \\ cA^{r-1} \end{pmatrix}}_{N_r} \underbrace{\begin{pmatrix} b & Ab & \cdots & A^{r-1}b \end{pmatrix}}_{R_r} = N_r R_r$$

Then

$$\boxed{\mathcal{H}_{i,j} = \eta_{i+j-1}(\infty), \qquad i,j = 1,\ldots,r}$$

So, $\mathcal{H}$ has moments $\eta_m(\infty)$ for $m = 1,\ldots,2r-1$ as its entries. Factorize

$$\boxed{\mathcal{H} = LU}$$

with $L \in \mathbb{R}^{r\times r}$ and $U \in \mathbb{R}^{r\times r}$ nonsingular.

Elegant solution for $s_0 = \infty$

Theorem

Define $V := R_r U^{-1}$ *and* $W^\top := L^{-1} N_r$. *Then*

- $W^\top V = I$ *and*
- *the reduced model*

$$\dot{x}_r = A_r x_r + B_r u, \quad y = C_r x$$

with $A_r = W^\top A V$, $B_r = W^\top B$, $C_r = CV$ *achieves*

$$\mu_m(\infty) = \mu_{r,m}(\infty), \quad m = 0, 1, \ldots, 2r - 1$$

i.e., it matches the first $2r$ *moments at infinity.*

Proof of previous result

Why? Interesting to understand the proof!!

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For reduced system, also define Hankel matrix

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Then, as before,

$$\widehat{\mathcal{H}}_{i,j} = \eta_{r,i+j-1}(\infty), \qquad i,j = 1,\ldots,r$$

So, it suffices to show that $\widehat{\mathcal{H}} = \mathcal{H}$.

Proof of previous result -ctd.

To see this, observe that

- $W^\top V = I$
  Indeed: $W^\top V = L^{-1} N_r R_r U^{-1} = L^{-1} L U U^{-1} = I.$
- $N_r V W^\top = N_r$
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  Similarly, $V W^\top R_r = R_r U^{-1} L^{-1} N_r R_r = R_r \mathcal{H}^{-1}\mathcal{H} = R_r.$
- $\hat{N}_r W^\top = N_r$
  Apply item 2 to the rows of $\hat{N}_r$ to see this (Try it!)
- $V \hat{R}_r = R_r$
  Apply item 3 to the columns of $\hat{R}_r$ to see this (Try it!)

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But then

$$\widehat{\mathcal{H}} = \hat{N}_r \hat{R}_r = \hat{N}_r W^\top V \hat{R}_r = N_r R_r = \mathcal{H}.$$

Krylov spaces

Definition

Let $A_0 \in \mathbb{R}^{n \times n}$ be a matrix and $b_0 \in \mathbb{R}^n$ be a vector. The rth order Krylov subspace defined by A_0 and b is

$$\mathcal{K}_r(A_0, b_0) := \text{span}\left(b_0, \quad A_0 b_0, \quad \cdots \quad A_0^{r-1} b_0\right)$$

- Usually obtained by iteratively applying A_0 to a vector.
- Krylov subspaces are linear subspaces of the state space.
- Efficiently computable
- Named after Aleksey Krylov.

Aleksey Nikolaevich Krylov

Aleksey Krylov (1863-1945) was a Russian mathematician and naval engineer and memoirist.

He developed theories for the oscillating motions of ships. Krylov subspaces were introduced in a paper publication of 1931.

click for more on his biography

A generalization

Theorem

Let $V \in \mathbb{R}^{n \times r}$ *be any matrix such that*

$$\operatorname{im}(V) = \mathcal{K}_r(A, b) = \operatorname{span}\left\{b, Ab, \cdots, A^{r-1}b\right\}.$$

Let $W \in \mathbb{R}^{n \times r}$ *be any matrix such that*

$$\operatorname{im}(W^\top) = \mathcal{K}_r(A^\top, c^\top) = \operatorname{span}\left\{c^\top, A^\top c^\top, \cdots, A^{\top r-1} c^\top\right\}.$$

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Then the r*th order model* $W^\top V\dot{x}_r = W^\top AVx_r + W^\top Bu$, $y = CVx_r$ *achieves*

$$\eta_{r,j}(\infty) = \eta_j(\infty) \qquad j = 0, \ldots, 2r-1$$

i.e., it matches the first $2r$ *moments at* $s_0 = \infty$.

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So, the subspaces $\mathcal{V}$ and $\mathcal{W}$ count, not their basis !

Another generalization for $s_0 \in \mathbb{C}$

Theorem

For $s_0 \in \mathbb{C}$, define $A_0 = (Is_0 - A)^{-1}$, $b_0 = (Is_0 - A)^{-1}b$ and $c_0 = c(Is_0 - A)^{-1}$.
Let $V \in \mathbb{R}^{n\times r}$ and $W \in \mathbb{R}^{n\times r}$ be any matrices such that

$$\operatorname{im}(V) = \mathcal{K}_r(A_0, b_0), \qquad \operatorname{im}(W^\top) = \mathcal{K}_r(A_0^\top, c_0^\top)$$

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Idea: Apply previous result to frequency shifted system (A_0, b_0, c_0)

Example 1

Given $H(s) = \frac{s^2-4s+5}{s^3-2s^2+3s-1}$. Approximate by order $r = 2$ system and match moments at $s_0 = \infty$.

- Step 1. Derive state space realization. For example,

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -3 & 2 \end{pmatrix}, \quad b = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad c = \begin{pmatrix} 5 & -4 & 1 \end{pmatrix}, \quad d = 0$$

- Step 2. Set

$$N_2 = \begin{pmatrix} c \\ cA \end{pmatrix} = \begin{pmatrix} 5 & -4 & 1 \\ 1 & 2 & -2 \end{pmatrix}, \quad R_2 = \begin{pmatrix} b & Ab \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 2 \end{pmatrix}$$

- Step 3. Factorize Hankel matrix

$$\mathcal{H} = N_2 R_2 = \begin{pmatrix} 1 & -2 \\ -2 & -2 \end{pmatrix} = LU$$

by nonsingular square matrices L and U. (see next slide).

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Example 1 – ctd

- (Step 3 ctd.) for example, set $L = \mathcal{H}$, $U = I$. Or $L = I$, $U = \mathcal{H}$. Or Matlab's permuted-lower–upper triangular decomposition

$$L = \begin{pmatrix} -1/2 & 1 \\ 1 & 0 \end{pmatrix}, \quad U = \begin{pmatrix} -2 & -2 \\ 0 & -3 \end{pmatrix}.$$

- Step 4. With the first choice of L and U define

$$V := R_2 U^{-1} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 2 \end{pmatrix} \in \mathbb{R}^{3\times 2}$$

$$W := N_2^\top L^{-\top} = \begin{pmatrix} 5 & 1 \\ -4 & 2 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} 1/3 & -1/3 \\ -1/3 & -1/6 \end{pmatrix} = \begin{pmatrix} 4/3 & -11/6 \\ -2 & 1 \\ 1 & 0 \end{pmatrix}$$

- Step 5. Defined 2nd order Petrov-Galerkin projected system

$$(W^\top AV, W^\top b, cV, d) = \left(\begin{bmatrix} 0 & -5/3 \\ 1 & 1/6 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & -2 \end{bmatrix}, 0 \right)$$

matches first $M = 4$ moments.

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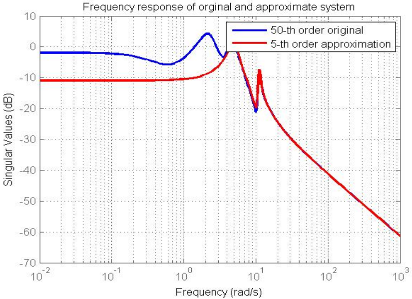
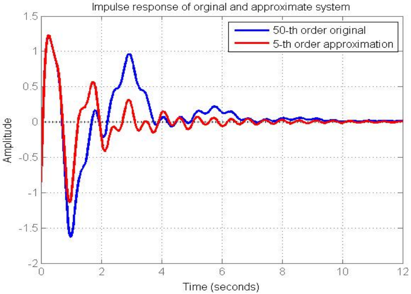
Example 2

Random LTI system of order $n = 50$ with its 5th order approximation that matches $M = 10$ moments at infinity.



frequency response



impulse response

Good approximation at large frequencies

Moment matching for LPV systems

Back to LPV systems:

Definition

The $m = (m_0, \ldots, m_d)$th moment of a transfer function $H(\mathsf{s}, \mathsf{p})$ at $(\mathsf{s}_0, \mathsf{p}_0) \in \mathbb{C} \times \mathbb{P}$ is

$$\eta_m(\mathsf{s}_0, \mathsf{p}_0) = \frac{\partial^{m_0} \partial^{m_1} \cdots \partial^{m_d}}{\partial \mathsf{s}^{m_0} \partial \mathsf{p}_1^{m_1} \cdots \partial \mathsf{p}_d^{m_d}} H(\mathsf{s}, \mathsf{p}) \bigg|_{(\mathsf{s},\mathsf{p})=(\mathsf{s}_0,\mathsf{p}_0)}$$

- We use multi-index notation $m = (m_0, \ldots, m_d)$.
- Cardinality of moment is $|m| = \sum_i m_i$.
- Numbers, vectors or matrices depending on dimensions u and y.
- Note that the gradient $\nabla H(\mathsf{s}_0, \mathsf{p}_0)$ consists of all moments $\eta_m(\mathsf{s}_0, \mathsf{p}_0)$ with $|m| = 1$.

The moment matching problem for LPV systems

Problem

Given LPV system $\Sigma(p)$ *of state dimension* n. *Let* $(s_0, p_0) \in \mathbb{C} \times \mathbb{P}$ *and* $r < n$.
Find reduced order system $\Sigma_r(p)$ *of state dimension* $r < n$ *such that*

$$\eta_m(s_0, p_0) = \eta_{r,m}(s_0, p_0) \quad \text{for } |m| = 0, \ldots, M$$

with M *maximal.*
Here $\eta_{r,m}(s_0, p_0)$ *is* m*th moment at* (s_0, p_0) *of* Σ_r.

Fit maximal number of coefficients in Taylor expansion of H in reduced order model H_r.

Moment matching in LPV SISO systems – A MBT !

Theorem (U. Baur, C. Beattie, P. Benner, S. Gugercin)

Suppose $H(s,p)$ *is a stable SISO system. Let* $(s_0, p_0) \in \mathbb{C} \times \mathbb{P}$*.*
Define $A_0 = (E(p_0)s_0 - A(p_0))^{-1}$ *and*

$$b_0 := (E(p_0)s_0 - A(p_0))^{-1} b(p_0), \quad c_0 := c(p_0)(E(p_0)s_0 - A(p_0))^{-1}$$

and let $V, W \in \mathbb{R}^{n \times 1}$ *be such that*

$$\operatorname{im} V = \mathcal{K}_1(A_0, b_0), \quad \operatorname{im}(W^\top) = \mathcal{K}_1(A_0^\top, c_0^\top)$$

Then the projected system $H_r(s,p)$ *with* $r = 1$ *achieves*

$$\eta_m(s_0, p_0) = \eta_{r,m}(s_0, p_0) \quad \text{for } |m| = 0, 1$$

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Then the projected system $H_r(s, p)$ with $r = 1$ achieves

$$\eta_m(s_0, p_0) = \eta_{r,m}(s_0, p_0) \quad \text{for } |m| = 0, 1$$

Thus, $H(s_0, p_0) = H_1(s_0, p_0)$ and the full gradient $\nabla H(s_0, p_0) = \nabla H_r(s_0, p_0)$

Moment matching in LPV SISO systems – A MBT !

Theorem (U. Baur, C. Beattie, P. Benner, S. Gugercin)

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Moment matching in LPV MIMO systems

Theorem (U. Baur, C. Beattie, P. Benner, S. Gugercin)

Suppose $H(s,p)$ *is a stable MIMO system. Let* $(s_0, p_0) \in \mathbb{C} \times \mathbb{P}$ *and* $(u_0, y_0) \in \mathbb{R}^{u+y}$. *Define* $A_0 = (E(p_0)s_0 - A(p_0))^{-1}$,

$$b_0 := (E(p_0)s_0 - A(p_0))^{-1} B(p_0) u_0, \quad c_0 := y_0^\top C(p_0)(E(p_0)s_0 - A(p_0))^{-1}$$

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Tangential interpolation conditions !!

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Tangential interpolation conditions !!
Matching for all multi-indices m with $|m| = 1$

Matching of multiple points in LPV systems

Take multiple points

$$\mathcal{S} := \{\mathsf{s}_1, \ldots, \mathsf{s}_K\}, \quad \mathcal{P} := \{\mathsf{p}_1, \ldots, \mathsf{p}_L\}$$

and address the problem to match moments

$$\eta_m(\mathsf{s}_k, \mathsf{p}_\ell) \;=\; \eta_{r,m}(\mathsf{s}_k, \mathsf{p}_\ell), \quad k = 1, \ldots, K,\; \ell = 1, \ldots, L$$

in multiple tangential input $\{\mathsf{u}_{k\ell}\}_{k=1,\ell=1}^{K,L}$ and output directions $\{\mathsf{y}_{k\ell}\}_{k=1,\ell=1}^{K,L}$.

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Thus, we wish to find reduced parametric model that achieves

$$\boxed{\mathsf{y}_{k\ell}^\top \eta_m(\mathsf{s}_k, \mathsf{p}_\ell)\mathsf{u}_{k\ell} \;=\; \mathsf{y}_{k\ell}^\top \eta_{r,m}(\mathsf{s}_k, \mathsf{p}_\ell)\mathsf{u}_{k\ell}}$$

for $k = 1, \ldots, K$; $\ell = 1, \ldots, L$; $m = 0, \ldots M$.

Matching of multiple points in LPV systems

Similar solution. Define

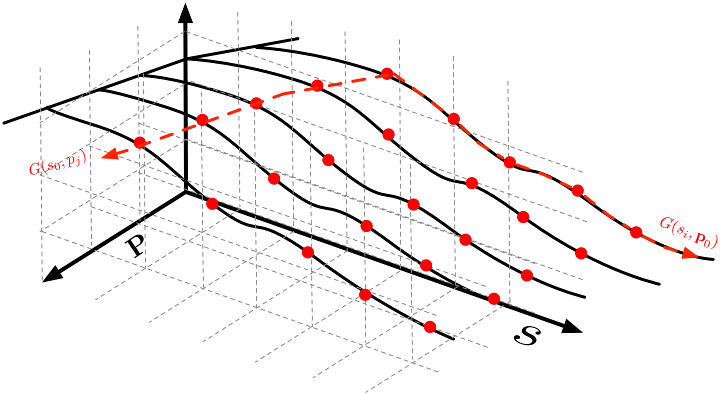
$$v_{k\ell} := (\mathsf{s}_k E(\mathsf{p}_\ell) - A(\mathsf{p}_\ell))^{-1} B(\mathsf{p}_\ell)\mathsf{u}_{k\ell}$$
$$w_{k\ell} := \mathsf{y}_{k\ell} C(\mathsf{p}_\ell)(\mathsf{s}_k E(\mathsf{p}_\ell) - A(\mathsf{p}_\ell))^{-1}$$

and set

$$V = \begin{pmatrix} v_{11} & \dots & v_{1L} & \dots & v_{K1} & \dots & v_{KL} \end{pmatrix}$$
$$W = \begin{pmatrix} w_{11} & \dots & w_{1L} & \dots & w_{K1} & \dots & w_{KL} \end{pmatrix}$$

Then projected LPV system $H_r(\mathsf{s},)$ with $r = KL$ will do.

Matching of multiple points in LPV systems



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Error analysis

Can view $H_r(\mathsf{s},\mathsf{p})$ as multi-variable Taylor series approximation of $H(\mathsf{s},\mathsf{p})$ at points in $\mathcal{S}\times\mathcal{P}$. Higher order moments define tensor with coefficients

$$[D^{|m|}H]_m := \frac{\partial^{m_0}\partial^{m_1}\cdots\partial^{m_d}}{\partial \mathsf{s}^{m_0}\partial \mathsf{p}_1^{m_1}\cdots\partial \mathsf{p}_d^{m_d}} H(\mathsf{s},\mathsf{p})\Big|_{(\mathsf{s},\mathsf{p})=(\mathsf{s}_0,\mathsf{p}_0)}$$

in the multi-index $m=(m_0,\ldots,m_d)$ and with cardinality $|m|=\sum_i m_i$.

Then $[D^{|m|}H]$ defines multi-linear functional $[D^{|m|}H]:(\mathbb{C}^{|m|})^{|m|}\to\mathbb{C}$ and we get

$$H_r(\mathsf{s},\mathsf{p}) = [D^0H] + [D^1H]\left(\begin{bmatrix}\mathsf{s}-\mathsf{s}_0\\ \mathsf{p}-\mathsf{p}_0\end{bmatrix}\right) + \cdots + \frac{1}{r!}[D^rH]\left(\begin{bmatrix}\mathsf{s}-\mathsf{s}_0\\ \mathsf{p}-\mathsf{p}_0\end{bmatrix},\cdots,\begin{bmatrix}\mathsf{s}-\mathsf{s}_0\\ \mathsf{p}-\mathsf{p}_0\end{bmatrix}\right)$$

as truncated multi-variable Taylor series expansion of $H(\mathsf{s},\mathsf{p})$ around $(\mathsf{s}_0,\mathsf{p}_0)$.

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Error analysis

Then error

$$H(\mathsf{s},\mathsf{p}) - H_r(\mathsf{s},\mathsf{p})$$

can be locally bounded through Mean-Value-Theorem:

$$|H(\mathsf{s},\mathsf{p}) - H_r(\mathsf{s},\mathsf{p})| \;\leq\; M \left| \underbrace{\begin{bmatrix} \mathsf{s}-\mathsf{s}_0 \\ \mathsf{p}-\mathsf{p}_0 \end{bmatrix} \otimes \cdots \otimes \begin{bmatrix} \mathsf{s}-\mathsf{s}_0 \\ \mathsf{p}-\mathsf{p}_0 \end{bmatrix}}_{r+1 \text{ factors}} \right|$$

for all (s,p) in a bounded neighborhood $\mathcal{B}_0$ of $(\mathsf{s}_0,\mathsf{p}_0)$.

Here,

$$M = \sup_{(\sigma,\rho)\in\mathcal{B}_0} \|D^{r+1}H(\sigma,\rho)\|$$

is the supremal induced norm of the tensor $[D^{r+1}H]$ over $\mathcal{B}_0$.

Outline

Time-varying parametrizations

Consider Linear Parameter-Varying (LPV) system:

$$\Sigma(\mathsf{p}) : \begin{cases} \dot{x}(t) = A(\mathsf{p})x(t) + B(\mathsf{p})u(t) \\ y(t) = C(\mathsf{p})x(t) \end{cases}$$

- Now $\mathsf{p} : \mathbb{T} \mapsto \mathbb{P} \subset \mathbb{R}^d$ is time-varying, unknown and only measurable.
- Projection framework as before.
- Transfer function does not exist !!!

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- Projection framework as before.
- Transfer function does not exist !!!
  Because, hey, what is

$$H(\mathsf{s}, \mathsf{p}) = C(\mathsf{p}(t)) \left[I\mathsf{s} - A(\mathsf{p}(t))\right]^{-1} B(\mathsf{p}(t))$$

supposed to mean? No way that $Y(\mathsf{s}, \mathsf{p}) = H(\mathsf{s}, \mathsf{p})U(\mathsf{s})$ ...

Model reduction by projection, time-varying parametrization

Parameter-varying projection: $V(\mathsf{p})$ with $\mathsf{p} : t \mapsto \mathsf{p}(t)$.

$$\Sigma_r(\mathsf{p}) : \begin{cases} \dot{x}_r = V^\top(\mathsf{p})(A(\mathsf{p})V(\mathsf{p}) - \frac{dV(\mathsf{p})}{dt})x_r + V^\top(\mathsf{p})B(\mathsf{p})u \\ y_r = C(\mathsf{p})V(\mathsf{p})x_r \end{cases}$$

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New projection scheme:

$$V(\mathsf{p}) \in \mathcal{V}_{\mathsf{p}} := \{V(\mathsf{p}) \in \mathbb{R}^{n\times r} \mid V^\top(\mathsf{p})V(\mathsf{p}) = I_r,\ V^\top(\mathsf{p})\frac{dV(\mathsf{p})}{dt} = \mathbf{0}\}$$

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Reduced-order model:

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The $\mathcal{H}_2$ norm

First recall the linear time-invariant (LTI) case:

$$\Sigma : \begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) \end{cases}$$

Defines transfer function (matrix):

$$H(s) = C(sI - A)^{-1}B.$$

$\mathcal{H}_2$ norm:

$$\|H\|_{\mathcal{H}_2} = \sqrt{\frac{1}{2\pi}\int_{-\infty}^{\infty} \text{trace}\left(H^*(i\omega)H(i\omega)\right)\ \mathrm{d}\omega}$$
$$= \sqrt{\text{trace}\left(C\mathcal{W}C^\top\right)} = \sqrt{\text{trace}\left(B^\top \mathcal{M} B\right)},$$

with $\mathcal{W}$ and $\mathcal{M}$ the reachability and observability Gramians.

Interpolatory $\mathcal{H}_2$ Approximation

Given H, the optimal $\mathcal{H}_2$ approximation amounts to finding a reduced-order system H_r which minimizes the $\mathcal{H}_2$ error, i.e.

$$H_r = \operatorname{argmin}_{\substack{\dim(\widetilde{H}_r)=r \\ \widetilde{H}_r \text{ stable}}} \left\| H - \widetilde{H}_r \right\|_{\mathcal{H}_2}$$

This problem is **nonconvex**. Common approach:
1st-order necessary condition (e.g. Iterative Rational Krylov Algorithm (IRKA)).

1st Order Optimality Condition

Let $H_r(s) = \sum_{k=1}^{r} \frac{1}{s-\widehat{\lambda}_k} c_k b_k^\top$ be the stable optimal $\mathcal{H}_2$ approximation of H. Then, for $k = 1, \ldots, r$,

$$(a)\ H(-\widehat{\lambda}_k)b_k = H_r(-\widehat{\lambda}_k)b_k,\quad (b)\ c_k^\top H^\top(-\widehat{\lambda}_k) = c_k^\top H_r^\top(-\widehat{\lambda}_k),$$

$$(c)\ c_k^\top H^\top(-\widehat{\lambda}_k)b_k = c_k^\top H_r^\top(-\widehat{\lambda}_k)b_k.$$

Formulation in terms of projection matrices:

Given full-order system Σ, the optimal $\mathcal{H}_2$ approximation problem amounts to finding

$$\Sigma_r : \begin{cases} \dot{x}_r(t) = V^\top A V x_r(t) + V^\top B u(t) \\ y_r(t) = C V x_r(t) \end{cases}$$

where

$$V = \operatorname{argmin}_{V^\top V = I_r} \ \text{trace}\left(\begin{pmatrix} C & -CV \end{pmatrix} \begin{pmatrix} \mathcal{W}_n & \mathcal{X} \\ \mathcal{X}^\top & \mathcal{W}_r \end{pmatrix} \begin{pmatrix} C^\top \\ -V^\top C^\top \end{pmatrix} \right)$$

is to minimize the $\mathcal{H}_2$ error, and the matrix $\begin{pmatrix} \mathcal{W}_n & \mathcal{X} \\ \mathcal{X}^\top & \mathcal{W}_r \end{pmatrix}$ is the reachability Gramian of the error system $\Sigma_e := \Sigma - \Sigma_r$.

Back to LPV

Assume parameter trajectory is known as p : $t \mapsto \rho(t)$ and LPV system has state-space representation

$$\Sigma_\rho(t) : \begin{cases} \dot{x}(t) = A_\rho(t)x(t) + B_\rho(t)u(t) \\ y(t) = C_\rho(t)x(t) \end{cases}$$

State Transition Matrix

$\Phi_\rho(\cdot,\cdot) : \mathbb{T} \times \mathbb{T} \to \mathbb{R}^{n\times n}$ is uniquely defined as the solution $\Phi_\rho(\cdot,\tau) = X(\cdot) : \mathbb{T} \to \mathbb{R}^{n\times n}$ of the homogeneous matrix differential equation

$$\dot{X}(t) = A_\rho(t)X(t) \quad \text{almost all } t \in \mathbb{T} \text{ with } X(\tau) = I.$$

Equivalently,

$$\frac{\partial}{\partial t}\Phi_\rho(t,\tau) = A_\rho(t)\Phi_\rho(t,\tau) \quad \text{almost all } t \in \mathbb{T} \text{ with } \Phi_\rho(\tau,\tau) = I.$$

Response Map and Impulse Response

Response map (input-output map):

$$y(t) = C_\rho(t)\Phi_\rho(t,\tau)x(\tau) + \int_\tau^t C_\rho(t)\Phi_\rho(t,\sigma)B_\rho(\sigma)u(\tau)d\sigma.$$

Impulse response or convolution kernel

$$h_\rho(t,\tau) = \begin{cases} 0, & t \leq \tau \\ C_\rho(t)\Phi_\rho(t,\tau)B_\rho(\tau), & t > \tau. \end{cases}$$

The impulse response energy is

$$\gamma_{\text{imp}} := \sqrt{\text{trace}\left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h_\rho^\top(t,\tau)h_\rho(t,\tau)d\tau dt\right)}.$$

$\mathcal{H}_2$ Norm of LPV Systems

$\mathcal{H}_2$ Norm

Let $H_\rho : (\omega_1, \omega_2) \mapsto H_\rho(i\omega_1, i\omega_2)$ denote the 2D Fourier transform of $h_\rho : (t, \tau) \mapsto h_\rho(t, \tau)$. The $\mathcal{H}_2$ norm of the system is defined as

$$\|H_\rho\|_{\mathcal{H}_2} := \frac{1}{2\pi}\sqrt{\text{trace}\left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} H_\rho^*(i\omega_1, i\omega_2)H_\rho(i\omega_1, i\omega_2)d\omega_1 d\omega_2\right)}.$$

Theorem

The $\mathcal{H}_2$ norm of the system equals to the impulse response energy, i.e.

$$\|H_\rho\|_{\mathcal{H}_2} = \gamma_{imp} = \sqrt{\mathit{trace}\left(\int_{-\infty}^{\infty} B_\rho^\top(t)\mathcal{M}_\rho(t)B_\rho(t)dt\right)}$$

$$= \sqrt{\mathit{trace}\left(\int_{-\infty}^{\infty} C_\rho(t)\mathcal{W}_\rho(t)C_\rho^\top(t)dt\right)}.$$

Gramians of LPV Systems

$\mathcal{W}_\rho(t)$ and $\mathcal{M}_\rho(t)$ are the time-varying reachability and observability Gramians and satisfy

- Reachability Gramian:

$$\mathcal{W}_\rho(t) = \int_{-\infty}^{t} \Phi_\rho(t,\tau)B_\rho(\tau)B_\rho^\top(\tau)\Phi_\rho^\top(t,\tau)d\tau,$$

$$\dot{\mathcal{W}}_\rho(t) = A_\rho(t)\mathcal{W}_\rho(t) + \mathcal{W}_\rho(t)A_\rho^\top(t) + B_\rho(t)B_\rho^\top(t),\ \ \mathcal{W}_\rho(-\infty) = 0.$$

- Observability Gramian:

$$\mathcal{M}_\rho(t) = \int_{t}^{\infty} \Phi_\rho^\top(\tau,t)C_\rho^\top(\tau)C_\rho(\tau)\Phi_\rho(\tau,t)d\tau,$$

$$-\dot{\mathcal{M}}_\rho(t) = A_\rho^\top(t)\mathcal{M}_\rho(t) + \mathcal{M}_\rho(t)A_\rho(t) + C_\rho^\top(t)C_\rho(t),\ \ \mathcal{M}(\infty) = 0.$$

Lang, Mena and Saak.
On the benefits of the $LDL^\top$ factorization for large-scale differential matrix equation solvers.
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Model Order Reduction for LPV Systems by Projection

Parameter-varying projection: $V(\mathbf{p})$ and $\mathbf{p} : t \mapsto \mathbf{p}(t)$.

$$\Sigma_r(\mathbf{p}) : \begin{cases} \dot{x}_r = V^\top(\mathbf{p})(A(\mathbf{p})V(\mathbf{p}) - \frac{dV(\mathbf{p})}{dt})x_r + V^\top(\mathbf{p})B(\mathbf{p})u \\ y_r = C(\mathbf{p})V(\mathbf{p})x_r + D_r(\mathbf{p})u. \end{cases}$$

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$\mathcal{H}_2$ LPV MOR optimization problem

Given LPV system $\Sigma(\mathbf{p})$ and a parameter trajectory $\mathbf{p} : t \mapsto \rho(t)$, find $V(\mathbf{p})$ by solving the optimization problem

$$\min_{V(\mathbf{p})} \frac{1}{2}\text{trace}\left(\int_{t_0}^{t_f} C_{e\rho}(t)\mathcal{W}_{e\rho}(t)C_{e\rho}^\top(t)dt\right)$$

$$\text{s.t. } \dot{\mathcal{W}}_{e\rho}(t) = A_{e\rho}(t)\mathcal{W}_{e\rho}(t) + \mathcal{W}_{e\rho}(t)A_{e\rho}^\top(t) + B_{e\rho}(t)B_{e\rho}^\top(t)$$

$$\dot{\mathcal{M}}_{e\rho}(t) + A_{e\rho}^\top(t)\mathcal{M}_{e\rho}(t) + \mathcal{M}_{e\rho}(t)A_{e\rho}(t) + C_{e\rho}^\top(t)C_{e\rho}(t) = 0$$

$$V^\top(\mathbf{p})V(\mathbf{p}) = I_r \text{ and } V^\top(\mathbf{p})\dot{V}(\mathbf{p}) = 0.$$

$\mathcal{H}_2$ LPV MOR optimization problem

Given LPV system $\Sigma(\mathbf{p})$ and a parameter trajectory $\mathbf{p} : t \mapsto \rho(t)$, find $V(\mathbf{p})$ by solving the optimization problem

$$\min_{V(\mathbf{p})} \frac{1}{2}\text{trace}\left(\int_{t_0}^{t_f} C_{e\rho}(t)\mathcal{W}_{e\rho}(t)C_{e\rho}^\top(t)dt\right)$$

$$\text{s.t. } \dot{\mathcal{W}}_{e\rho}(t) = A_{e\rho}(t)\mathcal{W}_{e\rho}(t) + \mathcal{W}_{e\rho}(t)A_{e\rho}^\top(t) + B_{e\rho}(t)B_{e\rho}^\top(t)$$

$$\dot{\mathcal{M}}_{e\rho}(t) + A_{e\rho}^\top(t)\mathcal{M}_{e\rho}(t) + \mathcal{M}_{e\rho}(t)A_{e\rho}(t) + C_{e\rho}^\top(t)C_{e\rho}(t) = 0$$

$$V^\top(\mathbf{p})V(\mathbf{p}) = I_r \text{ and } V^\top(\mathbf{p})\dot{V}(\mathbf{p}) = 0.$$

Here,

$$A_{e\rho}(t) = \begin{pmatrix} A_\rho(t) & 0 \\ 0 & V^\top(\rho(t))A_\rho(t)V(\rho(t)) \end{pmatrix}, \quad B_{e\rho}(t) = \begin{pmatrix} B_\rho(t) \\ V^\top(\rho(t))B_\rho(t) \end{pmatrix},$$

$$C_{e\rho}(t) = \begin{pmatrix} C_\rho(t) & -C_\rho(t)V(\rho(t)) \end{pmatrix}.$$

Approximate $V(\mathbf{p})$ by trial functions

Approximate the parameter-varying (PV) projection by

$$V(\mathbf{p}) \approx \sum_{k=0}^{K} \psi_k(\mathbf{p}) V_k,$$

where $V_k,\ k = 0, 1, 2, \ldots, K$ satisfy

$$V_k^\top V_l = \begin{cases} I_r, & k = l \\ \mathbf{0}, & k \neq l \end{cases}, \quad (K+1)r \leq n,$$

$$\sum_{k=0}^{K} \psi_k^2(\mathbf{p}) = 1, \ \forall \mathbf{p}.$$

Choose the scaled Fourier basis

$$\psi_k(\mathbf{p}) = \begin{cases} \sqrt{\frac{1}{1+K/2}} \cos(\frac{k}{2}\omega\mathbf{p}), & k \text{ is even;} \\ \sqrt{\frac{1}{1+K/2}} \sin(\frac{k+1}{2}\omega\mathbf{p}), & k \text{ is odd.} \end{cases}$$

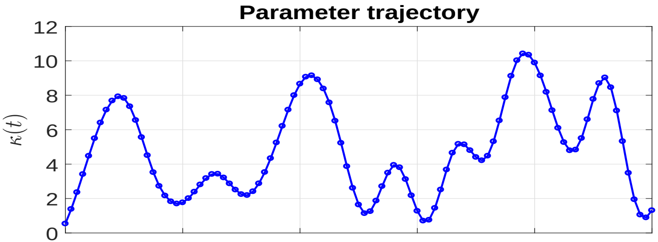
Consider the 1D heat equation with Dirichlet BC

$$\frac{\partial}{\partial t}T(x,t) = \kappa(t)\frac{\partial^2}{\partial x^2}T(x,t) + v(x,t), \;\; x \in [0,1].$$

The heat coefficient $\kappa(t) \in [0.1, 11]$ is a time-varying parameter.

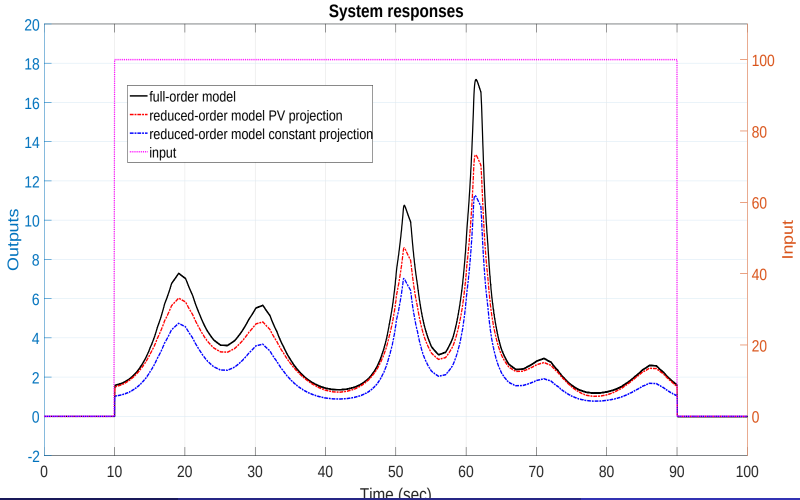
Semi-discretization – finite element model (dimension is $n = 10$)

$$\begin{aligned} \dot{T}_{\text{fem}}(t) &= \kappa(t) A T_{\text{fem}}(t) + B u_{\text{fem}}(t) \\ y_{\text{fem}}(t) &= C T_{\text{fem}}(t). \end{aligned}$$



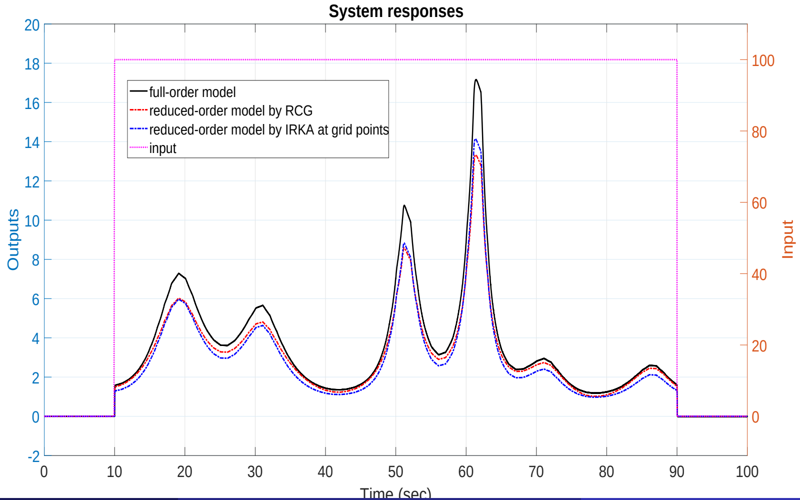
Constant vs. Parameter-Varying Projections

Reduction from $n = 10$ to $r = 3$: constant $(K = 0)$ vs parameter-varying $(K = 2)$ projections.



Parameter-Varying Projections vs. Point-wise IRKA

Reduction from $n = 10$ to $r = 3$: point-wise IRKA vs parameter-varying $(K = 2)$ projections.



Outline

Conclusions

- Model reduction for LPV systems need to distinguish time-varying and static parametrizations
- Very relevant for control, calibration, design, symbolic modeling
- Moment matching technique is well developed, computable, accurate.
- Adequate error bounds and error criteria remain research item.
- LPV for TV-LPV is hard but relevant problem.

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