Port-Hamiltonian Systems Theory: An Introductory Overview

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Dedicated to the memory of Jan C. Willems,
inspiring teacher and friend.

Contents

Abstract

An up-to-date survey of the theory of port-Hamiltonian systems is given, emphasizing novel developments and relationships with other formalisms. Port-Hamiltonian systems theory yields a systematic framework for network modeling of multi-physics systems. Examples from different areas show the range of applicability. While the emphasis is on modeling and analysis, the last part provides a brief introduction to control of port-Hamiltonian systems.

1

Introduction

1.1 Origins of port-Hamiltonian systems theory

The theory of port-Hamiltonian systems brings together different traditions in physical systems modeling and analysis.

Firstly, from a modeling perspective it originates in the theory of *port-based modeling* as pioneered by Henry Paynter in the late 1950s Paynter (1960); Breedveld (1984, 2009). Port-based modeling is aimed at providing a unified framework for the modeling of systems belonging to different physical domains (mechanical, electrical, hydraulic, thermal, etc.). This is achieved by recognizing *energy* as the 'lingua franca' between physical domains, and by identifying ideal system components capturing the main physical characteristics (energy-storage, energy-dissipation, energy-routing, etc.). Historically port-based modeling comes along with an insightful graphical notation emphasizing the structure of the physical system as a collection of ideal components linked by edges capturing the energy-flows between them. In analogy with chemical species these edges are called *bonds*, and the resulting graph is called a *bond graph*. Motivated by, among others, electrical circuit theory the energy flow along the bonds is represented by pairs of variables, whose product equals power. Typ-

ical examples of such pairs of variables (in different physical domains) are voltages and currents, velocities and forces, flows and pressures, etc.. A port-Hamiltonian formulation of bond graph models can be found in Golo et al. (2003). Port-based modeling can be seen to be a further abstraction of the theory of *across* and *through* variables (cf. MacFarlane (1970)) in the network modeling of physical systems[1].

A second origin of port-Hamiltonian systems theory is *geometric mechanics*; see e.g. Arnol'd (1978); Abraham & Marsden (1994); Marsden & Ratiu (1999); Bloch (2003); Bullo & Lewis (2004). In this branch of mathematical physics the Hamiltonian formulation of classical mechanics is formalized in a geometric way. The basic paradigm of geometric mechanics is to represent Hamiltonian dynamics in a coordinate-free manner using a state space (commonly the phase space of the system) endowed with a symplectic or Poisson structure, together with a Hamiltonian function representing energy. This geometric approach has led to an elegant and powerful theory for the analysis of the complicated dynamical behavior of Hamiltonian systems, displaying their intrinsic features, such as symmetries and conserved quantities, in a transparant way. Also infinite-dimensional Hamiltonian systems have been successfully cast into this framework Olver (1993).

Finally, a third pillar underlying the framework of port-Hamiltonian systems is *systems and control theory*, emphasizing dynamical systems as being open to interaction with the environment (e.g. via inputs and outputs), and as being susceptible to control interaction. The description and analysis of physical subclasses of control systems has roots in electrical network synthesis theory. Its geometric formulation was especially pioneered in Brockett (1977); see e.g. van der Schaft (1984, 1982a,b); Crouch (1981, 1984); Crouch & van der Schaft (1987); Nijmeijer & van der Schaft (1990); Maschke & van der Schaft (1992); Bloch (2003); Bullo & Lewis (2004) for some of the main developments, especially with regard to the anal-

[1] 'Abstraction' since the theory of across and through variables emphasizes the *balance laws* in the system; an aspect which is usually not emphasized in port-based modeling. In Chapter 12 and in Chapter 14 we will see how port-Hamiltonian systems can be also defined starting with the basic balance laws of the system.

ysis and control of nonlinear mechanical systems (e.g. with nonholonomic kinematic constraints).

A main difference of port-Hamiltonian systems theory with geometric mechanics lies in the fact that for port-Hamiltonian systems the underlying geometric structure is not necessarily the symplectic structure of the phase space, but in fact is determined by the *interconnection structure* of the system. In this sense port-Hamiltonian systems theory intrinsically merges *geometry* with *network theory*. The appropriate geometric object appears to be the notion of a *Dirac structure*, which was explored before in Weinstein (1983); Courant (1990); Dorfman (1993) as a geometric object generalizing at the same time symplectic and Poisson structures[2]. The usefulness of Dirac structures for a geometric theory of port-based modeling and analysis was first recognized in van der Schaft & Maschke (1995); Bloch & Crouch (1999); Dalsmo & van der Schaft (1999). Among others it has led to a theory of Hamiltonian *differential-algebraic equations*. Extensions to the distributed-parameter case were first explored in van der Schaft & Maschke (2002). A key property of Dirac structures is the fact that compositions of Dirac structures are again Dirac structures. This has the crucial consequence that the power-conserving interconnection of port-Hamiltonian systems (through their external ports) is again a port-Hamiltonian system; a fundamental property for network modeling and control.

Another main extension of port-Hamiltonian systems theory with respect to geometric mechanics is the inclusion of *energy-dissipating elements*, which are largely absent in classical Hamiltonian systems. This greatly broadens the range of applicability of port-Hamiltonian systems compared to that of Hamiltonian systems in analytical dynamics. In fact, the framework of port-based modeling and port-Hamiltonian systems emerges as a general theory for the modeling of complex physical systems as encountered in many areas of engineering[3]. Fur-

[2]The terminology 'Dirac structure' seems to be largely inspired by the 'Dirac bracket' introduced by Paul Dirac in order to cope with Hamiltonian systems subject to constraints due to degeneracy of the underlying Lagrangian function Dirac (1950, 1958). This was motivated in its turn by quantization theory.

[3]It should be added here that our emphasis in physical system modeling is on

thermore, because of its emphasis on *energy* and *power* as the lingua franca between different physical domains, port-Hamiltonian systems theory is ideally suited for a systematic mathematical treatment of *multi-physics systems*, i.e., systems containing subsystems from different physical domains (mechanical, electro-magnetic, hydraulic, chemical, etc.).

Apart from offering a systematic and insightful framework for modeling and analysis of multi-physics systems, port-Hamiltonian systems theory provides a natural starting point for control. Especially in the nonlinear case it is widely recognized that physical properties of the system (such as balance and conservation laws and energy considerations) should be exploited and/or respected in the design of control laws which are robust and physically interpretable. Port-Hamiltonian systems theory offers a range of concepts and tools for doing this, including the shaping of energy-storage and energy-dissipation, as well as the interpretation of controller systems as virtual system components. In this sense, port-Hamiltonian theory is a natural instance of a 'cyber-physical' systems theory: it admits the extension of physical system models with virtual ('cyber') system components, which may or may not mirror physical dynamics. From a broader perspective port-Hamiltonian systems theory is also related to multi-physics[4] network modeling approaches aimed at numerical *simulation*, such as 20-sim (based on bond graphs) and Modelica/Dymola.

1.2 Summary of contents

In these lecture notes we want to highlight a number of directions in port-Hamiltonian systems theory. Previous textbooks covering material on port-Hamiltonian systems are van der Schaft (2000) (Chapter 4), and Duindam et al. (2009). Especially Duindam et al. (2009) goes into more detail about a number of topics, and presents a wealth of

'modeling for control'. Since the addition of control will anyway modify the dynamical properties of the system the emphasis is on relatively simple models reflecting the main dynamical characteristics of the system.

[4] For specific physical domains (e.g., mechanical, electrical, chemical, hydraulic, ..) there are many network modeling and simulation software packages available.

material on various application domains. The current lecture notes present an up-to-date account of the basic theory, emphasizing novel developments.

Chapter 2 provides the basic definition of port-Hamiltonian systems and elaborates on the concept of a Dirac structure. Chapter 3 deals with Dirac structures on manifolds, and the resulting definition of port-Hamiltonian systems on manifolds. A brief discussion concerning integrability of Dirac structures is given, and the relation with the theory of kinematic constraints is provided. Chapter 4 details the special, but important, subclass of input-state-output port-Hamiltonian systems arising from the assumption of absence of algebraic constraints and the linearity of energy-dissipation relations. The resulting class of port-Hamiltonian systems is often taken as the starting point for the development of control theory for port-Hamiltonian systems.

With the general definition of port-Hamiltonian systems given in a geometric, coordinate-free, way, it is for many purposes important to represent the resulting dynamics in suitable coordinates, and in a form that is convenient for the system at hand. Chapter 5 shows how this amounts to finding a suitable representation of the Dirac structure, and how one can move from one representation to another. In Chapter 6 it is discussed how the power-conserving interconnection of port-Hamiltonian systems again defines a port-Hamiltonian system. This fundamental property of port-Hamiltonian system is based on the result that the composition of Dirac structures is another Dirac structure. Chapter 7 investigates the close connection of port-Hamiltonian systems with the concept of passivity, which is a key property for analysis and control. In Chapter 8 other structural properties of port-Hamiltonian systems are studied, in particular the existence of conserved quantities (Casimirs) and algebraic constraints.

Chapter 9 takes a step in a new direction by replacing the composition of the Dirac structure and the resistive structure by a general maximal monotone relation, leading to the novel class of incrementally port-Hamiltonian systems. In Chapter 10 the relation of port-Hamiltonian systems with the older class of input-output Hamilto-

nian systems is explored, and the key property of preservation of stability of input-output Hamiltonian systems under positive feedback (in contrast with negative feedback for port-Hamiltonian and passive systems) is discussed. Finally Chapter 11 makes the connection of port-Hamiltonian systems to another class of systems, namely the pseudo-gradient systems extending the Brayton-Moser equations of electrical circuits.

Chapter 12 deals with port-Hamiltonian systems on graphs, starting from the basic observation that the incidence structure of the graph defines a Poisson structure on the space of flow and effort variables associated to the vertices and edges of the graph. This is illustrated on a number of examples. In Chapter 13 the framework is extended to switching port-Hamiltonian systems, including a formulation of a jump rule generalizing the classical charge and flux conservation principle from electrical circuits with switches. Chapter 14 deals with the port-Hamiltonian formulation of distributed-parameter systems, based on the formulation of the Stokes-Dirac structure expressing the basic balance laws. Finally, Chapter 15 gives an introduction to the control theory of port-Hamiltonian systems, exploiting their basic properties such as passivity and existence of conserved quantities.

What is not in these lecture notes

The overview of port-Hamiltonian systems theory presented in this article is far from being complete: a number of topics are not treated at all, or only superficially. Notable omissions are the theory of scattering of port-Hamiltonian systems Stramigioli et al. (2002); van der Schaft (2009), treatment of symmetries and conservation laws of port-Hamiltonian systems van der Schaft (1998); Blankenstein & van der Schaft (2001), controllability and observability for input-output Hamiltonian systems and port-Hamiltonian systems van der Schaft (1984, 1982a,b); Maschke & van der Schaft (1992), realization theory of input-output Hamiltonian systems and port-Hamiltonian systems Crouch & van der Schaft (1987), port-Hamiltonian formulation of thermodynamical systems Eberard et al. (2007), model reduction of port-Hamiltonian systems Polyuga & van der Schaft (2011), well-posedness and stability

of distributed-parameter port-Hamiltonian systems Villegas (2007); Jacob & Zwart (2012), and structure-preserving discretization of distributed-parameter port-Hamiltonian systems Golo et al. (2004); Seslija et al. (2012). Furthermore, Chapter 15 on control of port-Hamiltonian systems only highlights a number of the developments in this area; for further information we refer to the extensive literature including Ortega et al. (2001a,b); Duindam et al. (2009); Ortega et al. (2008).

2

From modeling to port-Hamiltonian systems

2.1 Introduction

This chapter will provide the basic definition of port-Hamiltonian systems, starting from port-based network modeling. In order to motivate the general developments we start with a simple example, the ubiquitous *mass-spring* system.

Example 2.1 (Mass-spring system)**.** Consider a point mass with mass m, moving in one direction, without friction, under the influence of a spring force corresponding to a linear spring with spring constant k. The standard way of modeling the system is to start with the configuration $z \in \mathbb{R}$ of the mass, and to write down the second-order differential equation

$$m\ddot{z} = -k(z - z_0), \tag{2.1}$$

where z_0 is the rest length of the spring.

Port-based network modeling (as originating in particular in the work of Paynter, see Paynter (1960)) takes a different (but equivalent) point of view by regarding the mass-spring system as the *interconnection of two subsystems*, both of which store energy, namely the *spring system* storing potential energy and the *mass system* storing kinetic en-

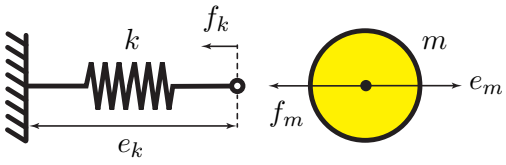


Figure 2.1: Mass-spring system as the interconnection of two subsystems.

ergy; see Figure 2.1. For the spring system the potential energy is expressed in terms of the elongation q of the spring. In case of a linear spring, satisfying Hooke's law, this potential energy is $\frac{1}{2}kq^2$. This leads to the system equations

$$\text{spring:} \begin{cases} \dot{q} = -f_k, \\ e_k = \dfrac{d}{dq}\left(\dfrac{1}{2}kq^2\right), \end{cases}$$

where[1] $-f_k$ denotes the velocity of the endpoint of the spring (where it is attached to the mass), and $e_k = kq$ denotes the spring force at this endpoint.

For the mass system we obtain similar equations using the kinetic energy $\frac{1}{2m}p^2$ expressed in terms of the *momentum* p of the mass

$$\text{mass:} \begin{cases} \dot{p} = -f_m, \\ e_m = \dfrac{d}{dp}\left(\dfrac{1}{2m}p^2\right), \end{cases}$$

where $-f_m$ denotes the force exerted on the mass, and $e_m = \frac{p}{m}$ is the velocity of the mass.

Finally, we *couple* the spring and the mass subsystems to each other through the interconnection element using Newton's third law (action $=-$reaction)

$$\text{interconnection:} \begin{cases} -f_k = e_m, \\ f_m = e_k, \end{cases}$$

[1]The reason for the minus sign in front of f_k is that we want the product $f_k e_k$ to be *incoming* power with respect to the interconnection. This sign convention will be adopted throughout.

leading to the final equations for the total system

$$\begin{bmatrix} \dot{q} \\ \dot{p} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} \dfrac{\partial H}{\partial q}(q,p) \\ \dfrac{\partial H}{\partial p}(q,p) \end{bmatrix}, \quad H(q,p) = \frac{1}{2}kq^2 + \frac{1}{2m}p^2, \tag{2.2}$$

which are the well-known *Hamiltonian equations* for the mass-spring system. Clearly, (2.2) is equivalent to the second-order model (2.1) by the correspondence $q = z - z_0$.

Although extremely simple, this example reflects some of the main characteristics of port-based network modeling: the system is regarded as the coupling of the energy-storing spring system with the energy-storing mass system through a power-conserving interconnection element which is routing the power from the mass system to the spring system and conversely. In the sequel we will see how this extends to general physical systems, replacing the above power-conserving interconnection element by the geometric notion of a Dirac structure, and by adding, next to the energy-storing elements, also energy-dissipating elements[2].

In general, in port-based modeling the, possibly large-scale, physical system is regarded as the interconnection of three types of ideal components[3]: (1) *energy-storing* elements, (2) *energy-dissipating* (resistive) elements, and (3) *energy-routing* elements.

Simplest examples of energy-storing elements are ideal inductors, capacitors, masses, and springs. Examples of energy-dissipating elements are resistors and dampers, while examples of energy-routing elements are transformers, gyrators and ideal constraints. Thus energy-dissipating elements are static (no dynamics involved), and energy-

[2] In this example already needed if we want to consider mass-spring-damper systems.

[3] These components do not necessarily match with the actual physical components. E.g., an inductor in a circuit may have, next to its energy-storing characteristics, also an energy-dissipation feature which needs to be taken into account for the modeling. This will mean that we model the inductor by an *ideal* inductor only reflecting the energy-storage *together* with an ideal resistor, accounting for the non-negligible energy-dissipation present in the physical inductor.

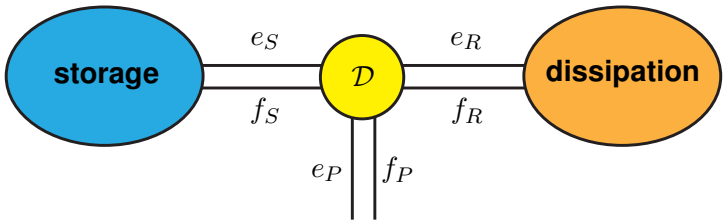


Figure 2.2: Port-Hamiltonian system.

routing elements are neither energy-storing or energy-dissipating but only redirect the power flow in the overall system.

For the port-Hamiltonian formulation (see also Golo et al. (2003)) the energy-storing elements will be grouped into a single object denoted by $\mathcal{S}$ ('storage'), and similarly the energy-dissipating elements are grouped into a single object denoted by $\mathcal{R}$ ('resistive'). Finally, the interconnection of all the energy-routing elements can be considered as one energy-routing structure[4] denoted by $\mathcal{D}$ (to be formalized by the geometric notion of a Dirac structure).

The essence of port-Hamiltonian systems modeling is thus represented in Fig. 2.2. The energy-storing elements $\mathcal{S}$ and the energy-dissipating (resistive) elements $\mathcal{R}$ are linked to a central interconnection (energy-routing) structure $\mathcal{D}$. This linking takes place via pairs (f, e) of equally dimensioned vectors of *flow* and *effort* variables. A pair (f, e) of vectors of flow and effort variables is called a *port*, and the total set of variables f, e is called the set of *port variables*. We refer to Appendix B for the physical meaning of efforts and flows in various physical domains.

Fig. 2.2 shows three ports: the port (f_S, e_S) linking to energy-storage, the port (f_R, e_R) corresponding to energy-dissipation, and the external port (f_P, e_P), by which the system interacts with its environment (including controller action). The scalar quantities $e_S^T f_S$, $e_R^T f_R$, and $e_P^T f_P$ denote the instantaneous powers transmitted through the links (the 'bonds' in bond graph terminology).

[4] Called *generalized junction structure* in bond graph terminology Breedveld (1984); Golo et al. (2003).

In the following sections, we will discuss the building blocks of port-Hamiltonian systems theory in more detail. We will start with the fundamental notion of a Dirac structure, then treat subsequently energy-storing elements, energy-dissipating (resistive) elements, and external ports, and finally end up with the basic geometric definition of a port-Hamiltonian system, together with a number of basic examples.

2.2 Port-based modeling and Dirac structures

Central in the definition of a port-Hamiltonian system is the notion of a *Dirac structure*, depicted in Fig. 2.2 by $\mathcal{D}$. In electrical circuit parlance, the Dirac structure acts as a 'printed circuit board' (without the energy-storing and energy-dissipating components), and provides the 'wiring' for the overall system.

Basic property of a Dirac structure is *power conservation*: the Dirac structure links the various port (flow and effort) variables f and e in such a way that the total power $e^T f$ is equal to zero. For the formal definition of a Dirac structure, we start with an abstract finite-dimensional linear space of *flows* $\mathcal{F}$.[5] The elements of $\mathcal{F}$ will be denoted by $f \in \mathcal{F}$, and are called *flow vectors*. The space of *efforts* is given by the *dual*[6] linear space $\mathcal{E} := \mathcal{F}^*$, and its elements are denoted by $e \in \mathcal{E}$. The *total space* of flow and effort variables is $\mathcal{F} \times \mathcal{E}$, and will be called the space of *port variables*. The *power* on the total space of port variables is defined by

$$P = < e \mid f >, \quad (f, e) \in \mathcal{F} \times \mathcal{E}, \tag{2.3}$$

where $< e \mid f >$ denotes the *duality product*, that is, the linear func-

[5] Usually one can take $\mathcal{F} = \mathbb{R}^k$. However, there are interesting cases where the coordinate-free viewpoint is really rewarding, e.g., in rigid body dynamics the space of flows is given as the space of twists $\mathcal{F} = \mathrm{se}(3)$, the Lie algebra of the matrix group $SE(3)$, while the space of efforts is given by the space of wrenches $\mathcal{E} = \mathrm{se}^*(3)$, the dual Lie algebra. We refer to Chapter 3 for some developments in this direction.

[6] The definition $\mathcal{E} = \mathcal{F}^*$ for the effort space is in some sense the minimal required structure. All definitions and results directly extend to the case that $\mathcal{F}$ has an *inner-product* structure. In this case we may take $\mathcal{E} = \mathcal{F}$ with the duality product $< e \mid f >$ replaced by the inner product $\langle e, f \rangle$.

tional $e \in \mathcal{E} = \mathcal{F}^*$ acting on $f \in \mathcal{F}$. In the usual case of $\mathcal{F} = \mathbb{R}^k$ this amounts to

$$< e \mid f >= e^T f,$$

where both $f \in \mathbb{R}^k$ and $e \in (\mathbb{R}^k)^*$ are represented as column vectors.

Definition 2.1. Consider a finite-dimensional linear space $\mathcal{F}$ with $\mathcal{E} = \mathcal{F}^*$. A subspace $\mathcal{D} \subset \mathcal{F} \times \mathcal{E}$ is a *Dirac structure* if

1. $< e \mid f >= 0$, for all $(f, e) \in \mathcal{D}$,
2. $\dim \mathcal{D} = \dim \mathcal{F}$.

Property (1) corresponds to *power-conservation*, and expresses the fact that the total power entering (or leaving) a Dirac structure is zero. It can be shown that the *maximal dimension* of any subspace $\mathcal{D} \subset \mathcal{F} \times \mathcal{E}$ satisfying Property (1) is equal to $\dim \mathcal{F}$. Instead of proving this directly, we will give an equivalent definition of a Dirac structure from which this claim immediately follows. Furthermore, this equivalent definition of a Dirac structure has the advantage that it generalizes to the case of an *infinite-dimensional* linear space $\mathcal{F}$, leading to the definition of an infinite-dimensional Dirac structure. This will be instrumental in the definition of a *distributed-parameter* port-Hamiltonian system in Chapter 14.

In order to give this equivalent characterization of a Dirac structure, we look more closely at the geometric structure of the total space of flow and effort variables $\mathcal{F} \times \mathcal{E}$. Related to the definition of power, there exists a canonically defined *bilinear form* $\ll, \gg$ on the space $\mathcal{F} \times \mathcal{E}$, defined as

$$\ll (f^a, e^a), (f^b, e^b) \gg := < e^a \mid f^b > + < e^b \mid f^a >, \tag{2.4}$$

with $(f^a, e^a), (f^b, e^b) \in \mathcal{F} \times \mathcal{E}$. Note that this bilinear form is *indefinite*, that is, $\ll (f, e), (f, e) \gg$ may be positive or negative. It is *non-degenerate* in the sense that $\ll (f^a, e^a), (f^b, e^b) \gg = 0$ for all (f^b, e^b) implies that $(f^a, e^a) = 0$.

Proposition 2.1 (Courant (1990); Dorfman (1993))**.** A *Dirac structure* on $\mathcal{F} \times \mathcal{E}$ is a subspace $\mathcal{D} \subset \mathcal{F} \times \mathcal{E}$ such that

$$\mathcal{D} = \mathcal{D}^{\perp\!\!\!\perp}, \tag{2.5}$$

where $\perp\!\!\!\perp$ denotes the orthogonal companion[7] with respect to the bilinear form $\ll, \gg$.

Alternatively, $\mathcal{D} \subset \mathcal{F} \times \mathcal{E}$, with $\mathcal{F}$ and $\mathcal{E}$ finite-dimensional, is a Dirac structure if and only if it satisfies Property 1 in Definition 2.1 and has *maximal dimension* with respect to this property, that is, if the subspace $\mathcal{D}'$ also satisfies Property 1 then $\dim \mathcal{D}' \leq \dim \mathcal{D}$. This maximal dimension is equal to $\dim \mathcal{F} = \dim \mathcal{E}$.

For the proof we refer to Appendix A.

From a mathematical point of view, there are a number of direct examples of Dirac structures $\mathcal{D} \subset \mathcal{F} \times \mathcal{E}$. We leave the simple proofs as an exercise to the reader.

1. Let $J : \mathcal{E} \to \mathcal{F}$ be a skew-symmetric linear mapping, that is, $J = -J^*$, where $J^* : \mathcal{E} \to \mathcal{E}^* = \mathcal{F}$ is the adjoint mapping. Then

$$\text{graph } J := \Big\{(f, e) \in \mathcal{F} \times \mathcal{E} \mid f = Je\Big\}$$

   is a Dirac structure.

2. Let $\omega : \mathcal{F} \to \mathcal{E}$ be a skew-symmetric linear mapping, then

$$\text{graph } \omega := \Big\{(f, e) \in \mathcal{F} \times \mathcal{E} \mid e = \omega f\Big\}$$

   is a Dirac structure.

3. Let $\mathcal{K} \subset \mathcal{F}$ be any subspace. Define

$$\mathcal{K}^\perp = \Big\{e \in \mathcal{E} \mid < e \mid \ f >= 0 \text{ for all } f \in \mathcal{K}\Big\} \tag{2.6}$$

   Then $\mathcal{K} \times \mathcal{K}^\perp \subset \mathcal{F} \times \mathcal{E}$ is a Dirac structure.

The last example of a Dirac structure is formalized as follows:

Definition 2.2. A Dirac structure $\mathcal{D} \subset \mathcal{F} \times \mathcal{E}$ is *separable* if

$$< e_a \mid f_b >= 0, \tag{2.7}$$

for all $(f_a, e_a), (f_b, e_b) \in \mathcal{D}$.

[7] A subspace $\mathcal{D}$ such that the bilinear form $\ll, \gg$ is zero restricted to this subspace (or equivalently $\mathcal{D} \subset \mathcal{D}^{\perp\!\!\!\perp}$) is sometimes called an *isotropic subspace*. In this terminology a Dirac structure is a *maximal isotropic subspace*.

Separable Dirac structures have the following simple geometric characterization (see Appendix A for a proof).

Proposition 2.2. Consider a separable Dirac structure $\mathcal{D} \subset \mathcal{F} \times \mathcal{E}$. Then,

$$\mathcal{D} = \mathcal{K} \times \mathcal{K}^{\perp}, \tag{2.8}$$

for some subspace $\mathcal{K} \subset \mathcal{F}$, where $\mathcal{K}^{\perp}$ is defined as in (2.6). Conversely, any subspace $\mathcal{D}$ as in (2.8) for some subspace $\mathcal{K} \subset \mathcal{F}$ is a separable Dirac structure.

Note that (2.7) can be regarded as a generalized statement of Tellegen's theorem for electrical circuits (with f denoting the vector of currents, and e denoting the vector of voltages).

A typical instance of a separable Dirac structure is the following.

Proposition 2.3. Let $A : \mathcal{V} \to \mathcal{W}$ be a linear map between the linear spaces $\mathcal{V}$ and $\mathcal{W}$ with adjoint mapping $A^* : \mathcal{W}^* \to \mathcal{V}^*$, that is,

$$< w^* \mid Av >=< A^* w^* \mid v >,$$

for all $v \in \mathcal{V}, w^* \in \mathcal{W}^*$ (where, as before, $< \cdot \mid \cdot >$ denotes the duality product between the dual spaces $\mathcal{W}$ and $\mathcal{W}^*$, respectively $\mathcal{V}$ and $\mathcal{V}^*$). Identify $(\mathcal{V} \times \mathcal{W})^* = \mathcal{V}^* \times \mathcal{W}^*$. Then,

$$\mathcal{D} := \left\{ (v, w, v^*, w^*) \in (\mathcal{V} \times \mathcal{W}) \times (\mathcal{V}^* \times \mathcal{W}^*) \mid Av = w, v^* = -A^* w^* \right\}$$

is a separable Dirac structure.

Remark 2.1. In some cases, e.g. 3D-mechanical systems, the above notion of Dirac structures on vector spaces will turn out not to be sufficient. In Chapter 3, we will discuss the extension of the definition of constant Dirac structures on vector spaces to that of *Dirac structures on manifolds*. Basically, a Dirac structure on a manifold will be the union of Dirac structures on the product of the tangent and cotangent space at every point of the manifold. As a result the Dirac structure will be *modulated* by the state.

A crucial property of Dirac structures is the fact that the *composition* of Dirac structures again defines a Dirac structure, see Chapter 6. This

has the consequence that we can interconnect all energy-routing elements to each other and that the resulting element (*generalized junction structure* in bond graph parlance) will again define a Dirac structure.

Finally, we will discuss a number of physical examples of Dirac structures.

2.2.1 Transformers, gyrators, ideal constraints, junctions

A *transformer*, see Paynter (1960), Breedveld (1984), is an element linking two scalar bonds with flow and effort variables $(f_1, e_1) \in \mathbb{R}^2$ and $(f_2, e_2) \in \mathbb{R}^2$ by

$$\begin{aligned} f_2 &= \alpha f_1, \\ e_1 &= -\alpha e_2, \end{aligned} \tag{2.9}$$

with α a constant, called the *transformer ratio*. The subspace defined by (2.9) is easily checked to be a separable Dirac structure. Also the vector version of (2.9)

$$\begin{aligned} f^b &= A f^a, \\ e^a &= -A^T e^b, \end{aligned} \tag{2.10}$$

with (f^a, e^a) and (f^b, e^b) pairs of column vectors of flow variables and effort variables of the same dimension, and A a matrix of appropriate dimensions, is immediately seen to define a Dirac structure.

Similarly, a *gyrator* is given by the relations

$$\begin{aligned} f_1 &= \beta e_2, \\ \beta e_1 &= -f_2, \end{aligned} \tag{2.11}$$

which again is defining a Dirac structure (but not a separable one). The resulting unit gyrator for $\beta = 1$ is called the *symplectic gyrator* Breedveld (1984). The multi-dimensional version is given as the Dirac structure defined by

$$\begin{aligned} f^a &= G e^b, \\ G^T e^a &= -f^b, \end{aligned} \tag{2.12}$$

where G is a matrix of appropriate dimensions.

Also ideal *effort and flow constraints* are examples of Dirac structures. Let (f, e) denote a (multi-dimensional) pair of flows and efforts.

Then, the ideal effort constraint

$$\mathcal{D} := \Big\{(f, e) \mid e = 0\Big\}$$

is defining a Dirac structure $\mathcal{D}$, and the same holds for the ideal flow constraint

$$\mathcal{D} := \Big\{(f, e) \mid f = 0\Big\}.$$

Finally, the equations of a so-called k-dimensional 0-*junction* (terminology from bond graph theory, cf. Paynter (1960); Breedveld (1984))

$$e_1 = e_2 = \cdots = e_k, \quad f_1 + f_2 + \cdots + f_k = 0,$$

and dually of a 1-*junction*

$$f_1 = f_2 = \cdots = f_k, \quad e_1 + e_2 + \cdots + e_k = 0,$$

are immediately seen to define separable Dirac structures.

2.2.2 Kirchhoff's laws as separable Dirac structures

Consider an electrical circuit with m branches (edges) and k nodes (vertices) where the current through the i-th branch is denoted by I_i and the voltage across the i-th branch is V_i. Collect the currents in an n-dimensional column vector I and the voltages in an n-dimensional column vector V. Then Kirchhoff's *current* laws can be written as

$$BI = 0, \tag{2.13}$$

with B the $k \times m$ *incidence matrix* of the circuit graph. Furthermore, Kirchhoff's *voltage* laws can be written as follows. All allowed vectors of voltages V in the circuit are given as

$$V = B^T\lambda, \tag{2.14}$$

with the vector λ ranging through $\mathbb{R}^k$. It is immediately seen that the total space of currents and voltages allowed by Kirchhoff's current and voltage laws,

$$\mathcal{D} := \Big\{(I, V) \mid BI = 0,\ V = B^T\lambda \text{ for some } \lambda\Big\}, \tag{2.15}$$

defines a separable Dirac structure. Consequently,

$$(V^a)^T I^b + (V^b)^T I^a = 0,$$

for all pairs $(I^a, V^a), (I^b, V^b) \in \mathcal{D}$. In particular, by taking V^a, I^b equal to zero, we obtain $(V^b)^T I^a = 0$ for all I^a satisfying (2.13) and all V^b satisfying (2.14). This is *Tellegen's theorem* from circuit theory.

Further theory regarding Kirchhoff's laws and electrical circuits can be found in Chapter 12.

2.2.3 Kinematic pairs

The equations describing a kinematic pair (e.g., a revolute or prismatic joint) in a three-dimensional mechanical system are, from the Dirac structure point of view, of the same type as Kirchhoff's laws.[8]

Indeed, the constraint forces F generated in a (frictionless and infinitely stiff) kinematic pair produce no power on the velocities V allowed by the kinematic pair, i.e.,

$$A^T V = 0, \quad F = A\lambda, \tag{2.16}$$

where the columns of A form a basis for the space of allowed reaction forces, and λ is the vector of reaction force Lagrange multipliers.

2.2.4 The principle of virtual work

The principle of virtual work can be formulated as

$$\sum_{i=1}^{n} F_i \delta q_i = 0, \tag{2.17}$$

where F_i are the impressed forces, and δq_i denotes the *virtual* displacements that are compatible with the kinematic constraints of the system. The expression $\sum_{i=1}^{n} F_i \delta q_i$ equals the infinitesimal work due to the impressed forces and an infinitesimal displacement. If the kinematic constraints of the system are given as $\mathcal{A}^T \delta q = 0$, with $\delta q =$

[8] However, for 3D mechanical systems the matrix A will often depend on the configuration coordinates; thus defining a Dirac structure on a manifold, see Chapter 3.

$(\delta q_1, \cdots, \delta q_n)^T$, then it follows that the impressed forces are given as $F = \mathcal{A}\lambda$, with $F = (F_1, \cdots, F_n)^T$, as in the previous subsection; see Chapter 3 for more details. We conclude that, like in the case of Kirchhoff's laws in the electrical domain, the principle of virtual work can be formulated as defining a separable Dirac structure on the product of the space of virtual displacements and impressed forces.

Originally, the principle of virtual work (2.17) is formulated as an *equilibrium* condition. Indeed, a system with configuration coordinates $q = (q_1, q_2, \ldots, q_n)^T$, which is subject to forces $F(q)$, is at equilibrium $\bar{q}$ if the virtual work $\sum_{i=1}^{n} F_i(\bar{q})\delta q_i$ corresponding to any admissible virtual displacement δq from $\bar{q}$ is equal to zero. In d'Alembert's principle this was extended by adding the *inertial forces* $\dot{p}$ to the impressed forces. This can be interpreted as linking the Dirac structure to *energy-storage* (in this case, kinetic energy).

2.3 Energy-storing elements

The energy-storing multi-port element $\mathcal{S}$ corresponds to the union of all the energy-storing elements of the system. The port variables of the Dirac structure associated with the energy-storing multi-port element will be denoted by (f_S, e_S), where f_S and e_S are vectors of equal dimension with their product $e_S^T f_S$ denoting the total power flowing into the Dirac structure from the energy storing elements (or, equivalently, minus the total power flowing into the storage elements). The total energy storage of the system is defined by a state space $\mathcal{X}$, together with a Hamiltonian function $H : \mathcal{X} \to \mathbb{R}$ denoting the energy. For now, we will assume that the state space $\mathcal{X}$ is *finite-dimensional* (in Chapter 14 we will discuss the extension to the infinite-dimensional case). In general, see Chapter 3, $\mathcal{X}$ will be a smooth (finite-dimensional) manifold, but in the present chapter $\mathcal{X}$ will be assumed to be a linear space.

The vector of flow variables of the energy-storing multi-port element is given by the *rate* $\dot{x}$ of the state $x \in \mathcal{X}$. Thus for any current state $x \in \mathcal{X}$ the flow vector $\dot{x}$ will be an element of the linear space $T_x\mathcal{X}$, the tangent space of $\mathcal{X}$ at $x \in \mathcal{X}$. By choosing local coordinates

$x = (x_1, \ldots, x_n)^T$ for $\mathcal{X}$ this means that the vector of flow variables is given by the vector $\dot{x} = (\dot{x}_1, \ldots, \dot{x}_n)^T$. In the case of a linear state space $\mathcal{X}$ the tangent space $T_x\mathcal{X}$ can be identified with $\mathcal{X}$, and we can take (global) linear coordinates for $\mathcal{X}$, thus identifying $\mathcal{X}$ with $\mathbb{R}^n$. Furthermore, the vector of effort variables of the energy-storing multi-port element is given by the gradient vector $\frac{\partial H}{\partial x}(x) \in T_x^*\mathcal{X}$, the dual space of the tangent space $T_x\mathcal{X}$. In coordinates $x = (x_1, \ldots, x_n)^T$ for $\mathcal{X}$ this means that the vector of effort variables is given by the vector $\frac{\partial H}{\partial x}(x)$ of partial derivatives of H with respect to $x_1, \ldots, x_n$ (which we throughout write as a *column* vector).

We obtain the following power-balance for the energy-storing multi-port element:

$$\frac{d}{dt}H =< \frac{\partial H}{\partial x}(x) \mid \dot{x} >= \frac{\partial^T H}{\partial x}(x)\dot{x}. \tag{2.18}$$

The interconnection of the energy-storing elements to the storage port (f_S, e_S) of the Dirac structure is accomplished by setting

$$f_S = -\dot{x} \quad \text{and} \quad e_S = \frac{\partial H}{\partial x}(x). \tag{2.19}$$

Hence, the power-balance (2.18) can be also written as

$$\frac{d}{dt}H = \frac{\partial^T H}{\partial x}(x)\dot{x} = -e_S^T f_S. \tag{2.20}$$

Remark 2.2. The minus sign in (2.19) is inserted in order to have a consistent power flow convention: $\frac{\partial^T H}{\partial x}(x)\dot{x}$ is the power flowing into the energy-storing elements, whereas $e_S^T f_S$ is the power flowing into the Dirac structure.

See Appendix B for details on how to set up the Hamiltonian for energy-storing elements of various physical domains.

2.4 Energy-dissipating (resistive) elements

The second multi-port element $\mathcal{R}$ corresponds to internal energy dissipation (due to friction, resistance, etc.), and its port variables are denoted by (f_R, e_R). These port variables are terminated on a static

energy-dissipating (resistive) relation $\mathcal{R}$. In general, a resistive relation will be a subset

$$\mathcal{R} \subset \mathcal{F}_R \times \mathcal{E}_R,$$

with the property that[9]

$$< e_R \mid f_R >= e_R^T f_R \leq 0, \tag{2.21}$$

for all $(f_R, e_R) \in \mathcal{R}$. We will call the subset $\mathcal{R}$ an *energy-dissipating relation*, or a *resistive structure*. Since the Dirac structure of a port-Hamiltonian system (without external port) satisfies the power-balance

$$e_S^T f_S + e_R^T f_R = 0, \tag{2.22}$$

this leads by substitution of the equations (2.20) and (2.21) to

$$\frac{d}{dt} H = -e_S^T f_S = e_R^T f_R \leq 0. \tag{2.23}$$

An important special case of energy-dissipating relations occurs when the resistive relation can be expressed as the graph of an *input-output* mapping, e.g.,

$$f_R = -F(e_R), \tag{2.24}$$

with $F : \mathbb{R}^m \to \mathbb{R}^{m_r}$ satisfying $e_R^T F(e_R) \geq 0$, for all $e_R \in \mathbb{R}^m$ (with m denoting the number of energy-dissipating elements). Sometimes the mapping F is derivable from a so-called *Rayleigh dissipation function* $D_R : \mathbb{R}^m \to \mathbb{R}$, in the sense that

$$F(e_R) = \frac{\partial D_R}{\partial e_R}(e_R).$$

For *linear* resistive elements, (2.24) specializes to

$$f_R = -R e_R, \tag{2.25}$$

for some positive semi-definite symmetric matrix $R = R^T \geq 0$.

[9] The sign of the inequality is due to the fact that $e_R^T f_R$ is the power flow associated to the Dirac structure, *not* to the energy-dissipating elements. Another way to resolve this sign problem would be to define an additional pair of flow and effort variables $(\bar{f}_R, \bar{e}_R)$ for the energy-dissipating elements satisfying $\bar{e}_R^T \bar{f}_R \geq 0$, and to interconnect them, similarly to the case of the energy-storing elements (see Remark 2.2), to the energy-dissipating port (f_R, e_R) of the Dirac structure by setting $f_R = -\bar{f}_R$ and $e_R = \bar{e}_R$.

Example 2.2. A linear damper in a mass-spring-damper system is modeled by an equation $f_R = -de_R$ with $d > 0$ the damping constant. An example of a nonlinear energy-dissipating relation is the cubic equation $f_R = -de_R^3$. Another type of example is provided by ideal Coulomb friction,which is modeled by the energy-dissipating relation

$$f_R = \begin{cases} -1 & \text{for} \quad e_R > 0, \\ \alpha & \text{for} \quad e_R = 0, \\ +1 & \text{for} \quad e_R < 0, \end{cases}$$

with $\alpha \in [-F_c, F_c]$ and $F_c > 0$ the Coulomb friction constant. Note that this does *not* correspond anymore to a function from e_R to f_R, or conversely.

In Chapter 4 we will further elaborate on linear energy-dissipating relations, and their geometric treatment. See also Chapter 9 for a further discussion on energy-dissipating and maximal monotone relations.

2.5 External ports

The external port (f_P, e_P) models the interaction of the system with its environment. This comprises different situations. One are the port variables which are accessible for *controller* action. Another type of external port variables corresponds to an *interaction port*. Typical example of the latter is a controlled robotic system interacting with its physical environment. Still another type of external port variables are variables corresponding to *sources*. For example, in an electrical circuit with voltage source the input is the voltage of the source, while the current through the source is the (resulting) output variable.

Taking the external port into account the power-balance (2.22) extends to

$$e_S^T f_S + e_R^T f_R + e_P^T f_P = 0, \tag{2.26}$$

whereby (2.23) extends to

$$\frac{d}{dt}H = e_R^T f_R + e_P^T f_P \leq e_P^T f_P, \tag{2.27}$$

since $e_R^T f_R \leq 0$. This inequality expresses the basic fact that the increase of the internally stored energy (the Hamiltonian) is always less than or equal to the externally supplied power.

2.6 Port-Hamiltonian dynamics

The dynamics of a port-Hamiltonian system is defined as follows.

Definition 2.3. Consider a state space $\mathcal{X}$ and a Hamiltonian

$$H : \mathcal{X} \to \mathbb{R},$$

defining energy-storage. A port-Hamiltonian system on $\mathcal{X}$ is defined by a Dirac structure

$$\mathcal{D} \subset T_x\mathcal{X} \times T_x^*\mathcal{X} \times \mathcal{F}_R \times \mathcal{E}_R \times \mathcal{F}_P \times \mathcal{E}_P,$$

having energy-storing port $(f_S, e_S) \in T_x\mathcal{X} \times T_x^*\mathcal{X}$ and a resistive structure

$$\mathcal{R} \subset \mathcal{F}_R \times \mathcal{E}_R,$$

corresponding to an energy-dissipating port $(f_R, e_R) \in \mathcal{F}_R \times \mathcal{E}_R$. Its dynamics is specified by

$$\begin{gathered}\left(-\dot{x}(t), \frac{\partial H}{\partial x}(x(t)), f_R(t), e_R(t), f_P, (t), e_P(t)\right) \in \mathcal{D}(x(t)), \\ (f_R(t), e_R)(t) \in \mathcal{R}(x(t)), \quad t \in \mathbb{R}.\end{gathered} \tag{2.28}$$

At the beginning of this chapter, we have already seen how a mass-spring system is modeled as a port-Hamiltonian system. The next example concerns a simple electrical circuit.

Example 2.3 (RL-circuit)**.** Consider an electrical circuit depicted in Fig. 2.3. The energy-storage port of the system is described as

$$\begin{aligned}\dot{\phi}_i &= -V_i, \\ I_i &= \frac{dH_i}{d\varphi_i}(\varphi_i),\end{aligned}$$

for $i = 1, 2$, where I_i are the currents through the inductors with flux-linkages ϕ_i, magnetic energy $H_i(\phi_1)$, and $-V_i$ the voltages across

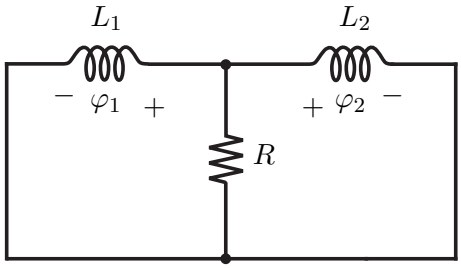


Figure 2.3: RL circuit.

them. The energy-dissipating relation corresponding to the (possible nonlinear) resistor is $V_R = -F(I_R)$, with $I_R F(I_R) \geq 0$. Kirchhoff's current and voltage laws define the Dirac structure expressed by the equations

$$I_1 + I_2 + I_R = 0, \quad V_1 = V_2 = V_R,$$

The resulting port-Hamiltonian system is given as

$$\dot{\varphi}_1 = F\left(-\frac{dH_1}{d\varphi_1}(\varphi_1) - \frac{dH_2}{d\varphi_2}(\varphi_2)\right),$$
$$\dot{\varphi}_2 = F\left(-\frac{dH_1}{d\varphi_1}(\varphi_1) - \frac{dH_2}{d\varphi_2}(\varphi_2)\right),$$

which for a linear resistor $V_R = -RI_R$, with $R > 0$, and linear inductors $I_i = \frac{\varphi_i}{L_i}$, for $i = 1, 2$, reduces to

$$\dot{\varphi}_1 = -\frac{R}{L_1}\varphi_1 - \frac{R}{L_2}\varphi_2,$$
$$\dot{\varphi}_2 = -\frac{R}{L_1}\varphi_1 - \frac{R}{L_2}\varphi_2.$$

In Chapter 12, we will elaborate on port-Hamiltonian models for general RLC circuits.

The following two examples of port-Hamiltonian systems emphasize port-based network modeling of *multi-physics* systems.

Example 2.4 (Levitated ball system)**.** Consider the dynamics of an iron ball that is levitated by the magnetic field of a controlled inductor

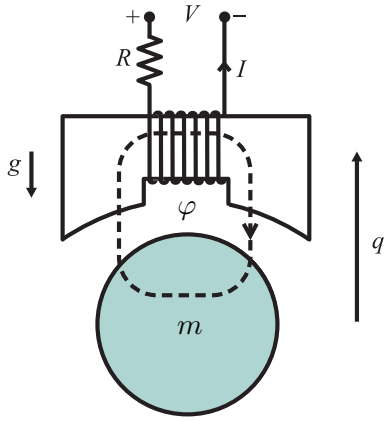


Figure 2.4: Magnetically levitated ball.

as schematically depicted in Fig. 2.4. The port-Hamiltonian description of this system (with q the height of the ball, p the vertical momentum, and φ the magnetic flux-linkage of the inductor) is given as

$$\begin{bmatrix} \dot{q} \\ \dot{p} \\ \dot{\varphi} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -R \end{bmatrix} \begin{bmatrix} \dfrac{\partial H}{\partial q} \\ \dfrac{\partial H}{\partial p} \\ \dfrac{\partial H}{\partial \varphi} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} V, \tag{2.29}$$

$$I = \frac{\partial H}{\partial \varphi}.$$

Although at first instance the mechanical and the magnetic part of the system look decoupled, they are actually coupled via the Hamiltonian

$$H(q, p, \varphi) = mgq + \frac{p^2}{2m} + \frac{\varphi^2}{2L(q)},$$

where the inductance $L(q)$ depends on the height q. In fact, the magnetic energy $\frac{\varphi^2}{2L(q)}$ depends both on the flux φ and the mechanical variable q. As a result the right-hand side of the second equation (describing the evolution of the mechanical momentum variable p) depends on the magnetic variable φ, and conversely the right-hand side of the

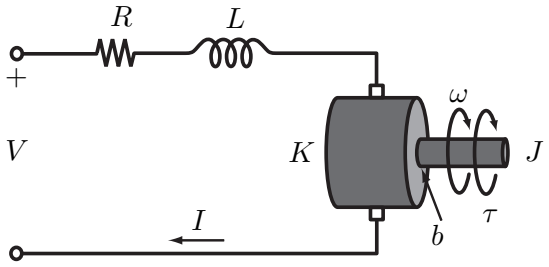


Figure 2.5: DC motor.

third equation (describing the evolution of the magnetic variable φ) depends on the mechanical variable q.

Example 2.5 (DC motor)**.** In the schematic model of a DC motor depicted in Figure 2.5, we can distinguish six interconnected subsystems:[10]
◦ two energy-storing elements with physical states ϕ, p: an ideal inductor L with state variable φ (flux-linkage), and rotational inertia J with state variable p (angular momentum);
◦ two energy-dissipating elements: the resistor R and the friction b;
◦ a gyrator K;
◦ an ideal voltage source V.
The energy-storing elements (here assumed to be linear) are given by

$$\text{Inductor:} \begin{cases} \dot{\varphi} = -V_L \\ I = \dfrac{d}{d\varphi}\left(\dfrac{1}{2L}\varphi^2\right) = \dfrac{\varphi}{L}, \end{cases}$$

$$\text{Inertia:} \begin{cases} \dot{p} = -\tau_J \\ \omega = \dfrac{d}{dp}\left(\dfrac{1}{2J}p^2\right) = \dfrac{p}{J}. \end{cases}$$

Hence, the corresponding total Hamiltonian reads $H(p,\phi) = \frac{1}{2L}\phi^2 + \frac{1}{2J}p^2$. The energy-dissipating relations (also assumed to be linear) are

[10] In this example it is obvious that the subsystems do not correspond to *actual* physical subsystems, but rather model the different physical phenomena present in the DC-motor.

given as

$$V_R = -RI, \quad \tau_b = -b\omega,$$

with $R, b > 0$, where τ_b is a damping torque. Furthermore, the equations of the gyrator (converting magnetic power into mechanical, or conversely) are

$$V_K = -K\omega, \quad \tau = KI.$$

for a certain positive constant K (the gyrator constant). Finally, the subsystems are interconnected by the equations

$$V_L + V_R + V_K + V = 0, \quad \tau_J + \tau_b + \tau = 0.$$

Note that the Dirac structure is defined by the above interconnection equations, together with the equations for the gyrator. Collecting all equations, we obtain the port-Hamiltonian model

$$\begin{aligned} \begin{bmatrix} \dot{\varphi} \\ \dot{p} \end{bmatrix} &= \begin{bmatrix} -R & -K \\ K & -b \end{bmatrix} \begin{bmatrix} \frac{\varphi}{L} \\ \frac{p}{J} \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} V, \\ I &= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{\varphi}{L} \\ \frac{p}{J} \end{bmatrix}. \end{aligned} \tag{2.30}$$

While in the previous example the coupling between the mechanical and magnetic domain was provided by the Hamiltonian (depending in a non-separable way on the mechanical state variables and the magnetic state variable), in this example the inter-domain coupling is given by the Dirac structure (through the gyrator constant K).

2.7 Port-Hamiltonian differential-algebraic equations

A usual feature of network modeling is the fact that the obtained model of the overall system consists of differential equations *and algebraic equations*. This stems from the fact that network modeling admits quite arbitrary interconnection equations between the subsystems of the overall system. The resulting set of differential and algebraic equations are commonly called *differential-algebraic systems* (DAEs). This is

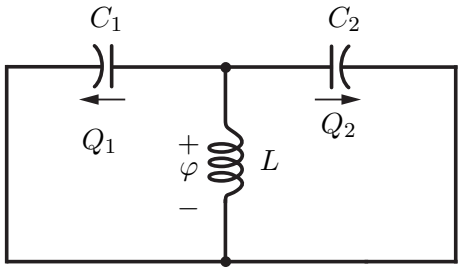


Figure 2.6: LC circuit.

in contrast with signal-flow diagram modeling where it is assumed that the external variables for the subsystems can be decomposed into inputs (free variables) and outputs (determined by the state and possibly the inputs) in such a way that the value of the inputs to one subsystem are equal to the value of the outputs of others. However, in physical systems modeling this is often *not* the case, and algebraic constraints due to interconnection constraints between the 'outputs' of the subsystems commonly arise.

In order to illustrate this issue, as well as to indicate how port-Hamiltonian systems theory yields a systematic approach of handling algebraic constraints, we consider a simple example from the realm of electrical circuits. The general theory of port-Hamiltonian DAEs will be touched upon in Chapter 8. For a general treatment of port-Hamiltonian DAEs we refer to van der Schaft (2013).

Example 2.6. Consider an LC-circuit consisting of two capacitors and one inductor as shown in Fig. 2.6. Naturally this system can be seen as the interconnection of three subsystems, the two capacitors and the inductor, interconnected by Kirchhoff's current and voltage laws. The capacitors (first assumed to be linear) are described by the following dynamical equations

$$\dot{Q}_i = -I_i,$$
$$V_i = \frac{Q_i}{C_i},$$

for $i = 1, 2$. Here I_i and V_i are the currents through, respectively the voltages across, the two capacitors, and C_i are their capacitances. Fur-

thermore, Q_i are the *charges* stored in the capacitors and are regarded as basic state variables.[11] Similarly, the linear inductor is described by the dynamical equations

$$\begin{aligned} \dot{\varphi} &= -V_L, \\ I_L &= \frac{\varphi}{L}, \end{aligned}$$

where I_L is the current through the inductor, and V_L is the voltage across the inductor. Here, the (magnetic) flux-linkage φ is taken as the state variable of the inductor, and L denotes its inductance.

Parallel interconnection of these three subsystems by Kirchhoff's laws amounts to the interconnection equations

$$V_1 = V_2 = V_L, \quad I_1 + I_2 + I_L = 0,$$

where the equation $V_1 = V_2$ gives rise to the algebraic constraint

$$\frac{Q_1}{C_1} = \frac{Q_2}{C_2}, \tag{2.31}$$

relating the two state variables Q_1, Q_2.

There are multiple ways to represent the dynamics of the total system. One is to regard either I_1 or I_2 as a *Lagrange multiplier* for the constraint (2.31). Indeed, by defining $\lambda = I_1$ one may write the total system as

$$\begin{aligned} \begin{bmatrix} \dot{Q}_1 \\ \dot{Q}_2 \\ \dot{\varphi} \end{bmatrix} &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} Q_1/C_1 \\ Q_2/C_2 \\ \varphi/L \end{bmatrix} + \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \lambda, \\ 0 &= \begin{bmatrix} -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} Q_1/C_1 \\ Q_2/C_2 \\ \varphi/L \end{bmatrix}, \end{aligned} \tag{2.32}$$

[11] In the port-Hamiltonian formulation there is a clear preference for taking the charges Q_i to be the state variables instead of the voltages V_i. This is due to the fact that the charges satisfy a conservation law, while the voltages do not. Furthermore, although the introduction of charge variables comes at the expense of extra variables, it will turn out to be very advantageous from a geometric point of view as well: the charge variables live in the 'right' state space.

Next one may *eliminate* the Lagrange multiplier λ by premultiplying the first three differential equations by the matrix

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Together with the algebraic constraint (2.31) this yields the *differential-algebraic system*

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{Q}_1 \\ \dot{Q}_2 \\ \dot{\varphi} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} Q_1/C_1 \\ Q_2/C_2 \\ \varphi/L \end{bmatrix}. \tag{2.33}$$

The two equivalent equational representations (2.32) and (2.33) result from two different representations of the Dirac structure of the system, namely

$$\mathcal{D} = \left\{ (f,e) \in \mathbb{R}^3 \times \mathbb{R}^3 \;\middle|\; f = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix}, e = \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix}, \right.$$
$$\left. \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix} + \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} = 0 \right\},$$

and

$$\mathcal{D} = \left\{ (f,e) \in \mathbb{R}^3 \times \mathbb{R}^3 \;\middle|\; \exists \lambda \text{ such that} \right.$$
$$\left. - \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \lambda,\ 0 = \begin{bmatrix} 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} \right\}.$$

Furthermore, the energy-storing relations are given by

$$f = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix} = - \begin{bmatrix} \dot{Q}_1 \\ \dot{Q}_2 \\ \dot{\varphi} \end{bmatrix}, \quad e = \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} = \begin{bmatrix} Q_1/C_1 \\ Q_2/C_2 \\ \varphi/L \end{bmatrix},$$

where the last vector is the gradient vector of the total stored energy

$$H(Q_1, Q_2, \varphi) := \frac{Q_1^2}{2C_1} + \frac{Q_2^2}{2C_2} + \frac{\varphi^2}{2L}. \tag{2.34}$$

From a DAE perspective, it may be noted that the algebraic constraint (2.31) is of *index one*. In fact, under reasonable assumptions on the Hamiltonian, this will turn out to be a general property of port-Hamiltonian differential-algebraic systems; see Chapter 8.

2.8 Detailed-balanced chemical reaction networks

The final section of this chapter illustrates how port-Hamiltonian modeling extends to physical systems outside the more traditional mechanical and electrical domain. Based on van der Schaft et al. (2013) it is shown how isothermal chemical reaction networks governed by mass-action kinetics and satisfying a thermodynamically justified assumption admit a natural port-Hamiltonian formulation. This treatment merges the geometric thermodynamic approach of Oster et al. (1973) with the line of research initiated in Horn & Jackson (1972); Feinberg (1987) based on the graph of chemical complexes.

Consider an isothermal chemical reaction network (under constant pressure) consisting of r reversible reactions involving m chemical species specified by a vector of concentrations $x \in \mathbb{R}^m_+ := \{x \in \mathbb{R}^m \mid x_i > 0, i = 1, \cdots, m\}$. The general form of the dynamics of the chemical reaction network (without inflows and outflows) is

$$\dot{x} = Sv(x),$$

with S the stoichiometric matrix, and $v(x) = \begin{bmatrix} v_1(x) & \cdots & v_r(x) \end{bmatrix}^T \in \mathbb{R}^r$ the vector of *reaction rates*. We assume that $v(x)$ is given by *mass action kinetics*; the most basic way of modeling reaction rates. Following van der Schaft et al. (2013) we will show how, under the assumption of existence of a thermodynamic equilibrium, the dynamics of the reaction network can be naturally modeled as a port-Hamiltonian system, with Hamiltonian given by the free Gibbs' energy.

In order to do so we first need to introduce some concepts and terminology. The collection of all the different left- and right-hand sides of the reactions are called the chemical complexes of the reaction network, or briefly, the *complexes*. Denoting the number of complexes by

c, the expression of the complexes in terms of the chemical species concentration vector $x \in \mathbb{R}^m_+$ is formalized by an $m \times c$ matrix Z, whose ρ-th column captures the expression of the ρ-th complex in the m chemical species. Note that by definition all elements of the matrix Z are non-negative integers.

The complexes can be naturally associated with the vertices of a *directed graph*, with edges corresponding to the reactions. The complex on the left-hand side of each reaction is called the *substrate* complex, and the one on the right-hand side the *product* complex. Formally, the reaction $\sigma \rightleftharpoons \pi$ between the σ-th and the π-th complex defines a directed edge with tail vertex being the σ-th complex and head vertex being the π-th complex. The resulting directed graph is called the *complex graph*, and is defined Bollobas (1998) by its $c \times r$ *incidence matrix* B. It is readily verified that the stoichiometric matrix S of the chemical reaction network is given as $S = ZB$.

Mass action kinetics for the reaction rate vector $v(x) \in \mathbb{R}^r$ is defined as follows. Consider first, as an example, the single reaction

$$X_1 + 2X_2 \rightleftharpoons X_3,$$

involving the three chemical species X_1, X_2, X_3 with concentrations x_1, x_2, x_3. In mass action kinetics the reaction is considered to be a combination of the *forward reaction* $X_1 + 2X_2 \rightharpoonup X_3$ with forward rate equation $v_1^+(x_1, x_2) = k^+ x_1 x_2^2$ and the *reverse reaction* $X_1 + 2X_2 \leftharpoondown X_3$, with rate equation $v^-(x_3) = k^- x_3$. The constants k^+, k^- are called respectively the *forward* and the *reverse reaction constants*. The net reaction rate is thus

$$v(x_1, x_2, x_3) = v^+(x_1, x_2) - v^-(x_3) = k^+ x_1 x_2^2 - k^- x_3.$$

In general, the mass action reaction rate of the j-th reaction of a chemical reaction network, from the substrate complex $\mathcal{S}_j$ to the product complex $\mathcal{P}_j$, is given as

$$v_j(x) = k_j^+ \prod_{i=1}^{m} x_i^{Z_{i\mathcal{S}_j}} - k_j^- \prod_{i=1}^{m} x_i^{Z_{i\mathcal{P}_j}}, \tag{2.35}$$

where $Z_{i\rho}$ is the (i, ρ)-th element of the matrix Z, and $k_j^+, k_j^- \geq 0$ are the forward/reverse reaction constants of the j-th reaction, respectively.

Eq. (2.35) can be rewritten in the following way. Let $Z_{\mathcal{S}_j}$ and $Z_{\mathcal{P}_j}$ denote the columns of Z corresponding to the substrate complex $\mathcal{S}_j$ and the product complex $\mathcal{S}_j$ of the j-th reaction. Defining the mapping $\mathrm{Ln} : \mathbb{R}^c_+ \to \mathbb{R}^c$ as the component-wise natural logarithm, (2.35) takes the form

$$v_j(x) = k_j^+ \exp\left(Z_{\mathcal{S}_j}^T \mathrm{Ln}(x)\right) - k_j^- \exp\left(Z_{\mathcal{P}_j}^T \mathrm{Ln}(x)\right). \tag{2.36}$$

A vector of concentrations $x^* \in \mathbb{R}^m_+$ is called a *thermodynamic equilibrium* if $v(x^*) = 0$. A chemical reaction network $\dot{x} = Sv(x)$ is called *detailed-balanced* if it admits a thermodynamic equilibrium $x^* \in \mathbb{R}^m_+$. Necessary and sufficient conditions for the existence of a thermodynamic equilibrium are usually referred to as the *Wegscheider conditions*, generalizing the classical results of Wegscheider (1902), and can be derived as follows Feinberg (1989); van der Schaft et al. (2013). Consider the j-th reaction from substrate $\mathcal{S}_j$ to product $\mathcal{P}_j$, described by the mass action rate equation (2.36). Then $x^* \in \mathbb{R}^m_+$ is a thermodynamic equilibrium if and only if

$$k_j^+ \exp\left(Z_{\mathcal{S}_j}^T \mathrm{Ln}(x^*)\right) = k_j^- \exp\left(Z_{\mathcal{P}_j}^T \mathrm{Ln}(x^*)\right), \quad j = 1, \ldots, r \tag{2.37}$$

The equations (2.37), referred to as the *detailed balance equations*, can be rewritten as follows. Define the *equilibrium constant* K_j^{eq} of the j-th reaction as (assuming $k_j^+ \neq 0$)

$$K_j^{eq} := \frac{k_j^+}{k_j^-} \tag{2.38}$$

Then the detailed balance equations (2.37) are equivalent to

$$K_j^{eq} = \exp\left(Z_{\mathcal{P}_j}^T \mathrm{Ln}\left(x^*\right) - Z_{\mathcal{S}_j}^T \mathrm{Ln}\left(x^*\right)\right), \quad j = 1, \ldots, r \tag{2.39}$$

Collecting all reactions, and making use of the incidence matrix B of the complex graph, this amounts to the vector equation

$$K^{eq} = \mathrm{Exp}\left(B^T Z^T \mathrm{Ln}\left(x^*\right)\right) = \mathrm{Exp}\left(S^T \mathrm{Ln}\left(x^*\right)\right), \tag{2.40}$$

where K^{eq} is the r-dimensional vector with j-th element $K_j^{eq}, j = 1, \ldots, r$. It follows that there exists a thermodynamic equilibrium

$x^* \in \mathbb{R}^m_+$ if and only if $k_j^+ > 0, k_j^- > 0$, for all $j = 1, \ldots, r$, and furthermore

$$\mathrm{Ln}\ (K^{eq}) \in \mathrm{im}\, S^T \tag{2.41}$$

It also follows that once a thermodynamic equilibrium x^* is given, the set of *all* thermodynamic equilibria is given by

$$\mathcal{E} := \{x^{**} \in \mathbb{R}^m_+ \mid S^T \mathrm{Ln}\ (x^{**}) = S^T \mathrm{Ln}\ (x^*)\} \tag{2.42}$$

Let now $x^* \in \mathbb{R}^m_+$ be a thermodynamic equilibrium. Consider the rewritten form (2.39) of the detailed-balance equations, and define the 'conductance' $\kappa_j(x^*) > 0$ of the j-th reaction as the common value of the forward and reverse reaction rate at thermodynamic equilibrium x^*, i.e.,

$$\kappa_j(x^*) := k_j^+ \exp\left(Z_{\mathcal{S}_j}^T \mathrm{Ln}\,(x^*)\right) = k_j^- \exp\left(Z_{\mathcal{P}_j}^T \mathrm{Ln}\,(x^*)\right), \tag{2.43}$$

for $j = 1, \cdots, r$. Then the mass action reaction rate (2.36) of the j-th reaction can be rewritten as

$$v_j(x) = \kappa_j(x^*) \left[\exp\left(Z_{\mathcal{S}_j}^T \mathrm{Ln}\left(\frac{x}{x^*}\right)\right) - \exp\left(Z_{\mathcal{P}_j}^T \mathrm{Ln}\left(\frac{x}{x^*}\right)\right)\right],$$

where for any vectors $x, z \in \mathbb{R}^m$ the quotient vector $\frac{x}{z} \in \mathbb{R}^m$ is defined element-wise. Defining the $r \times r$ diagonal matrix of conductances as

$$\mathcal{K} := \mathrm{diag}(\kappa_1(x^*), \cdots, \kappa_r(x^*)), \tag{2.44}$$

it follows that the mass action reaction rate vector $v(x)$ of a balanced reaction network equals

$$v(x) = -\mathcal{K} B^T \mathrm{Exp}\left(Z^T \mathrm{Ln}\left(\frac{x}{x^*}\right)\right),$$

and thus the dynamics of a balanced reaction network takes the form

$$\dot{x} = -ZB\mathcal{K}B^T \mathrm{Exp}\left(Z^T \mathrm{Ln}\left(\frac{x}{x^*}\right)\right), \quad \mathcal{K} > 0. \tag{2.45}$$

The matrix $\mathcal{L} := B\mathcal{K}B^T$ in (2.45) defines a *weighted Laplacian matrix* for the complex graph, with weights given by the conductances $\kappa_1(x^*), \cdots, \kappa_r(x^*)$. Note that $\mathcal{K}$, and therefore the Laplacian matrix

$\mathcal{L} = B\mathcal{K}B^T$, is *dependent* on the choice of the thermodynamic equilibrium x^*. However, this dependence is minor: for a connected complex graph the matrix $\mathcal{K}$ is, *up to a positive multiplicative factor*, independent of the choice of x^*, cf. van der Schaft et al. (2013).

How does this define a port-Hamiltonian system ? Define first the Hamiltonian (up to a constant the Gibbs' free energy, cf. van der Schaft et al. (2013); Oster et al. (1973)) as

$$G(x) = x^T \mathrm{Ln}\left(\frac{x}{x^*}\right) + (x^* - x)^T \mathbb{1}_m,$$

where $\mathbb{1}_m$ denotes a vector of dimension m with all ones. It is immediately checked $\frac{\partial G}{\partial x}(x) = \mathrm{Ln}\,(\frac{x}{x^*}) = \mu(x)$, where μ is (up to a constant) known as the vector of chemical potentials. Then the mass action reaction dynamics (2.45) is obtained from considering the auxiliary port-Hamiltonian system

$$\begin{aligned} \dot{x} &= Z f_R, \\ e_R &= Z^T \frac{\partial G}{\partial x}(x), \end{aligned} \tag{2.46}$$

with inputs $f_R \in \mathbb{R}^c$ and outputs $e_R \in \mathbb{R}^c$, together with the energy-dissipating relation

$$f_R = -B\mathcal{K}B^T \mathrm{Exp}\,(e_R). \tag{2.47}$$

Indeed, by using the properties of the Laplacian matrix $B\mathcal{K}B^T$ and the fact that the exponential function is strictly increasing, it can be shown that van der Schaft et al. (2013)

$$\gamma^T B\mathcal{K}B^T \mathrm{Exp}\,(\gamma) \geq 0 \text{ for all } \gamma, \tag{2.48}$$

with equality if and only if $B^T\gamma = 0$. Hence (2.47) defines a true *energy-dissipating* relation, that is, $e_R^T f_R \leq 0$ for all $e_R \in \mathbb{R}^c$ and $f_R \in \mathbb{R}^c$ satisfying (2.47). Therefore the mass action kinetics detailed-balanced chemical reaction network is a port-Hamiltonian system with Hamiltonian G and energy-dissipating relation (2.47).

The consequences of the port-Hamiltonian modeling of detailed-balanced mass action kinetics reaction networks for the analysis of the reaction network are explored in van der Schaft et al. (2013). In particular, it follows that *all* equilibria are in fact thermodynamic equilib-

ria, and a Lyapunov analysis using the Gibbs' free energy (the Hamiltonian) shows that that starting from any initial state in the positive orthant the system will converge to a unique thermodynamic equilibrium (at least under the assumption of *persistence* of the reaction network: the vector of concentrations does not approach the boundary of the positive orthant $\mathbb{R}^m_+$), cf. van der Schaft et al. (2013) for details.[12]

[12]For an extension of these results to *complex-balanced* mass action kinetics reaction networks we refer to Rao et al. (2013).

3

Port-Hamiltonian systems on manifolds

3.1 Modulated Dirac structures

For quite a few system classes, in particular those with 3-D mechanical components, the Dirac structure is *modulated* by the state variables. Furthermore, the state space $\mathcal{X}$ is not necessarily anymore a linear space but instead a (differentiable[1]) *manifold*. As before, the flows $f_S = -\dot{x}$ corresponding to energy-storage are elements of the *tangent space* $T_x\mathcal{X}$ at the state $x \in \mathcal{X}$, while the efforts $e_S = \frac{\partial H}{\partial x}(x)$ are elements of the *co-tangent space* $T_x^*\mathcal{X}$. The modulation of the Dirac structure is usually intimately related to the underlying geometry of the system.

Example 3.1 (Spinning rigid body)**.** Consider a rigid body spinning around its center of mass in the absence of gravity. The energy variables are the three components of the body angular momentum p along the three principal axes: $p = (p_x, p_y, p_z)^T$, and the energy is the kinetic energy

$$H(p) = \frac{1}{2}\left(\frac{p_x^2}{I_x} + \frac{p_y^2}{I_y} + \frac{p_z^2}{I_z}\right),$$

[1] 'Manifold' will always mean 'differentiable manifold'.

where I_x, I_y, I_z are the principal moments of inertia. Euler's equations describing the dynamics are

$$\begin{bmatrix} \dot{p}_x \\ \dot{p}_y \\ \dot{p}_z \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & -p_z & p_y \\ p_z & 0 & -p_x \\ -p_y & p_x & 0 \end{bmatrix}}_{J(p)} \begin{bmatrix} \frac{\partial H}{\partial p_x} \\ \frac{\partial H}{\partial p_y} \\ \frac{\partial H}{\partial p_z} \end{bmatrix}. \tag{3.1}$$

The Dirac structure is given as the graph of the skew-symmetric matrix $J(p)$, i.e., modulated by the state variables p. In this example, the state space $\mathcal{X}$ is still a linear space. In fact, $\mathcal{X} = so^*(3)$, the dual of the Lie algebra $so(3)$ of the matrix group $SO(3)$.

Modulated Dirac structures often arise as a result of *ideal constraints* imposed on the generalized velocities of the mechanical system by its environment, called *kinematic constraints*. In many cases, these constraints will be configuration dependent, yielding a Dirac structure modulated by the configuration variables.

Consider a mechanical system with n degrees of freedom, locally described by n configuration variables $q = (q_1, \ldots, q_n)$. Expressing the kinetic energy as $\frac{1}{2}\dot{q}^T M(q)\dot{q}$, with $M(q) > 0$ being the generalized mass matrix, we define in the usual way the Lagrangian function $L(q, \dot{q})$ as the *difference* of kinetic energy and potential energy $U(q)$, i.e.

$$L(q, \dot{q}) = \frac{1}{2}\dot{q}^T M(q)\dot{q} - U(q). \tag{3.2}$$

Suppose now that there are constraints on the generalized velocities $\dot{q}$, described as

$$A^T(q)\dot{q} = 0, \tag{3.3}$$

with $A(q)$ an $n \times k$ matrix of rank k everywhere (that is, there are k independent kinematic constraints). Classically, the constraints (3.3) are called *holonomic* if it is possible to find new configuration coordinates $\overline{q} = (\overline{q}_1, \ldots, \overline{q}_n)$ such that the constraints are equivalently expressed as

$$\dot{\overline{q}}_{n-k+1} = \dot{\overline{q}}_{n-k+2} = \cdots = \dot{\overline{q}}_n = 0, \tag{3.4}$$

in which case one may eliminate the configuration variables $\overline{q}_{n-k+1}, \ldots, \overline{q}_n$, since the kinematic constraints (3.4) are equivalent to

the *geometric* constraints

$$\overline{q}_{n-k+1} = c_{n-k+1}, \ldots, \overline{q}_n = c_n, \tag{3.5}$$

for certain constants $c_{n-k+1}, \ldots, c_n$ determined by the initial conditions. Then the system reduces to an *unconstrained* system in the $(n-k)$ remaining configuration coordinates $(\overline{q}_1, \ldots, \overline{q}_{n-k})$. If it is *not* possible to find coordinates $\overline{q}$ such that (3.4) holds (that is, if we are not able to *integrate* the kinematic constraints as above), then the constraints are called *nonholonomic*.

The equations of motion for the mechanical system with Lagrangian $L(q, \dot{q})$ and constraints (3.3) are given by the Euler-Lagrange equations Neimark & Fufaev (1972)

$$\begin{aligned} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} &= A(q)\lambda + B(q)u,\ \lambda \in \mathbb{R}^k,\ u \in \mathbb{R}^m, \\ A^T(q)\dot{q} &= 0, \end{aligned} \tag{3.6}$$

where $B(q)u$ are the external forces (controls) applied to the system, for some $n \times m$ matrix $B(q)$, while $A(q)\lambda$ are the *constraint forces*. The Lagrange multipliers $\lambda(t)$ are uniquely determined by the requirement that the constraints $A^T(q(t))\dot{q}(t) = 0$ have to be satisfied for all times t.

Defining the generalized momenta

$$p = \frac{\partial L}{\partial \dot{q}} = M(q)\dot{q}, \tag{3.7}$$

the constrained Euler-Lagrange equations (3.6) transform into *constrained Hamiltonian equations*

$$\begin{aligned} \dot{q} &= \frac{\partial H}{\partial p}(q,p) \\ \dot{p} &= -\frac{\partial H}{\partial q}(q,p) + A(q)\lambda + B(q)u \\ y &= B^T(q)\frac{\partial H}{\partial p}(q,p) \\ 0 &= A^T(q)\frac{\partial H}{\partial p}(q,p) \end{aligned} \tag{3.8}$$

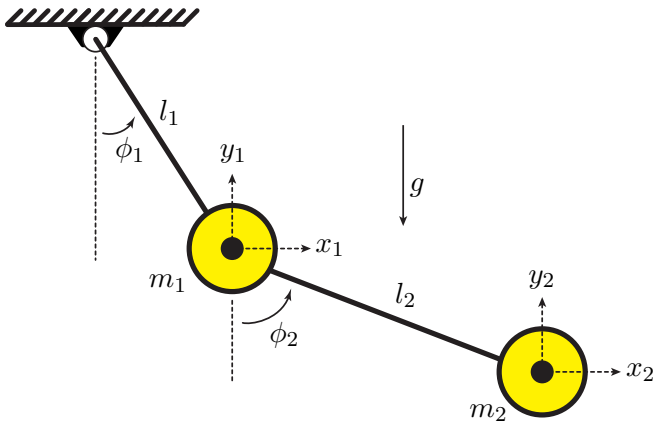


Figure 3.1: Double pendulum.

with $H(q,p) = \frac{1}{2}p^T M^{-1}(q)p + U(q)$ the total energy. The constrained Hamiltonian equations (3.8) define a port-Hamiltonian system, with respect to the modulated Dirac structure

$$\mathcal{D} = \Big\{(f_S, e_S, f_P, e_P) \mid 0 = A^T(q)e_S,\ e_P = B^T(q)e_S, \tag{3.9}$$
$$-f_S = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix} e_S + \begin{bmatrix} 0 \\ A(q) \end{bmatrix} \lambda + \begin{bmatrix} 0 \\ B(q) \end{bmatrix} f_P,\ \lambda \in \mathbb{R}^k \Big\}.$$

Example 3.2 (The double pendulum)**.** One way of modeling a double pendulum as depicted in Figure 3.1 (and mechanisms in general) is to regard the system as the interconnection of two single pendula.

Consider two ideal pendula with length $l_i, i = 1, 2$, in the vertical plane; described by the Cartesian coordinates $(x_i, y_i), i = 1, 2$, of their upper end, together with an angle (with respect to the vertical axis) $\phi_i, i = 1, 2$. For simplicity assume that the masses m_i of the pendula are concentrated at the lower ends of the pendula. Then the energies (kinetic and potential) of the pendula are given by

$$H_i(x_i, y_i, \phi_i, p_{x_i}, p_{y_i}, p_{\phi_i}) = \frac{1}{2m_i}p_{x_i}^2 + \frac{1}{2m_i}p_{y_i}^2 + \frac{1}{2I_i}p_{\phi_i}^2 + m_i g(y_i - l_i \cos\phi_i),$$

for $i = 1, 2$, where $p_{x_i} := m_i\dot{x}_i, p_{y_i} := m_i\dot{y}_i$ are the Cartesian momenta, and $p_{\phi_i} := I_i\dot{\phi}_i, I_i = m_i l_i^2$ are the angular momenta, $i = 1, 2$. The dou-

ble pendulum system is obtained by imposing the following *geometric constraints*

$x_1 = y_1 = 0$ (fixing the first pendulum at the top),

$x_2 = x_1 + l_1 \sin\phi_1, y_2 = y_1 - l_1 \cos\phi_1$ (attaching 2nd pendulum to 1st).

Differentiating these geometric constraints one obtains the *kinematic constraints*

$$\dot{x}_1 = \dot{y}_1 = 0,\ \dot{x}_2 = l_1\dot{\phi}_1 \cos\phi_1,\ \dot{y}_2 = l_1\dot{\phi}_1 \sin\phi_1.$$

The corresponding Dirac structure in this case is given as in (3.9), with $n = 4$, $B = 0$ and A given by

$$A^T(\phi_1) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & l_1\cos\phi_1 & -1 & 0 & 0 \\ 0 & 0 & l_1\sin\phi_1 & 0 & -1 & 0 \end{bmatrix}.$$

Furthermore, the total Hamiltonian H is given as $H_1 + H_2$.

Note that by differentiating the geometric constraints to kinematic constraints some information is lost (for example, $\dot{x}_1 = 0$ only implies $x_1 =$ constant instead of $x_1 = 0$). It turns out that the Casimirs of the Dirac structure $\mathcal{D}$ (see Chapter 8) still encode this loss of information; in fact x_1 is a Casimir of the Dirac structure.

Example 3.3 (Rolling euro)**.** Let x, y be the Cartesian coordinates of the point of contact of the coin with the plane; see Figure 3.2. Furthermore, φ denotes the heading angle, and θ the angle of King Willem-Alexander's head[2]. With all constants set to unity, the constrained Lagrangian equations of motion are

$$\begin{aligned} \ddot{x} &= \lambda_1, \\ \ddot{y} &= \lambda_2, \\ \ddot{\theta} &= -\lambda_1\cos\varphi - \lambda_2\sin\varphi + u_1, \\ \ddot{\varphi} &= u_2, \end{aligned} \tag{3.10}$$

with u_1 the control torque about the rolling axis, and u_2 the control torque about the vertical axis.

[2]On the Dutch version of the Euro.

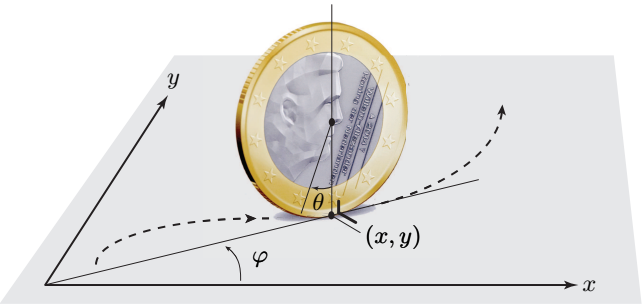


Figure 3.2: The geometry of the rolling Euro.

The total energy is $H = \frac{1}{2}p_x^2 + \frac{1}{2}p_y^2 + \frac{1}{2}p_\theta^2 + \frac{1}{2}p_\varphi^2$. The rolling constraints are $\dot{x} = \dot{\theta}\cos\varphi$ and $\dot{y} = \dot{\theta}\sin\varphi$, i.e., rolling without slipping, which can be written in the form (3.3) by defining

$$A^T(x, y, \theta, \phi) = \begin{bmatrix} 1 & 0 & -\cos\phi & 0 \\ 0 & 1 & -\sin\phi & 0 \end{bmatrix}. \tag{3.11}$$

While in the previous example of the double pendulum the kinematic constraints are derived from geometric constraints, this is *not* possible in the current example: the kinematic constraints $A^T(q)\dot{q} = 0$ for $A^T(q)$ given by (3.11) can*not* be integrated to geometric constraints $\Phi(q) = 0$. Such kinematic constraints are called *non-holonomic*.

The foregoing motivates to extend the definition of a *constant* Dirac structure $\mathcal{D} \subset \mathcal{F} \times \mathcal{E}$ (with $\mathcal{F}$ a linear space, and $\mathcal{E}$ its dual) to *Dirac structures on manifolds*.

Definition 3.1. Let $\mathcal{X}$ be a manifold. A Dirac structure $\mathcal{D}$ on $\mathcal{X}$ is a vector sub-bundle of the Whitney sum[3] $T\mathcal{X} \oplus T^*\mathcal{X}$ such that

$$\mathcal{D}(x) \subset T_x\mathcal{X} \times T_x^*\mathcal{X}$$

is for every $x \in \mathcal{X}$ a constant Dirac structure as before.

[3]The Whitney sum of two vector bundles with the same base space is defined as the vector bundle whose fiber above each element of this common base space is the product of the fibers of each individual vector bundle.

Simply put, a Dirac structure on a manifold $\mathcal{X}$ is point-wise (that is, for every $x \in \mathcal{X}$) a constant Dirac structure $\mathcal{D}(x) \subset T_x\mathcal{X} \times T_x^*\mathcal{X}$.

If, next to the energy storage port, there are additional ports (such as energy-dissipating and external ports) with total set of port variables $f \in \mathcal{F}$ and $e \in \mathcal{E} = \mathcal{F}^*$, then a modulated Dirac structure is point-wise (i.e., for every $x \in \mathcal{X}$) specified by a Dirac structure

$$\mathcal{D}(x) \subset T_x\mathcal{X} \times T_x^*\mathcal{X} \times \mathcal{F} \times \mathcal{E}. \tag{3.12}$$

Remark 3.1. For a full geometric definition of the above Dirac structure we refer to Dalsmo & van der Schaft (1999); Blankenstein & van der Schaft (2001), and especially to Merker (2009), where a formulation in terms of Courant algebroids is given.

3.2 Integrability

A key issue in the case of modulated Dirac structures is that of *integrability*. Loosely speaking, a Dirac structure is *integrable* if it is possible to find local coordinates for the state space manifold such that the Dirac structure expressed in these coordinates is a *constant* Dirac structure, that is, it is *not* modulated anymore by the state variables.

First let us consider modulated Dirac structures which are given for every $x \in \mathcal{X}$ as the *graph* of a skew-symmetric mapping $J(x)$ from the co-tangent space $T_x^*\mathcal{X}$ to the tangent space $T_x\mathcal{X}$.

Integrability in this case means that the structure matrix J satisfies the conditions

$$\sum_{l=1}^{n} \left[J_{lj}(x)\frac{\partial J_{ik}}{\partial x_l}(x) + J_{li}(x)\frac{\partial J_{kj}}{\partial x_l}(x) + J_{lk}(x)\frac{\partial J_{ji}}{\partial x_l}(x) \right] = 0, \tag{3.13}$$

for $i, j, k = 1, \ldots, n$. In this case we may find, by Darboux's theorem (see e.g. Weinstein (1983); Marsden & Ratiu (1999); Arnol'd (1978); Nijmeijer & van der Schaft (1990)) around any point x_0 where the rank of the matrix $J(x)$ is constant, local coordinates $x = (q, p, r)$ in which the matrix $J(x)$ becomes the constant skew-symmetric matrix

$$\begin{bmatrix} 0 & -I_k & 0 \\ I_k & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{3.14}$$

Such coordinates are called *canonical*. A skew-symmetric matrix $J(x)$ satisfying (3.13) defines a *Poisson bracket* on $\mathcal{X}$, given for every $F, G : \mathcal{X} \to \mathbb{R}$ as

$$\{F, G\}(x) = \frac{\partial^T F}{\partial x}(x) J(x) \frac{\partial G}{\partial x}(x). \tag{3.15}$$

Indeed, by (3.13) the Poisson bracket satisfies the *Jacobi-identity*

$$\{F, \{G, K\}\} + \{G, \{K, F\}\} + \{K, \{F, G\}\} = 0, \tag{3.16}$$

for all functions F, G, K. Conversely, satisfaction of (3.16) for all F, G, K implies (3.13). (Take $G = x_i, F = x_j$ and $K = x_k$.)

The choice of coordinates $x = (q, p, r)$ for the state space manifold also induces a basis for $T_x\mathcal{X}$ and a dual basis for $T_x^*\mathcal{X}$. Denoting the corresponding splitting for the flows by $f = (f_q, f_p, f_r)$ and for the efforts by $e = (e_q, e_p, e_r)$, the Dirac structure defined by J in canonical coordinates is seen to be given by

$$\mathcal{D} = \{(f_q, f_p, f_r, e_q, e_p, e_r) \mid f_q = -e_p, f_p = e_q, f_r = 0\}. \tag{3.17}$$

A similar story can be told for the case of a Dirac structure given as the graph of a skew-symmetric mapping $\omega(x)$ from the tangent space $T_x\mathcal{X}$ to the co-tangent space $T_x^*\mathcal{X}$. In this case the integrability conditions take the (slightly simpler) form

$$\frac{\partial \omega_{ij}}{\partial x_k}(x) + \frac{\partial \omega_{ki}}{\partial x_j}(x) + \frac{\partial \omega_{jk}}{\partial x_i}(x) = 0, \quad i, j, k = 1, \ldots, n. \tag{3.18}$$

The skew-symmetric matrix $\omega(x)$ can be regarded as the coordinate representation of a *differential two-form* ω on the manifold $\mathcal{X}$, that is $\omega = \sum_{i=1,j=1}^{n} dx_i \wedge dx_j$, and the integrability condition (3.18) corresponds to the *closedness* of this two-form ($d\omega = 0$). The differential two-form ω is called a *pre-symplectic structure*, and a *symplectic structure* if the rank of $\omega(x)$ is equal to the dimension of $\mathcal{X}$. If (3.18) holds, then again by a version of Darboux's theorem we may find, around any point x_0 where the rank of the matrix $\omega(x)$ is constant, local coordinates $x = (q, p, s)$ in which the matrix $\omega(x)$ becomes the constant skew-symmetric matrix

$$\begin{bmatrix} 0 & I_k & 0 \\ -I_k & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{3.19}$$

The choice of canonical coordinates $x = (q, p, s)$ induces a basis for $T_x\mathcal{X}$ and a dual basis for $T_x^*\mathcal{X}$. Denoting the corresponding splitting for the flows by $f = (f_q, f_p, f_s)$ and for the efforts by $e = (e_q, e_p, e_s)$, the Dirac structure corresponding to ω in canonical coordinates is seen to be given by

$$\mathcal{D} = \{(f_q, f_p, f_s, e_q, e_p, e_s) \mid f_q = -e_p, f_p = e_q, e_s = 0\}. \tag{3.20}$$

In case of a symplectic structure the variables s are absent and the Dirac structure reduces to

$$\mathcal{D} = \{(f_q, f_p, e_q, e_p) \mid f_q = -e_p, f_p = e_q\}, \tag{3.21}$$

which is the standard *symplectic gyrator*.

For general Dirac structures, integrability is defined in the following way.

Definition 3.2. Dorfman (1993); Courant (1990) A Dirac structure $\mathcal{D}$ on $\mathcal{X}$ is *integrable* if for arbitrary pairs of smooth vector fields and differential one-forms $(X_1, \alpha_1), (X_2, \alpha_2), (X_3, \alpha_3) \in \mathcal{D}$ there holds

$$< L_{X_1}\alpha_2 \mid X_3 > + < L_{X_2}\alpha_3 \mid X_1 > + < L_{X_3}\alpha_1 \mid X_2 >= 0, \tag{3.22}$$

with L_{X_i} denoting the Lie-derivative.

Remark 3.2 (Pseudo-Dirac structures)**.** In the usual definition of Dirac structures on manifolds (see Courant (1990); Dorfman (1993)), the *integrability* condition is *included* in the definition. Dirac structures that do *not* satisfy this integrability condition are therefore sometimes (but not in this chapter) called *pseudo*-Dirac structures.

The above integrability condition for Dirac structures generalizes properly the closedness of symplectic forms and the Jacobi identity for Poisson brackets as discussed before. In particular, for Dirac structures given as the graph of a symplectic or Poisson structure, the notion of integrability is equivalent to the Jacobi-identity or closedness condition as discussed above (see e.g. Courant (1990); Dorfman (1993); Dalsmo & van der Schaft (1999) for details).

Note that a *constant* Dirac structure trivially satisfies the integrability condition. Conversely, a Dirac structure satisfying the integrability

condition together with an additional constant rank condition can be represented *locally* as a *constant* Dirac structure. The precise form of the constant rank condition can be stated as follows. For any Dirac structure $\mathcal{D}$, we may define the distribution

$$G_{\mathcal{D}}(x) = \{X \in T_x\mathcal{X} \mid \exists \alpha \in T_x^*\mathcal{X} \text{ s.t. } (X, \alpha) \in D(x)\}.$$

Dually we may define the co-distribution

$$P_{\mathcal{D}}(x) = \{\alpha \in T_x^*\mathcal{X} \mid \exists X \in T_x\mathcal{X} \text{ s.t. } (X, \alpha) \in D(x)\}.$$

We call x_0 a *regular* point for the Dirac structure if both the distribution $G_{\mathcal{D}}$ and the co-distribution $P_{\mathcal{D}}$ have constant dimension around x_0.

If the Dirac structure is integrable and x_0 is a regular point, then, again by a version of Darboux's theorem, we can choose local coordinates $x = (q, p, r, s)$ for $\mathcal{X}$ (with $\dim q = \dim p$), such that, in the resulting bases for (f_q, f_p, f_r, f_s) for $T_x\mathcal{X}$ and (e_q, e_p, e_r, e_s) for $T_x^*\mathcal{X}$, the Dirac structure on this coordinate neighborhood is given as

$$\begin{cases} f_q = -e_p, \ f_p = e_q \\ f_r = 0, \quad\ e_s = 0. \end{cases} \tag{3.23}$$

Coordinates $x = (q, p, r, s)$ as above are again called *canonical*. Note that the choice of canonical coordinates for a Dirac structure satisfying the integrability condition encompasses the choice of canonical coordinates for a Poisson structure and for a (pre-)symplectic structure as above.

Example 3.4 (Kinematic constraints)**.** Recall that the modulated Dirac structure corresponding to an actuated mechanical system subject to kinematic constraints $A^T(q)\dot{q} = 0$ is given by

$$\begin{aligned} \mathcal{D} = \Big\{ (f_S, e_S, f_C, e_C) \mid 0 = \begin{bmatrix} 0 & A^T(q) \end{bmatrix} e_S, \ e_C = \begin{bmatrix} 0 & B^T(q) \end{bmatrix} e_S, \\ -f_S = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix} e_S + \begin{bmatrix} 0 \\ A(q) \end{bmatrix} \lambda + \begin{bmatrix} 0 \\ B(q) \end{bmatrix} f_c, \ \lambda \in \mathbb{R}^k \Big\}. \end{aligned}$$

Complete necessary and sufficient conditions for integrability of this Dirac structure have been derived in van der Schaft & Maschke

(1994); Dalsmo & van der Schaft (1999). Here we only state a slightly simplified version of this result, detailed in Dalsmo & van der Schaft (1999). We assume that the actuation matrix $B(q)$ has the special form (often encountered in examples) that its j-th column is given as

$$\begin{bmatrix} 0 \\ \dfrac{\partial C_j}{\partial q}(q) \end{bmatrix}.$$

for some function $C_j(q)$ only depending on the configuration variables q, $j = 1, \ldots, m$. In this case, the Dirac structure $\mathcal{D}$ is integrable *if and only if the kinematic constraints are holonomic*. Thus the Dirac structure corresponding to the Double pendulum example is integrable, while the Dirac structure corresponding to the Rolling euro example is not integrable.

4

Input-state-output port-Hamiltonian systems

An important subclass of port-Hamiltonian systems occurs if (1) there are no algebraic constraints between the state variables, (2) the external port variables can be split into input and output variables, and (3) the resistive structure is linear and of input-output form. This class of systems, in the usual input-state-output format $\dot{x} = f(x, u)$, $y = h(x, u)$, also provides a natural starting point for the development of control strategies; see Chapter 15.

In the present chapter we will explore the special properties of this subclass of port-Hamiltonian systems, while in the last section we discuss the relationships with classical Hamiltonian dynamical systems.

4.1 Linear resistive structures

In quite a few cases of interest the energy-dissipating elements can be assumed to be *linear*. This allows for a geometric interpretation, which is especially interesting when combined with the geometric notion of a Dirac structure. Linear energy-dissipation in the port-variables f_R, e_R can be modeled as a subspace (possibly modulated by the state vari-

ables $x \in \mathcal{X}$)

$$\mathcal{R}(x) = \{(f_R, e_R) \in \mathcal{F}_R \times \mathcal{E}_\mathcal{R} \mid R_f(x) f_R + R_e(x) e_R = 0\}, \tag{4.1}$$

with $\mathcal{E}_R = \mathcal{F}_R^*$, satisfying the energy-dissipation property

$$R_f(x) R_e^T(x) = R_e(x) R_f^T(x) \geq 0, \tag{4.2}$$

together with the dimensionality condition

$$\operatorname{rank} \begin{bmatrix} R_f(x) & R_e(x) \end{bmatrix} = \dim f_R, \quad x \in \mathcal{X}. \tag{4.3}$$

Indeed, by (4.3) and the equality in (4.2) we can equivalently rewrite the kernel representation (4.1) as an image representation

$$f_R = R_e^T(x)\lambda, \quad e_R = -R_f^T(x)\lambda. \tag{4.4}$$

That is, any pair (f_R, e_R) satisfying (4.1) also satisfies (4.4) for some λ, and conversely, every (f_R, e_R) satisfying (4.4) for some λ also satisfies (4.1). Hence by (4.2) for all (f_R, e_R) satisfying (4.1)

$$e_R^T f_R = -\left(R_f^T(x)\lambda\right)^T R_e^T(x)\lambda = -\lambda^T R_f(x) R_e^T(x)\lambda \leq 0,$$

thus defining a true energy-dissipating relation. We will call $\mathcal{R}$ a *linear resistive structure*.

A linear resistive structure $\mathcal{R}$ defined on the state space manifold $\mathcal{X}$ can be regarded as a geometric object having properties which are somewhat opposite to those of a Dirac structure. Recall that on any space $\mathcal{F} \times \mathcal{E}$, with $\mathcal{E} = \mathcal{F}^*$, we can define the indefinite form, cf. (2.4), $\ll (f^a, e^a), (f^b, e^b) \gg = < e^a \mid f^b > + < e^b \mid f^a >$, with $^{\perp\!\!\!\perp}$ denoting orthogonal companion.

Proposition 4.1. Let $\mathcal{R}(x) \subset \mathcal{F}_R \times \mathcal{E}_R$ defined by (4.1) be a linear resistive structure on $\mathcal{X}$, that is, satisfying (4.2) and (4.3). Then $\mathcal{R}(x)^{\perp\!\!\!\perp} = \{(f_R, e_R) \in \mathcal{F}_R \times \mathcal{E}_R \mid R_f(x) f_R - R_e(x) e_R = 0\} =: (-\mathcal{R})(x)$.

We leave the proof as an exercise. As a direct result we obtain

Proposition 4.2. Let $\mathcal{R}(x) \subset \mathcal{F}_R \times \mathcal{E}_R$ be a linear resistive structure on $\mathcal{X}$ and let $\mathcal{D}(x) \subset T_x\mathcal{X} \times T_x^*\mathcal{X} \times \mathcal{F}_R \times \mathcal{E}_R \times \mathcal{F}_P \times \mathcal{E}_P$ be a Dirac structure

on $\mathcal{X}$. Define the composition $\mathcal{D}(x) \circ \mathcal{R}(x) \subset T_x\mathcal{X} \times T_x^*\mathcal{X} \times \mathcal{F}_P \times \mathcal{E}_P$ as

$$\mathcal{D}(x) \circ \mathcal{R}(x) = \Big\{(f_S, e_S, f_P, e_P) \mid \exists (f_R, e_R) \in \mathcal{R}(x) \text{ s.t.} \\ (f_S, e_S, f_R, e_R, f_P, e_P) \in \mathcal{D}(x)\Big\}.$$

Then,

$$(\mathcal{D}(x) \circ \mathcal{R}(x))^{\perp\!\!\!\perp} = \mathcal{D}(x) \circ (-\mathcal{R})\,(x).$$

In the next section we will furthermore use the following simple result.

Proposition 4.3. Consider a Dirac structure $\mathcal{D}(x) \subset T_x\mathcal{X} \times T_x^*\mathcal{X} \times \mathcal{F}_R \times \mathcal{E}_R \times \mathcal{F}_P \times \mathcal{E}_P$ and a linear resistive structure $\mathcal{R}(x) \subset \mathcal{F}_R \times \mathcal{E}_R$. Suppose that the composition $\mathcal{D}(x) \circ \mathcal{R}(x) \subset T_x\mathcal{X} \times T_x^*\mathcal{X} \times \mathcal{F}_P \times \mathcal{E}_P$ can be written as the graph of a linear mapping from $T_x^*\mathcal{X} \times \mathcal{E}_P$ to $T_x\mathcal{X} \times \mathcal{F}_P$ given as

$$\begin{bmatrix} e_S \\ e_P \end{bmatrix} \mapsto \begin{bmatrix} f_S \\ f_P \end{bmatrix} = K(x) \begin{bmatrix} e_S \\ e_P \end{bmatrix}. \tag{4.5}$$

Then, by factorizing $K(x)$ into its skew-symmetric part $K_{ss}(x)$ and its symmetric part $K_s(x)$ we have

$$K(x) = K_{ss}(x) + K_s(x),\ K_{ss}^T(x) = -K_{ss}(x), K_s^T(x) = K_s(x) \geq 0. \tag{4.6}$$

Proof. For all $(f_S, e_S, f_P, e_P) \in \mathcal{D}(x) \circ \mathcal{R}(x)$ there exists $(f_R, e_R) \in \mathcal{R}(x)$ such that $(f_S, e_S, f_R, e_R, f_P, e_P) \in \mathcal{D}(x)$. Hence

$$\begin{bmatrix} e_S^T & e_P^T \end{bmatrix} (x) K_s(x) \begin{bmatrix} e_S \\ e_P \end{bmatrix} = \begin{bmatrix} e_S^T & e_P^T \end{bmatrix} K(x) \begin{bmatrix} e_S \\ e_P \end{bmatrix} = \\ e_S^T f_S + e_P^T f_P = -e_R^T f_R \geq 0,$$

implying $K_s(x) \geq 0$. ∎

4.2 Input-state-output port-Hamiltonian systems

Consider now a port-Hamiltonian system where the composition of the Dirac structure $\mathcal{D}$ and the linear resistive structure $\mathcal{R}$ satisfies the

conditions of Proposition 4.3, and thus is given as the graph of a mapping (4.5) satisfying (4.6). Write out correspondingly

$$K_{ss}(x) = \begin{bmatrix} -J(x) & -g(x) \\ g^T(x) & M(x) \end{bmatrix}, \quad K_s(x) = \begin{bmatrix} R(x) & P(x) \\ P^T(x) & S(x) \end{bmatrix}, \tag{4.7}$$

where $J^T(x) = -J(x), M^T(x) = -M(x)$ and $R^T(x) = R(x), S^T(x) = S(x)$. Then it follows from Proposition 4.3 that the matrices $R(x), P(x), S(x)$ satisfy

$$\begin{bmatrix} R(x) & P(x) \\ P^T(x) & S(x) \end{bmatrix} \geq 0. \tag{4.8}$$

Denoting $u := e_P$ and $y := f_P$ it follows that

$$e_S^T f_S + u^T y = \begin{bmatrix} e_S^T & u^T \end{bmatrix} \begin{bmatrix} f_S \\ y \end{bmatrix} = \begin{bmatrix} e_S^T & u^T \end{bmatrix} \begin{bmatrix} R(x) & P(x) \\ P^T(x) & S(x) \end{bmatrix} \begin{bmatrix} e_S \\ u \end{bmatrix} \geq 0.$$

Hence, together with the energy-storage relations $e_S = \frac{\partial H}{\partial x}(x), f_S = -\dot{x}$, we obtain the port-Hamiltonian system

$$\begin{aligned} \dot{x} &= [J(x) - R(x)] \frac{\partial H}{\partial x}(x) + [g(x) - P(x)] u, \\ y &= \left[g^T(x) + P(x)\right] \frac{\partial H}{\partial x}(x) + [M(x) + S(x)] u, \end{aligned} \tag{4.9}$$

called an *input-state-output port-Hamiltonian system with feedthrough term*. Along the trajectories of the system we recover the fundamental power-balance

$$\frac{d}{dt} H(x) = -e_S^T f_S = u^T y - \begin{bmatrix} e_S^T & u^T \end{bmatrix} \begin{bmatrix} R(x) & P(x) \\ P^T(x) & S(x) \end{bmatrix} \begin{bmatrix} e_S \\ u \end{bmatrix} \leq u^T y.$$

Example 4.1. In case of feedthrough terms the skew-symmetric matrix J may also depend on the parameters of energy-dissipation, as the following example shows. Consider the linear electrical circuit depicted in Figure 4.1. The dynamics of the circuit takes the port-

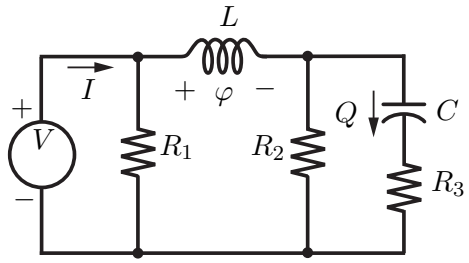


Figure 4.1: Circuit for Example 4.1.

Hamiltonian form

$$\begin{bmatrix} \dot{\varphi} \\ \dot{Q} \end{bmatrix} = \begin{bmatrix} -\dfrac{R_2 R_3}{R_2+R_3} & -\dfrac{R_3}{R_2+R_3} \\ \dfrac{R_3}{R_2+R_3} & -\dfrac{1}{R_2+R_3} \end{bmatrix} \begin{bmatrix} \dfrac{\partial H}{\partial \varphi} \\ \dfrac{\partial H}{\partial Q} \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} V$$
$$I = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} \dfrac{\partial H}{\partial \varphi} \\ \dfrac{\partial H}{\partial Q} \end{bmatrix} + \frac{V}{R_1},$$

with $H(\varphi, Q) = \frac{1}{2L}\varphi^2 + \frac{1}{2C}Q^2$. This defines a port-Hamiltonian input-state-output system with feedthrough specified by

$$J = \begin{bmatrix} 0 & -\dfrac{R_3}{R_2+R_3} \\ \dfrac{R_3}{R_2+R_3} & 0 \end{bmatrix}, \; R = \begin{bmatrix} \dfrac{R_2 R_3}{R_2+R_3} & 0 \\ 0 & \dfrac{1}{R_2+R_3} \end{bmatrix},$$

$M = 0$, $P = 0$, and $S = 1/R_1$.

In the special case $S(x) = 0$, implying by (4.8) $P(x) = 0$, and additionally assuming $M(x) = 0$, we obtain from (4.9) the system description

$$\begin{aligned} \dot{x} &= [J(x) - R(x)] \frac{\partial H}{\partial x}(x) + g(x)u, \\ y &= g^T(x) \frac{\partial H}{\partial x}(x). \end{aligned} \tag{4.10}$$

This is simply called an *input-state-output port-Hamiltonian system*.

Alternatively, (4.10) can be obtained by first considering the port-Hamiltonian system without energy dissipation

$$\begin{aligned}\dot{x} &= J(x)\frac{\partial H}{\partial x}(x) + g(x)u + g_R(x)f_R,\\ y &= g^T(x)\frac{\partial H}{\partial x}(x),\\ e_R &= g_R^T(x)\frac{\partial H}{\partial x}(x),\end{aligned}$$

with the (open) port f_R, e_R, and then to terminate this port by the energy-dissipation relation $f_R = -\tilde{R}(x)e_R$ where $\tilde{R}^T(x) = \tilde{R}(x) \geq 0$. This yields the input-state-output port-Hamiltonian system (4.10) with $R(x) = g_R(x)\tilde{R}(x)g_R^T(x)$. For details we refer to van der Schaft (2009).

4.3 Memristive dissipation

Another type of resistive relationship is given by the memristor. The memristor, a contraction of memory and resistance that refers to a resistor with memory, was postulated in the early seventies Chua (1971) to complete the family of existing fundamental electrical circuit elements: the resistors, inductor, and capacitor.[1] In the port-Hamiltonian framework, a memristive port is described as follows. Let $f_M \in \mathcal{F}_M$ and $e_M \in \mathcal{E}_M$, with $\mathcal{E}_M = \mathcal{F}_M^*$, denote the flows and efforts associated to the memristive port, and let $x_{f_M} \in \mathcal{X}_M$ and $x_{e_M} \in \mathcal{X}_M^*$ the corresponding time-integrals of f_M and e_M, respectively. Then, the relationship $x_{e_M} = -\Phi_M(x_{f_M})$, with Φ_M some differentiable mapping from $\mathcal{X}_M$ to $\mathcal{X}_M^*$, constitutes a x_{f_M}–controlled memristive port

$$e_M = -R_M(x_{f_M})f_M,\ R_M(x_{f_M}) = \frac{\partial \Phi_M}{\partial x_{f_M}}(x_{f_M}),$$

[1] From a mathematical perspective, the behavior of a resistor, inductor, and a capacitor, whether linear or nonlinear, is described by relationships between two of the four basic electrical variables: voltage, current, charge, and flux linkage. Indeed, a resistor is described by the relationship of current and voltage; an inductor by that of current and flux linkage, and a capacitor by that of voltage and charge. But what about the relationship between charge and flux linkage? This missing relationship defines the memristor.

and we define the associated memristive structure as

$$\mathcal{M} = \Big\{ (f_M, e_M) \in \mathcal{F}_M \times \mathcal{E}_M \;\Big|\; \dot{x}_{f_M} - f_M = 0, \\ e_M + R_M(x_{f_M}) f_M = 0 \Big\}.$$

Note that the memory effect stems from the fact that the memristor 'remembers' the amount of flow that has passed through it via $\dot{x}_{f_M} = f_M$.

Now, locally around $x_{f_M} \in \mathcal{X}_M$, the memristive structure $\mathcal{M}$ defines a port-Hamiltonian system with a direct feedthrough term. Indeed, let $H_M : \mathcal{X}_M \to \mathbb{R}$ be the *zero* function, then the dynamics on $\mathcal{M}$ locally take the from

$$\begin{aligned} \dot{x}_{f_M} &= f_M, \\ e_M &= \frac{\partial H_M}{\partial x_{f_M}}(x_{f_M}) - R_M(x_{f_M}) f_M. \end{aligned} \tag{4.11}$$

The fact that $H_M(x_{f_M}) = 0$, for all $x_{f_M} \in \mathcal{X}_M$, together with the fact that $e_M \equiv 0$ whenever $f_M \equiv 0$ regardless of the internal state x_{f_M}, clearly underscores the 'no energy discharge property' as discussed in Chua (1971). A dual representation can be obtained starting from the x_{e_M}–controlled relationship $x_{f_M} = -\Phi^*_M(x_{e_M})$.

The concept of the memristor and its generalizations can be useful in modelling a wide variety of phenomena, including thermistors, Josephson junctions, discharge tubes, and even ionic systems like the Hodgkin-Huxley model of a neuron; see Jeltsema & van der Schaft (2010) and Jeltsema & Doria (2012) for a further discussion, some illustrative examples, and the inclusion of so-called meminductors and memcapacitors, the memory equivalents of inductors and capacitors, in the port-Hamiltonian framework.

4.4 Relation with classical Hamiltonian systems

In this section we recall the classical framework of Lagrangian and Hamiltonian differential equations as originating from analytical mechanics, and indicate how it naturally extends to input-state-output port-Hamiltonian systems as dealt with in the previous section.

Recall that the *Euler-Lagrange equations*, see e.g. Arnol'd (1978); Marsden & Ratiu (1999); Abraham & Marsden (1994), are given as

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}(q,\dot{q})\right) - \frac{\partial L}{\partial q}(q,\dot{q}) = \tau, \tag{4.12}$$

where $q = (q_1, \ldots, q_k)^T$ are generalized configuration coordinates for the system with k degrees of freedom, the Lagrangian L equals the *difference* $K - P$ between kinetic energy[2] $K(q,\dot{q})$ and potential energy $P(q)$, and $\tau = (\tau_1, \ldots, \tau_k)^T$ is the vector of generalized forces acting on the system. Furthermore, $\frac{\partial L}{\partial \dot{q}}$ denotes the column-vector of partial derivatives of $L(q,\dot{q})$ with respect to the generalized velocities $\dot{q}_1, \ldots, \dot{q}_k$, and similarly for $\frac{\partial L}{\partial q}$. In standard mechanical systems the kinetic energy K is of the form

$$K(q,\dot{q}) = \frac{1}{2}\dot{q}^T M(q)\dot{q}, \tag{4.13}$$

where the $k \times k$ inertia (generalized mass) matrix $M(q)$ is symmetric and positive definite for all q. In this case the vector of generalized *momenta* $p = (p_1, \ldots, p_k)^T$, defined for any Lagrangian L as $p = \frac{\partial L}{\partial \dot{q}}$, is simply given by

$$p = M(q)\dot{q}, \tag{4.14}$$

and by defining the state vector $(q_1, \ldots, q_k, p_1, \ldots, p_k)^T$ the k second-order equations (4.12) transform into $2k$ first-order equations

$$\begin{aligned} \dot{q} &= \frac{\partial H}{\partial p}(q,p) \qquad (= M^{-1}(q)p), \\ \dot{p} &= -\frac{\partial H}{\partial q}(q,p) + \tau, \end{aligned} \tag{4.15}$$

where

$$H(q,p) = \frac{1}{2}p^T M^{-1}(q)p + P(q) \quad (= \frac{1}{2}\dot{q}^T M(q)\dot{q} + P(q)) \tag{4.16}$$

is the total energy of the system. The equations (4.15) are called the *Hamiltonian equations* of motion, and H is called the *Hamiltonian*. The

[2]Strictly speaking, $K(q,\dot{q})$ is the kinetic *co-energy*. However, whenever the kinetic co-energy is *quadratic* as in (4.13) then its value is equal to the value of the corresponding true kinetic energy $\frac{1}{2}p^T M^{-1}(q)p$.

state space of (4.16) with local coordinates (q, p) is usually called the *phase space*.

The following *power-balance* immediately follows from (4.15):

$$\frac{d}{dt}H = \frac{\partial^T H}{\partial q}(q,p)\dot{q} + \frac{\partial^T H}{\partial p}(q,p)\dot{p} = \frac{\partial^T H}{\partial p}(q,p)\tau = \dot{q}^T\tau, \qquad (4.17)$$

expressing that the increase in energy of the system is equal to the supplied work (*conservation of energy*). Hence by defining the *input* to be $u = \tau$ and the *output* to be $y := \dot{q}$ we obtain $\frac{d}{dt}H = y^T u$. In particular, if the Hamiltonian $H(q, p)$ is assumed to be the sum of a positive kinetic energy and a potential energy which is nonnegative, then it follows that the system (4.15) with inputs $u = \tau$ and outputs $y := \dot{q}$ is *passive* (in fact, *lossless*) with storage function $H(q, p)$.

System (4.15) with inputs $u = \tau$ and outputs $y := \dot{q}$ is an example of a Hamiltonian system with *collocated* inputs and outputs, which more generally is given in the following form

$$\begin{aligned} \dot{q} &= \frac{\partial H}{\partial p}(q,p), \\ \dot{p} &= -\frac{\partial H}{\partial q}(q,p) + B(q)u, \quad u \in \mathbb{R}^m, \\ y &= B^T(q)\frac{\partial H}{\partial p}(q,p) \quad (= B^T(q)\dot{q}),\ y \in \mathbb{R}^m, \end{aligned} \qquad (4.18)$$

where $q = (q_1, \ldots, q_k)^T$ and $p = (p_1, \ldots, p_k)^T$, and $B(q)$ is the input force matrix, with $B(q)u$ denoting the generalized forces resulting from the control inputs $u \in \mathbb{R}^m$. (In case $m < k$ we speak of an *underactuated* system. If $m = k$ and the matrix $B(q)$ is invertible for all q, then the system is fully actuated.) Again we obtain the energy balance

$$\frac{dH}{dt}(q(t), p(t)) = u^T(t)y(t). \qquad (4.19)$$

A further generalization of the class of Hamiltonian systems (4.18) with collocated inputs and outputs consists in considering systems which are described in local coordinates as

$$\begin{aligned} \dot{x} &= J(x)\frac{\partial H}{\partial x}(x) + g(x)u, \quad x \in \mathcal{X},\ u \in \mathbb{R}^m \\ y &= g^T(x)\frac{\partial H}{\partial x}(x), \quad y \in \mathbb{R}^m, \end{aligned} \qquad (4.20)$$

where $J(x)$ is an $n \times n$ matrix with entries depending smoothly on x, which is assumed to be *skew-symmetric*

$$J(x) = -J^T(x). \tag{4.21}$$

Indeed, because of (4.21) the energy-balance $\frac{dH}{dt}(x(t)) = y^T(t)u(t)$ continues to hold. The system (4.20) with J satisfying (4.21) is an input-state-output port-Hamiltonian system with Dirac structure determined by $J(x)$ and $g(x)$, with Hamiltonian H, and with zero resistive structure. Note that (4.18) (and hence (4.15)) is a particular case of (4.20) with $x = (q, p)$, and $J(x)$ being given by the constant skew-symmetric matrix $J = \begin{bmatrix} 0 & I_k \\ -I_k & 0 \end{bmatrix}$, and $g(q,p) = \begin{bmatrix} 0 \\ B(q) \end{bmatrix}$.

Finally adding linear energy-dissipation will then lead to the input-state-output port-Hamiltonian systems as defined in the beginning of this chapter.

We note that the generalization of classical Hamiltonian systems (4.15) to systems

$$\dot{x} = J(x)\frac{\partial H}{\partial x}(x),$$

with $J(x)$ satisfying (4.21) is common in geometric mechanics; see e.g. Arnol'd (1978); Abraham & Marsden (1994); Marsden & Ratiu (1999). In fact, in many situations the formulation $\dot{x} = J(x)\frac{\partial H}{\partial x}(x)$ can be inferred from the classical Hamiltonian formulation by *symmetry* considerations. The most classical example of this are the Euler equations for the angular momenta of a spinning rigid body, cf. eqn. (3.1), which is derivable from the classical 6-dimensional Hamiltonian equations for the motion of a spinning rigid body by symmetry of the Hamiltonian (equal to the kinetic energy) under the action of the matrix group $SO(3)$. In all these cases, the matrix $J(x)$ will satisfy, on top of its skew-symmetry property, an *integrability condition* guaranteeing the existence of *canonical coordinates*, cf. Chapter 3. Note however that these integrability conditions are *not* essential for the definition of input-state-output port-Hamiltonian systems.

5

Representations of Dirac structures

A fundamental concept in the geometric, coordinate-free, definition of a port-Hamiltonian system in Chapter 2 is the geometric notion of a Dirac structure. For many purposes, including simulation and control, it is useful to obtain *coordinate representations* of port-Hamiltonian systems, and to be able to convert one type of representation into another. The key for doing so is to study coordinate representations of Dirac structures. Specifically, once a basis for the space of the flows and the dual space of efforts is taken, it is possible to give several matrix representations of a Dirac structure. Once a specific representation of the Dirac structure has been obtained the coordinate representation of the corresponding port-Hamiltonian system, generally in the form of a set of *differential-algebraic equations*, follows.

In the next sections, we discuss a number of representations of Dirac structures and the resulting form of port-Hamiltonian systems. For the corresponding proofs and further information, the reader is referred to van der Schaft (2009). The chapter is concluded by representing a Dirac structure in terms of the (pure) spinor formalism stemming from exterior algebra.

5.1 Kernel and image representations

Every Dirac structure $\mathcal{D} \subset \mathcal{F} \times \mathcal{E}$, with $\mathcal{E} = \mathcal{F}^*$, can be represented in *kernel* representation as

$$\mathcal{D} = \{(f, e) \in \mathcal{F} \times \mathcal{E} \mid Ff + Ee = 0\}$$

for linear maps $F : \mathcal{F} \to \mathcal{V}$ and $E : \mathcal{E} \to \mathcal{V}$ satisfying

$$\begin{array}{ll} (i) & EF^* + FE^* = 0, \\ (ii) & \mathrm{rank}(F + E) = \dim \mathcal{F}, \end{array} \tag{5.1}$$

where $\mathcal{V}$ is a linear space with the same dimension as $\mathcal{F}$, and where $F^* : \mathcal{V}^* \to \mathcal{E}$ and $E^* : \mathcal{V}^* \to (\mathcal{F}^*)^* = \mathcal{F}$ are the adjoint maps of F and E, respectively.

It follows from (5.1) that $\mathcal{D}$ can be also written in *image* representation as

$$\mathcal{D} = \{(f, e) \in \mathcal{F} \times \mathcal{E} \mid f = E^*\lambda,\ e = F^*\lambda,\ \lambda \in \mathcal{V}^*\}.$$

Sometimes it will be useful to relax the requirements on the linear mappings F and E by allowing $\mathcal{V}$ to be a linear space of dimension greater than the dimension of $\mathcal{F}$. In this case we shall speak of *relaxed* kernel and image representations.

Matrix kernel and image representations are obtained by choosing linear coordinates for $\mathcal{F}$, $\mathcal{E}$ and $\mathcal{V}$. Indeed, take any basis $f_1, \cdots, f_n$ for $\mathcal{F}$ and the *dual* basis $e_1 = f_1^*, \cdots, e_n = f_n^*$ for $\mathcal{E} = \mathcal{F}^*$, where dim $\mathcal{F} = n$. Furthermore, take any set of linear coordinates for $\mathcal{V}$. Then the linear maps F and E are represented by $n \times n$ matrices F and E satisfying

$$\begin{array}{ll} (i) & EF^T + FE^T = 0, \\ (ii) & \mathrm{rank}[F \mid E] = \dim \mathcal{F}. \end{array}$$

In the case of a relaxed kernel and image representation F and E will be $n' \times n$ matrices with $n' \geq n$.

5.2 Constrained input-output representation

Every Dirac structure $\mathcal{D} \subset \mathcal{F} \times \mathcal{E}$ can be represented as

$$\mathcal{D} = \{(f, e) \in \mathcal{F} \times \mathcal{E} \mid f = Je + G\lambda,\ G^T e = 0\}, \tag{5.2}$$

for a skew-symmetric mapping $J : \mathcal{F} \to \mathcal{E}$ and a linear mapping G, such that

$$\begin{aligned} \operatorname{im} G &= \{f \in \mathcal{F} \mid (f, 0) \in \mathcal{D}\}, \\ \ker J &= \{e \in \mathcal{E} \mid (0, e) \in \mathcal{D}\}. \end{aligned}$$

Conversely, for every G and skew-symmetric J equation (5.2) defines a Dirac structure.

We have already encountered constrained input-output representations in the case of electrical circuits and mechanical systems with kinematic constraints in Section 2.2.

5.3 Hybrid input-output representation

As we have seen, the graph of a skew-symmetric map from $\mathcal{F}$ to $\mathcal{E}$, or from $\mathcal{E}$ to $\mathcal{F}$, is a Dirac structure, but not every Dirac structure can be represented this way. On the other hand, by exchanging part of the flow variables with effort variables, any Dirac structure *can* be represented as the graph of a mapping. Specifically, let $\mathcal{D}$ be given in matrix kernel representation by square matrices E and F as in (5.1). Suppose rank $F = m$ $(\leq n)$. Select m independent columns of F, and group them into a matrix F_1. Write (possibly after permutations) $F = [F_1 \mid F_2]$, and correspondingly $E = [E_1 \mid E_2]$,

$$f = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}, \text{ and } e = \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}.$$

Then, cf. Bloch & Crouch (1999), the matrix $[F_1 \mid E_2]$ is invertible, and

$$\mathcal{D} = \left\{ \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} \in \mathcal{F}, \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} \in \mathcal{E} \;\middle|\; \begin{bmatrix} f_1 \\ e_2 \end{bmatrix} = J \begin{bmatrix} e_1 \\ f_2 \end{bmatrix} \right\},$$

with $J := -[F_1 \mid E_2]^{-1}[F_2 \mid E_1]$ skew-symmetric.

It follows that any Dirac structure can be written as the graph of a skew-symmetric map. The vectors e_1 and f_2 can be regarded as input vectors, while the complementary vectors f_1 and e_2 can be seen as output vectors[1].

[1]The hybrid input-output representation of a Dirac structure is similar to the

5.4 Canonical coordinate representation

Consider a constant Dirac structure $\mathcal{D} \subset \mathcal{F} \times \mathcal{E}$. Then, cf. Courant (1990), there exist coordinates (q, p, r, s) for $\mathcal{F}$ and corresponding dual coordinates for $\mathcal{E}$ such that (f, e), when partitioned as $(f_q, f_p, f_r, f_s, e_q, e_p, e_r, e_s)$, is contained in $\mathcal{D}$ if and only if

$$\begin{aligned} f_q &= -e_p, \\ f_p &= e_q, \\ f_r &= 0, \\ e_s &= 0. \end{aligned}$$

For a non-constant Dirac structure on a manifold (cf. Chapter 3) it is possible to construct such canonical coordinates locally around a regular point of the Dirac structure if and only if the Dirac structure is *integrable*; see Chapter 3 and Courant (1990) for details.

The representation of a Dirac structure by canonical coordinates is very close to the classical Hamiltonian equations of motion, as recalled in Section 4.4. Indeed, for a system without energy-dissipating relations and without external ports, the dynamics with respect to the canonical coordinate representation of a Dirac structure and an arbitrary Hamiltonian $H(q, p, r, s)$ takes the form

$$\begin{aligned} \dot{q} &= \frac{\partial H}{\partial p}(q, p, r, s), \\ \dot{p} &= -\frac{\partial H}{\partial q}(q, p, r, s), \\ \dot{r} &= 0, \\ 0 &= \frac{\partial H}{\partial s}(q, p, r, s), \end{aligned}$$

where the third line of equations correspond to conserved quantities (any function of r is conserved), and the fourth line of equations represent the algebraic constraints which are present in the system. Note

multi-port description of a passive linear circuit, where it is known that, although it is not always possible to describe the port as an admittance or as an impedance, it is possible to describe it as a hybrid admittance/impedance transfer matrix, for a suitable selection of input voltages and currents and complementary output currents and voltages Belevitch (1968).

that if this last set of equations can be solved for $s = s(q, p, r)$ as a function of the remaining variables, then the system can be reduced to the unconstrained canonical equations

$$\begin{aligned}\dot{q} &= \frac{\partial \bar{H}}{\partial p}(q, p, r),\\ \dot{p} &= -\frac{\partial \bar{H}}{\partial q}(q, p, r),\\ \dot{r} &= 0,\end{aligned}$$

for the Hamiltonian $\bar{H}(q, p, r) := H(q, p, r, s(q, p, r)$. More about elimination of algebraic constraints in port-Hamiltonian systems can be found in van der Schaft (2013); see also Chapter 8.

5.5 Spinor representation

We close this chapter by briefly outlining a recently proposed spinor representation of a Dirac structure in Maks (2010). The spinor concept has been discovered by Cartan in 1913 and the combined theory of spinors and Clifford algebras has played an important role in mathematical physics since Pauli and Dirac presented their equations of the electron in quantum mechanics in 1927 and 1928, respectively.

Let $\mathcal{W}$ be a real vector space of suitable dimension equipped with a non-gegenerate quadratic form Q. There exists a well-established theory of the so-called Clifford algebra $Cl(\mathcal{W}, Q)$ associated to $(\mathcal{W}, Q)$, which is closely connected to the existence of a spinor space $\mathcal{S}$ of $(\mathcal{W}, Q)$. Without going into the details of the Clifford algebra $Cl(\mathcal{W}, Q)$, the main idea is to represent $\mathcal{W}$ as a linear space of operators that act on $\mathcal{S}$ in the following way. Let $\rho : \mathcal{W} \to \text{End}(\mathcal{S})$, with $\text{End}(\mathcal{S})$ (endomorphisms of $\mathcal{S}$) denoting the set of linear operators on $\mathcal{S}$, be a linear mapping subject to the condition $\rho^2(w) = Q(w)1$, for each $w \in \mathcal{W}$.[2]. Then, the elements of $\mathcal{S}$ are called *spinors*.

Now, focussing on the particular structure $\mathcal{W} = \mathcal{F} \times \mathcal{E}$, with $\mathcal{E} = \mathcal{F}^*$, and denoting the elements of $\mathcal{W}$ as (f, e), with $f \in \mathcal{F}$ and $e \in \mathcal{E}$

[2]Here '1' should be understood as the identity operator. Note that $\rho^2(w) = Q(w)1$ is often replace by the less formal condition $w^2 = Q(w)$, where $\rho(w)$ is simply identified with w and $Q(w)$ is understood to mean $Q(w)1$.

as before, the quadratic form Q associated to the bilinear form $\ll, \gg$ is given by

$$Q((f,e)) =< f \mid e > .$$

A well-known model for a spinor space is the *exterior algebra* of the dual of $\mathcal{F}$, i.e., $\mathcal{S} \cong \wedge\mathcal{E}$.

Definition 5.1. The (linear) action of $\mathcal{W} = \mathcal{F} \times \mathcal{E}$ on the spinor space $\wedge\mathcal{E}$ is defined by

$$\rho((f,e))s = i_f s + e \wedge s,$$

for each $s \in \wedge\mathcal{E}$, i.e., by the sum of the *interior* product $i_f s$ and the *exterior* product $e \wedge s$ by the flows f and efforts e, respectively.

Consider a port-Hamiltonian system without energy-dissipation satisfying

$$\left(-\dot{x}, \frac{\partial H}{\partial x}(x), f_P, e_P\right) \in \mathcal{D}. \tag{5.3}$$

According to Definition 5.1, the spinor representation of $\mathcal{D}$ is associated to the vector space $\mathcal{F} \times \mathcal{E} = T_x\mathcal{X} \times T_x^*\mathcal{X} \times \mathcal{F}_P \times \mathcal{E}_P$, and naturally translates (5.3) in the language of spinors into the algebraic identity

$$i_{(-\dot{x},f_P)}s + \left(\frac{\partial H}{\partial x}(x), e_P\right) \wedge s = 0, \tag{5.4}$$

The strength of the representation (5.4) is that it contains all the relevant information of the port-Hamiltonian system in a single algebraic identity, namely the flows $(-\dot{x}, f_P)$, the efforts $(\frac{\partial H}{\partial x}(x), e_P)$, and the spinor s encoding the interconnection structure of the system. For a more detailed exposition, references to Clifford algebra and exterior calculus, and some illustrative examples, the reader is referred to Maks (2010).

6

Interconnection of port-Hamiltonian systems

Crucial feature of network modeling is 'interconnectivity' or 'compositionality', meaning that complex, large-scale, systems can be built up from simpler parts, and that certain properties of the complex system can be studied in terms of its constituent parts and the way they are interconnected. As we will see in this chapter, port-Hamiltonian systems completely fit within this paradigm, in the sense that the power-conserving interconnection of port-Hamiltonian systems defines another port-Hamiltonian system.

The theory underlying the compositionality of port-Hamiltonian systems concerns the *composition of Dirac structures*. It will be shown that the composition of Dirac structures through a number of pairs of shared flow and effort variables leads to another Dirac structure, defined in terms of the remaining flow and effort variables. Once this has been shown, the rest of the story is easy: the Hamiltonian of the interconnected port-Hamiltonian system will be the sum of the Hamiltonians of its subsystems, while similarly the energy-dissipation relation of the interconnected system is obtained as the union of the energy-dissipating relations of the subsystems.

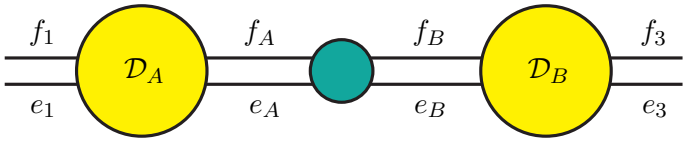


Figure 6.1: The composition of $\mathcal{D}_A$ and $\mathcal{D}_B$.

6.1 Composition of Dirac structures

Physically it is plausible that the interconnection of a number of power-conserving elements is again power-conserving. We will show how this can be formalized within the framework of Dirac structures, and how this leads to the stronger statement that the composition of Dirac structures defines a Dirac structure.

Without loss of generality we consider the composition of *two* Dirac structures with partially shared variables. Once we have shown that the composition of two Dirac structures is again a Dirac structure, it is immediate that the power-conserving interconnection of any number of Dirac structures is again a Dirac structure. Thus consider a Dirac structure $\mathcal{D}_A \subset \mathcal{F}_1 \times \mathcal{F}_2 \times \mathcal{E}_1 \times \mathcal{E}_2$ with $\mathcal{E}_i = \mathcal{F}_i^*, i = 1, 2$, and another Dirac structure $\mathcal{D}_B \subset \mathcal{F}_2 \times \mathcal{F}_3 \times \mathcal{E}_2 \times \mathcal{E}_3$, with $\mathcal{E}_3 = \mathcal{F}_3^*$. The linear space $\mathcal{F}_2$ is the space of shared flow variables, and $\mathcal{E}_2 = \mathcal{F}_2^*$ the space of shared effort variables; see Figure 6.1.

In order to compose $\mathcal{D}_A$ and $\mathcal{D}_B$, a problem arises with regard to the *sign* convention for the power flow corresponding to the power variables $(f_2, e_2) \in \mathcal{F}_2 \times \mathcal{E}_2$. Indeed, if we take the convention[1] that $< e \mid f >$ denotes *incoming* power, then for

$$(f_1, e_1, f_A, e_A) \in \mathcal{D}_A \subset \mathcal{F}_1 \times \mathcal{E}_1 \times \mathcal{F}_2 \times \mathcal{E}_2,$$

the term $< e_A \mid f_A >$ denotes the incoming power in $\mathcal{D}_A$ due to the power variables $(f_A, e_A) \in \mathcal{F}_2 \times \mathcal{E}_2$, while for

$$(f_B, e_B, f_3, e_3) \in \mathcal{D}_B \subset \mathcal{F}_2 \times \mathcal{E}_2 \times \mathcal{F}_3 \times \mathcal{E}_3,$$

the term $< e_B \mid f_B >$ denotes the incoming power *in* $\mathcal{D}_B$. Clearly, the *incoming* power in $\mathcal{D}_A$ due to the power variables in $\mathcal{F}_2 \times \mathcal{E}_2$ should

[1] In physics it seems more common to take the opposite sign convention: positive power is *outgoing*. However, the same argument remains to hold.

equal the *outgoing* power from $\mathcal{D}_B$. Thus we cannot simply equate the flows f_A and f_B and the efforts e_A and e_B, but instead we define the interconnection constraints as

$$f_A = -f_B \in \mathcal{F}_2, \quad e_A = e_B \in \mathcal{E}_2. \tag{6.1}$$

Therefore, the *composition* of the Dirac structures $\mathcal{D}_A$ and $\mathcal{D}_B$, denoted $\mathcal{D}_A \circ \mathcal{D}_B$, is defined as

$$\mathcal{D}_A \circ \mathcal{D}_B := \Big\{(f_1, e_1, f_3, e_3) \in \mathcal{F}_1 \times \mathcal{E}_1 \times \mathcal{F}_3 \times \mathcal{E}_3 \mid \exists (f_2, e_2) \in \mathcal{F}_2 \times \mathcal{E}_2$$
$$\text{s.t. } (f_1, e_1, f_2, e_2) \in \mathcal{D}_A \text{ and } (-f_2, e_2, f_3, e_3) \in \mathcal{D}_B\Big\}.$$

The next theorem is proved (in different ways) in Cervera et al. (2007); van der Schaft (1999); Dalsmo & van der Schaft (1999), and Narajanan (2002).

Theorem 6.1. Let $\mathcal{D}_A \subset \mathcal{F}_1 \times \mathcal{E}_1 \times \mathcal{F}_2 \times \mathcal{E}_2$ and $\mathcal{D}_B \subset \mathcal{F}_2 \times \mathcal{E}_2 \times \mathcal{F}_3 \times \mathcal{E}_3$ be Dirac structures. Then $\mathcal{D}_A \circ \mathcal{D}_B \subset \mathcal{F}_1 \times \mathcal{E}_1 \times \mathcal{F}_3 \times \mathcal{E}_3$ is a Dirac structure.

Furthermore, the following explicit expression can be given for the composition of two Dirac structures in terms of their matrix kernel/image representation; see Cervera et al. (2007) for a proof.

Theorem 6.2. Let $\mathcal{F}_i, i = 1, 2, 3$, be finite-dimensional linear spaces with $\dim \mathcal{F}_i = n_i$. Consider Dirac structures $\mathcal{D}_A \subset \mathcal{F}_1 \times \mathcal{E}_1 \times \mathcal{F}_2 \times \mathcal{E}_2$, $n_A = \dim \mathcal{F}_1 \times \mathcal{F}_2 = n_1 + n_2$, $\mathcal{D}_B \subset \mathcal{F}_2 \times \mathcal{E}_2 \times \mathcal{F}_3 \times \mathcal{E}_3$, $n_B = \dim \mathcal{F}_2 \times \mathcal{F}_3 = n_2 + n_3$, given by relaxed matrix kernel/image representations $(F_A, E_A) = ([F_1 \mid F_{2A}], [E_1 \mid E_{2A}])$, with F_A and E_A $n'_A \times n_A$ matrices, $n'_A \geq n_A$, respectively $(F_B, E_B) = ([F_{2B} \mid F_3], [E_{2B} \mid E_3])$, with F_B and E_B $n'_B \times n_B$ matrices, $n'_B \geq n_B$. Define the $(n'_A + n'_B) \times 2n_2$ matrix

$$M = \begin{bmatrix} F_{2A} & E_{2A} \\ -F_{2B} & E_{2B} \end{bmatrix}, \tag{6.2}$$

and let L_A and L_B be $m \times n'_A$, respectively $m \times n'_B$, matrices ($m := \dim \ker M^T$), with

$$L = [L_A \mid L_B], \quad \ker L = \operatorname{im} M. \tag{6.3}$$

Then,

$$F = \begin{bmatrix} L_A F_1 \mid L_B F_3 \end{bmatrix}, \quad E = \begin{bmatrix} L_A E_1 \mid L_B E_3 \end{bmatrix}, \tag{6.4}$$

is a relaxed matrix kernel/image representation of $\mathcal{D}_A \circ \mathcal{D}_B$.

Separable Dirac structures turn out to have the following special compositional property (van der Schaft & Maschke (2013)):

Proposition 6.1. Let $\mathcal{D}_A \subset \mathcal{F}_2 \times \mathcal{E}_1 \times \mathcal{F}_2 \times \mathcal{E}_2$ and $\mathcal{D}_B \subset \mathcal{F}_2 \times \mathcal{E}_2 \times \mathcal{F}_3 \times \mathcal{E}_3$ be two separable Dirac structures given as

$$\mathcal{D}_A = \mathcal{K}_A \times \mathcal{K}_A^\perp, \quad \mathcal{D}_B = \mathcal{K}_B \times \mathcal{K}_B^\perp$$

where $\mathcal{K}_A \subset \mathcal{F}_1 \times \mathcal{F}_2$ and $\mathcal{K}_B \subset \mathcal{F}_2 \times \mathcal{F}_3$ Define the composition

$$\mathcal{K}_A \circ \mathcal{K}_B = \Big\{ (f_1, f_3) \in \mathcal{F}_1 \times \mathcal{F}_3 \mid \exists f_2 \in \mathcal{F}_2 \\ \text{s.t. } (f_1, f_2) \in \mathcal{K}_A, (-f_2, f_3) \in \mathcal{K}_B \Big\}$$

Then the composition $\mathcal{D}_A \circ \mathcal{D}_B$ is the separable Dirac structure

$$\mathcal{D}_A \circ \mathcal{D}_B = (\mathcal{K}_A \circ \mathcal{K}_B) \times (\mathcal{K}_A \circ \mathcal{K}_B)^\perp \tag{6.5}$$

6.2 Interconnection of port-Hamiltonian systems

The composition theory of Dirac structures has the following consequence for interconnection of port-Hamiltonian systems. Consider k port-Hamiltonian systems with state spaces $\mathcal{X}_i$, Hamiltonians H_i, energy-dissipating relations $\mathcal{R}_i$, external port flow and effort spaces $\mathcal{F}_i \times \mathcal{E}_i$, and Dirac structures $\mathcal{D}_i$, $i = 1, \ldots, k$. Furthermore, consider an *interconnection* Dirac structure

$$\mathcal{D}_I \subset \mathcal{F}_1 \times \cdots \times \mathcal{F}_k \times \mathcal{E}_1 \times \cdots \times \mathcal{E}_k \times \mathcal{F} \times \mathcal{E}, \tag{6.6}$$

with $\mathcal{F} \times \mathcal{E}$ the new space of external flow and effort port variables, cf. Figure 6.2. Obviously, the direct product $\mathcal{D}_1 \times \cdots \times \mathcal{D}_k$ is again a Dirac structure on the resulting state space

$$\mathcal{X} := \mathcal{X}_1 \times \cdots \times \mathcal{X}_k.$$

Therefore by the theory of the previous section the subspace

$$\mathcal{D} := (\mathcal{D}_1 \times \cdots \times \mathcal{D}_k) \circ \mathcal{D}_I$$

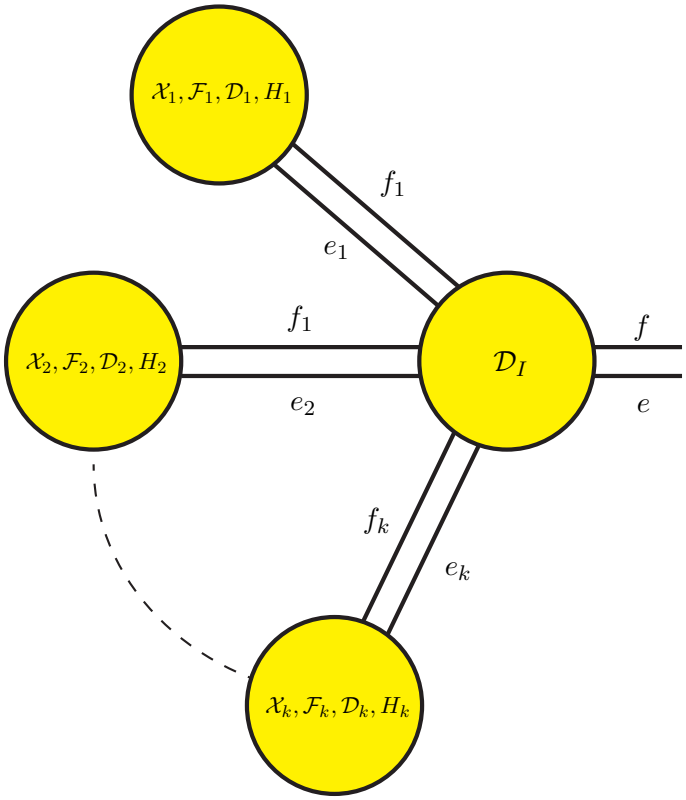


Figure 6.2: Interconnection of port-Hamiltonian systems.

is again a Dirac structure on $\mathcal{X}$.

As a result, the interconnection of the k port-Hamiltonian systems through the interconnection Dirac structure $\mathcal{D}_I$ defines another port-Hamiltonian system with Dirac structure $\mathcal{D}$, Hamiltonian H being the sum

$$H := H_1 + \cdots + H_k,$$

and with resistive structure $\mathcal{R}$ being the direct product of the resistive structures $\mathcal{R}_i, i = 1, \ldots, k$. This is a key result in the theory of port-Hamiltonian systems, allowing to build up complex port-Hamiltonian systems models from simple ones.

Finally we mention that the theory of composition of Dirac structures and the interconnection of port-Hamiltonian systems can be also

extended to *infinite-dimensional* Dirac structures and port-Hamiltonian systems, see e.g. Golo (2002); Villegas (2007); Kurula et al. (2010); Jacob & Zwart (2012). This implies that also distributed-parameter port-Hamiltonian subsystems (cf. Chapter 14) can be included into the overall port-Hamiltonian description.

7

Port-Hamiltonian systems and passivity

Passivity is a fundamental property that constitutes a cornerstone for major developments in systems and control theory; see Willems (1972a,b); Hill & Moylan (1976); van der Schaft (2000), and the references therein. For linear systems, passivity can be characterized in the frequency-domain by the notion of a positive-real transfer function. In the time-domain, both for linear and nonlinear systems, passivity is characterized by a dissipation inequality Willems (1972a), which in the linear case reduces to a set of Linear Matrix Inequalities (LMIs) Willems (1972b), and in the nonlinear case to a set of (in-)equalities usually referred to as the Hill-Moylan conditions Hill & Moylan (1976).

A system $\dot{x} = f(x,u)$, $y = h(x,u)$, where $x \in \mathcal{X}$ and $u, y \in \mathbb{R}^m$, is called *passive* if there exists a differentiable *storage function* $S : \mathcal{X} \to \mathbb{R}$ with $S(x) \geq 0, x \in \mathcal{X}$, satisfying the *differential dissipation inequality*

$$\frac{d}{dt}S(x(t)) \leq u^T(t)y(t), \tag{7.1}$$

along all solutions $x(\cdot)$ corresponding to input functions $u(\cdot)$. For physical systems, the right-hand side $u^T y$ is usually interpreted as the *supplied power*, and $S(x)$ as the *stored energy* of the system when be-

ing in state x. Furthermore, the system is called *lossless* if (7.1) holds with equality. Hence, a passive system cannot store more energy than it is supplied with, and in the lossless case the stored energy is exactly equal to the supplied one.

Passivity is intimately related to (Lyapunov) stability. Indeed, if S has a strict minimum at a certain state x^*, then it follows from (7.1) with $u = 0$ that x^* is a equilibrium of the unforced dynamics $\dot{x} = f(x, 0)$ with Lyapunov function $S(x)$; implying *stability* of the equilibrium state x^*.

The differential dissipation inequality (7.1) can be restated as

$$\frac{\partial^T S}{\partial x}(x) f(x, u) \leq u^T h(x, u), \tag{7.2}$$

for all x, u. For affine nonlinear systems $\dot{x} = f(x) + g(x)u$, $y = h(x)$, with $g(x)$ an $n \times m$ matrix, this is easily seen to reduce to

$$\frac{\partial^T S}{\partial x}(x) f(x) \leq 0, \quad h(x) = g^T(x) \frac{\partial S}{\partial x}(x). \tag{7.3}$$

An *integral form* of the differential dissipation inequality (7.1) is provided by

$$S(x(t_1)) - S(x(t_0)) \leq \int_{t_1}^{t_2} u^T(t) y(t) dt, \tag{7.4}$$

for all time instants $t_1 \leq t_2$, all states $x(t_0)$, and all input functions $u : [t_0, t_1] \to \mathbb{R}^m$, where $x(t_1)$ denotes the state at time t_1 resulting from initial state $x(t_0)$ at time t_0. This integral form allows one to relax the requirement of differentiability of S. Clearly (7.1) implies (7.4), while if S is differentiable then conversely (7.4) can be seen to imply (7.1).

For port-Hamiltonian systems, passivity can be directly inferred from the power-balance (2.27), and thus is a direct consequence of the properties of the Dirac structure and the energy-dissipating relation. Indeed, since by definition of the energy-dissipating relation, the term $e_R^T f_R$ is always less than or equal to zero, the power-balance (2.27) can be written as

$$\frac{d}{dt} H = e_R^T f_R + e_P^T f_P \leq e_P^T f_P. \tag{7.5}$$

Under the additional assumption that H is greater or equal than zero[1], the latter inequality implies that any port-Hamiltonian system is passive with respect to the port variables f_P, e_P and storage function H. Furthermore, if $e_R^T f_R = 0$, the system is *lossless*.

On the other hand, not every passive system is necessarily a port-Hamiltonian system, as is illustrated by the following example.

Example 7.1. Consider the nonlinear system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ -x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u,$$
$$y = x_1^2 x_2,$$

which is passive (in fact, it lossless) with storage function $H(x_1, x_2) = \frac{1}{2}x_1^2 x_2^2$. However, it is readily observed that there does *not* exist a 2×2 matrix $J(x_1, x_2) = -J^T(x_1, x_2)$, with entries depending smoothly on x_1 and x_2, such that

$$\begin{bmatrix} x_1 \\ -x_2 \end{bmatrix} = J(x_1, x_2) \begin{bmatrix} x_1 x_2^2 \\ x_1^2 x_2 \end{bmatrix}.$$

Indeed, such a $J(x_1, x_2)$ will necessarily have a singularity at $(0, 0)$.

In the next section we will we show that, under an extra condition, any passive *linear* system

$$\begin{aligned} \dot{x} &= Ax + Bu, \\ y &= Cx + Du, \end{aligned} \tag{7.6}$$

where A is an $n \times n$ matrix, B an $n \times m$ matrix, C an $m \times n$ matrix, and D an $m \times m$ matrix, *can* be written as a port-Hamiltonian system.

7.1 Linear port-Hamiltonian systems

Consider the linear version of the input-state-output port-Hamiltonian system with feedthrough term (4.9)

$$\begin{aligned} \dot{x} &= [J - R]Qx + [G - P]u, \\ y &= [G + P]^T Qx + [M + S]u, \end{aligned} \tag{7.7}$$

[1]Note that it is sufficient to assume that H is *bounded from below*, i.e., $H(x) \geq c$, for some real constant c. Then $S(x) := H(x) - c$ defines a storage function.

where $J = -J^T$ is an $n \times n$ matrix and $M = -M^T$ is an $m \times m$ matrix, both reflecting the interconnection structure. The Hamiltonian of the system is given by the quadratic function $H(x) = \frac{1}{2}x^T Q x$, where $Q = Q^T$ is an $n \times n$ matrix referred to as the energy matrix. Furthermore, $R = R^T$ is an $n \times n$ matrix and $S = S^T$ is an $m \times m$ matrix, both reflecting the linear resistive structure, and G and P are $n \times m$ matrices, satisfying

$$\begin{bmatrix} R & P \\ P^T & S \end{bmatrix} \geq 0. \tag{7.8}$$

In particular, if $P = 0$, then the latter condition reduces to the condition that $R \geq 0$ and $S \geq 0$.

Theorem 7.1. The following properties hold:
1. If the system (7.6) is passive, with quadratic storage function $\frac{1}{2}x^T Q x$, satisfying $Q \geq 0$, and[2] $\ker Q \subset \ker A$, then it allows a port-Hamiltonian representation of the form (7.7).
2. If $Q \geq 0$, then the port-Hamiltonian system (7.7) is passive.

Proof. Because of the condition $\ker Q \subset \ker A$, it follows from linear algebra that there exists a matrix Σ such that

$$\begin{bmatrix} A & B \\ -C & -D \end{bmatrix} = \Sigma \begin{bmatrix} Q & 0 \\ 0 & I \end{bmatrix}. \tag{7.9}$$

In fact, if $Q > 0$ then such a Σ is uniquely defined as

$$\Sigma := \begin{bmatrix} AQ^{-1} & B \\ -CQ^{-1} & -D \end{bmatrix}.$$

Passivity of the system (7.6), with quadratic storage function $\frac{1}{2}x^T Q x$, amounts to the differential dissipation inequality $x^T Q \dot{x} \leq u^T y$, for all x and u. Substituting $\dot{x} = Ax + Bu$ and $y = Cx + Du$, and making use of (7.9), the differential dissipation inequality can be rewritten as

$$\begin{bmatrix} x^T u^T \end{bmatrix} \begin{bmatrix} Q & 0 \\ 0 & I \end{bmatrix} \Sigma \begin{bmatrix} Q & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} \leq 0,$$

[2] Note that the condition $\ker Q \subset \ker A$ is automatically satisfied if $Q > 0$.

for all x and u, or equivalently

$$\begin{bmatrix} Q & 0 \\ 0 & I \end{bmatrix} \left(\Sigma + \Sigma^T\right) \begin{bmatrix} Q & 0 \\ 0 & I \end{bmatrix} \leq 0.$$

It follows from linear algebra that we can choose Σ satisfying (7.9) in such a way that $\Sigma + \Sigma^T \leq 0$. Hence, if we write $\Sigma = \bar{J} - \bar{R}$, $\bar{J} = -\bar{J}^T$, and $\bar{R} = \bar{R}^T$, then $\bar{R} \geq 0$. Now, denote

$$\bar{J} = \begin{bmatrix} J & G \\ -G^T & -M \end{bmatrix}, \quad \bar{R} = \begin{bmatrix} R & P \\ P^T & S \end{bmatrix},$$

with $J = -J^T$, $M = -M^T$, $R = R^T$, and $S = S^T$, then (7.6) can be written as

$$\begin{bmatrix} \dot{x} \\ -y \end{bmatrix} = \left(\begin{bmatrix} J & G \\ -G^T & -M \end{bmatrix} - \begin{bmatrix} R & P \\ P^T & S \end{bmatrix} \right) \begin{bmatrix} Qx \\ u \end{bmatrix},$$

which readily can be seen to coincide with (7.7).

Secondly, to prove that linear port-Hamiltonian systems (7.7) are passive with storage function $H(x) = \frac{1}{2} x^T Q x$, with $Q \geq 0$, we need to show that, along the trajectories of (7.7), the differential dissipation inequality (7.1) holds. Indeed, using skew-symmetry of J and M, and the condition (7.8), we obtain

$$\begin{aligned} \frac{d}{dt} H &= -x^T Q R Q x + y^T u - u^T S u - 2 x^T Q P u \\ &= - \begin{bmatrix} (Qx)^T u^T \end{bmatrix} \begin{bmatrix} R & P \\ P^T & S \end{bmatrix} \begin{bmatrix} Qx \\ u \end{bmatrix} + y^T u \leq y^T u. \end{aligned}$$

■

7.2 Available and required storage

Although using the Hamiltonian H as a storage function may suggest that the storage function is unique, we know from Willems (1972a,b) that generally a passive system admits *infinitely many* possible storage functions. First of all, let us define the non-negative function

$$S_a(x) = \sup_{\substack{u(\cdot) \\ \tau \geq 0}} - \int_0^\tau u^T(t) y(t) dt \tag{7.10}$$

for $x(0) = x$. It can be shown Willems (1972a); van der Schaft (2000), that the system is passive *if and only if* the function S_a is well-defined for all $x \in \mathcal{X}$, that is, the righthand side of (7.10) is finite for all $x \in \mathcal{X}$. It is easy to see, cf. van der Schaft (2000), that, whenever the system is *reachable* from a certain state x^*, the function S_a is well-defined (and thus the system is passive) if and only if $S_a(x^*) < \infty$.

The quantity $S_a(x)$ represents the maximal amount of energy that can be extracted from the system starting from the initial state $x(0) = x$, and is therefore called the *available storage*. Moreover, it is the *smallest* of all possible storage functions: $S_a(x) \leq S(x), x \in \mathcal{X}$, for all other storage functions S.

Furthermore, if we assume that the system is reachable from a certain state x^* then there exists a *largest* storage function in the following sense. Define the expression

$$S_r(x) = \inf_{\substack{u(\cdot) \\ \tau \geq 0}} - \int_{-\tau}^{0} u^T(t) y(t) dt, \tag{7.11}$$

where $x(-\tau) = x^*$ and $x(0) = x$. Then it follows from the assumed passivity of the system that there exists a constant $\kappa > -\infty$ such that $S_r(x) \geq \kappa$, and such that the function $S_r(x) - \kappa$ defines a storage function. Moreover, for any storage function S it holds that

$$S(x) \leq S_r(x) + S(x^*),$$

and $S_r(x) + S(x^*)$ is a storage function.

The quantity $S_r(x)$ represents the *required supply* to reach x at $t = 0$ starting from x^*. For lossless systems it can be shown that $S_a = S_r$, and thus that storage functions are *unique* up to a positive constant.

Example 7.2. Consider a port-Hamiltonian input-state-output system

$$\begin{aligned} \dot{x} &= [J(x) - R(x)] \frac{\partial H}{\partial x}(x) + g(x)u, \\ y &= g^T(x) \frac{\partial H}{\partial x}(x), \end{aligned}$$

with $R(x) \geq 0$ and $H(x) \geq 0$ for all x. The system is passive with storage function given by its Hamiltonian H. Let $\tilde{H}(x)$ be another storage

function. Then usually (see however the possible obstructions indicated in Example 7.1) there exist matrices $\tilde{J}(x)$ and $\tilde{R}(x)$ such that

$$[J(x) - R(x)]\frac{\partial H}{\partial x}(x) = [\tilde{J}(x) - \tilde{R}(x)]\frac{\partial \tilde{H}}{\partial x}(x).$$

Hence, the system is port-Hamiltonian with respect to a *different* Hamiltonian and a *different* Dirac structure and energy-dissipating relation specified by $\tilde{J}$ and $\tilde{R}$.

7.3 Shifted port-Hamiltonian systems and passivity

Consider a port-Hamiltonian system given by a Dirac structure (see Chapter 2)

$$\mathcal{D} \subset T_x\mathcal{X} \times T_x^*\mathcal{X} \times \mathcal{F}_R \times \mathcal{E}_R \times \mathcal{F}_P \times \mathcal{E}_P,$$

a resistive structure $\mathcal{R} \subset \mathcal{F}_R \times \mathcal{E}_R$, and a Hamiltonian $H : \mathcal{X} \to \mathbb{R}$, with resulting dynamics (2.28). Let us assume that both the Dirac structure $\mathcal{D}$ and the resistive structure $\mathcal{R}$ are *constant*; that is, not depending on x. Furthermore, assume that the resistive structure $\mathcal{R}$ is *linear* (see Chapter 4).

Consider now the situation of a *steady-state* x^*, corresponding to steady-state values $f_R^*, e_R^*, f_P^*, e_P^*$, i.e.,

$$\left(0, \frac{\partial H}{\partial x}(x^*), f_R^*, e_R^*, f_P^*, e_P^*\right) \in \mathcal{D}, \quad (f_R^*, e_R^*) \in \mathcal{R}. \tag{7.12}$$

Then, by using the linearity of $\mathcal{D}$ and $\mathcal{R}$, we can subtract (7.12) from (2.28), so as to obtain

$$\begin{aligned}\Big(-\dot{x}(t), \frac{\partial H}{\partial x}(x(t)) - \frac{\partial H}{\partial x}(x^*), f_R(t) - f_R^*, e_R(t) - e_R^*, \\ f_P(t) - f_P^*, e_P(t) - e_P^*\Big) \in \mathcal{D}, \\ (f_R(t) - f_R^*, e_R(t) - e_R^*) \in \mathcal{R}.\end{aligned} \tag{7.13}$$

This defines a *shifted* port-Hamiltonian system as follows. Define as in Jayawardhana et al. (2007) the *shifted* Hamiltonian corresponding to

the steady-state x^* as[3]

$$\tilde{H}(x) := H(x) - (x - x^*)^T \frac{\partial H}{\partial x}(x^*) - H(x^*), \tag{7.14}$$

then we observe that

$$\frac{\partial \tilde{H}}{\partial x}(x) = \frac{\partial H}{\partial x}(x) - \frac{\partial H}{\partial x}(x^*).$$

Theorem 7.2. Consider a port-Hamiltonian system with constant Dirac structure $\mathcal{D}$ and constant linear resistive structure $\mathcal{R}$. Furthermore, assume the existence of a steady-state x^* satisfying (7.12). Then, by defining the shifted Hamiltonian (7.14), we have obtained a port-Hamiltonian system (7.13) with the same state space $\mathcal{X}$, same Dirac structure $\mathcal{D}$ and resistive structure $\mathcal{R}$, but with shifted Hamiltonian $\tilde{H}$ and shifted external port variables $f_P - f_P^*, e_P - e_P^*$.

Corollary 7.3. Assume additionally that the Hamiltonian H is *convex*. Then the shifted port-Hamiltonian system (7.13) with shifted Hamiltonian $\tilde{H}$ is passive with respect to the shifted external port variables $f_P - f_P^*, e_P - e_P^*$.

Proof. By convexity of H the shifted Hamiltonian $\tilde{H}$ has a minimum at x^*, and thus the shifted port-Hamiltonian system is passive with respect to $(f_P - f_P^*, e_P - e_P^*)$. ∎

In Chapter 9 we will further generalize the idea of shifted port-Hamiltonian systems and shifted passivity by exploiting the concept of (maximal) monotone relations.

[3] In thermodynamics this is called the *availability function*, cf. Kennan (1951). In convex analysis it is also known as the *Bregman function*, cf. Bürger et al. (2013).

8

Conserved quantities and algebraic constraints

In Chapter 7 it is shown that passivity is a key property for stability analysis since the Hamiltonian H may serve as a Lyapunov function. Indeed, from (7.1) it is readily observed that for an autonomous port-Hamiltonian system, with dynamics specified by

$$\left(-\dot{x}, \frac{\partial H}{\partial x}(x), f_R, e_R\right) \in \mathcal{D} \tag{8.1}$$

and $(f_R, e_R) \in \mathcal{R}$, the power-balance reduces to

$$\frac{d}{dt}H = e_R^T f_R \leq 0, \tag{8.2}$$

which implies that if $H(x^*) = 0$ and $H(x) > 0$, for every $x \neq x^*$, then x^* is a stable equilibrium.[1] However, the point where the Hamiltonian is minimal (which typically coincides with the zero state) is often not the one of practical interest for set-point regulation, in which case the Hamiltonian alone can not be employed as a Lyapunov function.

[1]The equilibrium x^* is asymptotically stable if the dissipation term $e_R^T f_R < 0$ for all $x \neq x^*$, or alternatively if a detectability condition is satisfied, guaranteeing asymptotic stability by the use of LaSalle's Invariance principle.

8.1 Casimirs of conservative port-Hamiltonian systems

A well-known approach in Hamiltonian systems, see e.g. Marsden & Ratiu (1999), is to consider, next to the Hamiltonian function, additional *conserved quantities* which may be present in the system. For, consider functions $C : \mathcal{X} \to \mathbb{R}$ such that $\frac{d}{dt}C = 0$ (dynamical invariance) along the trajectories of the system. The main idea then is to search for functions C such that $V := H + C$ has a minimum at the equilibrium x^*, and, consequently, that we are able to infer (asymptotic) stability by replacing (8.2) with

$$\frac{d}{dt}V = e_R^T f_R \leq 0,$$

using V as a Lyapunov function.

Functions that are conserved quantities of the system for *every* Hamiltonian are called *Casimir* functions or briefly *Casimirs*. Casimirs are completely characterized by the Dirac structure $\mathcal{D}$. Indeed, for every port-Hamiltonian system without energy-dissipation and external ports, and with specified dynamics $(-\dot{x}, \frac{\partial H}{\partial x}(x)) \in \mathcal{D}$, the function $C : \mathcal{X} \to \mathbb{R}$ is a Casimir if and only if the gradient vector $e = \frac{\partial C}{\partial x}(x)$ satisfies $e^T f_S = 0$, for all f_S for which there exists e_S such that $(f_S, e_S) \in \mathcal{D}$, or, equivalently,

$$\frac{d}{dt}C = \frac{\partial^T C}{\partial x}(x(t))\dot{x}(t) = -\frac{\partial^T C}{\partial x}(x(t))f_S = -e^T f_S = 0. \tag{8.3}$$

By the property $\mathcal{D} = \mathcal{D}^{\perp\!\!\!\perp}$ of the Dirac structure $\mathcal{D}$, this is readily seen to be equivalent to the requirement that $e = \frac{\partial C}{\partial x}(x)$ satisfies $(0, e) \in \mathcal{D}$.

Example 8.1. For any Hamiltonian dynamics

$$\dot{x} = J(x)\frac{\partial H}{\partial x}(x),$$

with $J(x) = -J^T(x)$, the corresponding Casimirs are solutions to the set of PDE's

$$\frac{\partial^T C}{\partial x}(x)J(x) = 0.$$

The well-known Casimir for the spinning rigid body of Example 3.1, is the total angular momentum $p_x^2 + p_y^2 + p_z^2$, whose vector of partial derivatives is in the kernel of the matrix $J(p)$ of (3.1).

8.2 Linear resistive structures and the dissipation obstacle

Similarly, a Casimir for a port-Hamiltonian system (8.1) with linear resistive structure $\mathcal{R}$, cf. Chapter 4, is defined as a function $C : \mathcal{X} \to \mathbb{R}$ satisfying

$$\left(0, \frac{\partial C}{\partial x}(x), 0, 0\right) \in \mathcal{D}. \tag{8.4}$$

Indeed, for every port-Hamiltonian system with the same Dirac structure this implies the identity (8.3). Although the above definition of a Casimir function suggests to hold only for a specific linear resistive structure $R_f f_R + R_e e_R = 0$, where the square matrices R_f and R_e satisfy the condition $R_f R_e^T = R_e R_f^T \geq 0$, together with rank$[R_f \mid R_e] =$ dimf_R, it can be shown Pasumarthy & van der Schaft (2007) that a conserved quantity for one resistive relation actually is a conserved quantity for *all* linear resistive structures.

The fact that a Casimir for one linear resistive structure is a Casimir for all linear resistive structures is closely related to the so-called *dissipation obstacle* for the existence of Casimir functions in the case of input-state-output port-Hamiltonian systems. Indeed, for autonomous systems of the form

$$\dot{x} = [J(x) - R(x)] \frac{\partial H}{\partial x}(x),$$

with $J(x) = -J^T(x)$, $R(x) = R^T(x)$, and $R(x) \geq 0$ for all $x \in \mathcal{X}$, the corresponding Casimirs are solutions to the set of PDE's

$$\frac{\partial^T C}{\partial x}(x) \left[J(x) - R(x)\right] = 0.$$

Multiplying from the right by $\frac{\partial C}{\partial x}(x))$, and using skew-symmetry of $J(x)$ and positive semi-definiteness of $R(x)$, this is seen to be equivalent to

$$\frac{\partial^T C}{\partial x}(x) R(x) = 0, \quad \frac{\partial^T C}{\partial x}(x) J(x) = 0.$$

Hence, Casimirs are necessarily independent of those state-space coordinates that are directly affected by physical damping. We come back to the dissipation obstacle in Chapter 15.

8.3 Algebraic constraints

Algebraic constraints on the state variables are to a large extent determined by the Dirac structure. Indeed, let us first consider a port-Hamiltonian system without external and resistive ports, described by a Dirac structure $\mathcal{D}$ and a Hamiltonian H. Define for every $x \in \mathcal{X}$ the subspace

$$P_{\mathcal{D}}(x) := \{\alpha \in T_x^*\mathcal{X} \mid \exists X \in T_x\mathcal{X} \text{ such that } (\alpha, X) \in \mathcal{D}(x)\}.$$

(This defines a *co-distribution* on the manifold $\mathcal{X}$.) Then the definition of the port-Hamiltonian system implies that

$$\frac{\partial H}{\partial x}(x) \in P_{\mathcal{D}}(x).$$

In general, this imposes algebraic constraints on the state variables $x \in \mathcal{X}$. For example, if the Dirac structure is given in image representation (see Chapter 5) as

$$\mathcal{D}(x) = \{(X, \alpha) \in T_x\mathcal{X} \times T_x^*\mathcal{X} \mid X = E^T(x)\lambda, \alpha = F^T(x)\lambda\},$$

then it follows that

$$\frac{\partial H}{\partial x}(x) \in \operatorname{im} F^T(x),$$

which leads in general (depending on the Hamiltonian H) to algebraic constraints on the state variables x. Similarly, if the Dirac structure is given in constrained input-output form (5.2) then the algebraic constraints are given as

$$G^T(x)\frac{\partial H}{\partial x}(x) = 0.$$

The resulting dynamics is thus a combination of differential and algebraic equations, called port-Hamiltonian differential equations (DAEs).

In the case of resistive and/or external ports, the algebraic constraints on the state variables x may also depend on the resistive and/or external port variables. A special case arises for resistive ports. Consider a Dirac structure

$$\{(X, \alpha, f_R, e_R) \in \mathcal{D}(x) \subset T_x\mathcal{X} \times T_x^*\mathcal{X} \times \mathcal{F}_R \times \mathcal{E}_R\},$$

with the resistive flow and effort variables satisfying a relation

$$R(f_R, e_R) = 0.$$

Then, the gradient of the Hamiltonian has to satisfy the condition

$$\frac{\partial H}{\partial x}(x) \in \{\alpha \in T_x^*\mathcal{X} \mid \exists X, f_R, e_R \in T_x\mathcal{X} \times \mathcal{F}_R \times \mathcal{E}_R \\ \text{such that } (X, \alpha, f_R, e_R) \in \mathcal{D}(x), R(f_R, e_R) = 0\}.$$

Depending on the resistive relation $R(f_R, e_R) = 0$, this may again induce algebraic constraints on the state variables x.

8.4 Elimination of algebraic constraints

An important problem concerns the possibility to solve for the *algebraic constraints* of a port-Hamiltonian differential-algebraic system. We will focus on the case that the Dirac structure is given in constrained input-output form (5.2) and thus the algebraic constraints are explicitly given as

$$G^T(x)\frac{\partial H}{\partial x}(x) = 0. \tag{8.5}$$

The precise way this will constrain the state variables x depends on $G(x)$ as well as on the properties of the Hamiltonian H. For example, if the Hamiltonian H is such that its gradient $\frac{\partial H}{\partial x}(x)$ happens to be contained in the kernel of the matrix $G^T(x)$ for all x, then the algebraic constraints (8.5) are automatically satisfied, and actually the state variables are not constrained at all.

In general, under constant rank assumptions, the set

$$\mathcal{X}_c := \{x \in \mathcal{X} \mid G^T(x)\frac{\partial H}{\partial x}(x) = 0\}$$

will define a submanifold of the total state space $\mathcal{X}$, called the *constrained state space*. In order that this constrained state space qualifies as the state space for a port-Hamiltonian system *without* further algebraic constraints, one needs to be able to restrict the dynamics of the port-Hamiltonian system to the constrained state space. This is always possible under the condition that the matrix

$$G^T(x)\frac{\partial^2 H}{\partial x^2}(x)G(x) \tag{8.6}$$

has full rank. Indeed, in this case the differentiated constraint equation

$$0 = \frac{d}{dt}\left(G^T(x)\frac{\partial H}{\partial x}(x)\right) = * + G^T(x)\frac{\partial^2 H}{\partial x^2}(x)G(x)\lambda$$

(with $*$ denoting unspecified terms) can be uniquely solved for λ, leading to a uniquely defined dynamics on the constrained state space $\mathcal{X}_c$. Hence the set of *consistent states* for the port-Hamiltonian differential-algebraic system (the set of initial conditions for which the system has a unique ordinary solution) is equal to the constrained state space $\mathcal{X}_c$. Using terminology from the theory of DAEs, the condition that the matrix in (8.6) has full rank ensures that the *index* of the DAEs specified by the port-Hamiltonian system is equal to one. This can be summarized as

Proposition 8.1. Consider the port-Hamiltonian differential-algebraic system represented as in (5.2), with algebraic constraints $G^T(x)\frac{\partial H}{\partial x}(x) = 0$. Suppose that the matrix $G^T(x)\frac{\partial^2 H}{\partial x^2}(x)G(x)$ has full rank for all $x \in \mathcal{X}_c$. Then the system has index one, and the set of consistent states is equal to $\mathcal{X}_c$.

If the matrix in (8.6) does not have full rank, then the index of the port-Hamiltonian differential-algebraic system will be larger than one, and it will be necessary to further constrain the space $\mathcal{X}_c$ by considering apart from the 'primary' algebraic constraints (8.5), also their (repeated) time-derivatives (called *secondary constraints*). We refer to van der Schaft (1987); Nijmeijer & van der Schaft (1990) for a detailed treatment and conditions for reducing the port-Hamiltonian DAE system to a system without algebraic constraints in case $J(x)$ corresponds to a symplectic structure. For the linear case, and the relation with the theory of matrix pencils see van der Schaft (2013).

A particular elegant representation of the algebraic constraints arises from the *canonical coordinate representation*. We will only consider the case of a system without energy-dissipation and external ports. If the Dirac structure $\mathcal{D}$ on the state space manifold is *integrable*, cf. Chapter 3, then there exist local coordinates $x = (q, p, r, s)$ for $\mathcal{X}$ in which the system (without energy-dissipation and external ports) takes the

form

$$\begin{aligned}\dot{q} &= \frac{\partial H}{\partial p}(q,p,r,s)\\ \dot{p} &= -\frac{\partial H}{\partial q}(q,p,r,s)\\ \dot{r} &= 0\\ 0 &= \frac{\partial H}{\partial s}(q,p,r,s)\end{aligned} \tag{8.7}$$

Hence the Casimirs are all state functions only depending on r, while the algebraic constraints take the simple form $\frac{\partial H}{\partial s} = 0$.

The condition that the matrix in (8.6) has full rank is in the canonical coordinate representation equivalent to the partial Hessian matrix $\frac{\partial^2 H}{\partial s^2}$ being invertible. Solving, by the Implicit Function theorem, the algebraic constraints $\frac{\partial H}{\partial s} = 0$ for s as a function $s(q,p,r)$ reduces the DAEs (8.7) to the ODEs

$$\begin{aligned}\dot{q} &= \frac{\partial \bar{H}}{\partial p}(q,p.r)\\ \dot{p} &= -\frac{\partial \bar{H}}{\partial q}(q,p,r)\\ \dot{r} &= 0\end{aligned} \tag{8.8}$$

where $\bar{H}(q,p,r) := H(q,p,r,s(q,p,r))$.

9

Incrementally port-Hamiltonian systems

Recall the definition of a port-Hamiltonian system, cf. Chapter 2. It is defined by a Dirac structure $\mathcal{D} \subset \mathcal{F}_S \times \mathcal{E}_S \times \mathcal{F}_P \times \mathcal{E}_P \times \mathcal{F}_R \times \mathcal{E}_R$, where the flow and effort variables $(f_R, e_R) \in \mathcal{F}_R \times \mathcal{E}_R$ are terminated by an energy-dissipating relation (resistive structure) $\mathcal{R} \subset \mathcal{F}_R \times \mathcal{E}_R$ satisfying the property

$$e_R^T f_R \leq 0 \text{ for all } (f_R, e_R) \in \mathcal{R}.$$

It follows that the *composition* of the Dirac structure $\mathcal{D}$ with the resistive structure $\mathcal{R}$, defined as

$$\begin{aligned} \mathcal{D} \circ \mathcal{R} := \{(f_S, e_S, f_P, e_P) \in \mathcal{F})S \times \mathcal{E}_S \times \mathcal{F}_P \times \mathcal{E}_P \mid \\ \exists (f_R, e_R) \in \mathcal{R} \text{ s.t. } (f_S, e_S, f_P, e_P, f_R, e_R) \in \mathcal{D}\} \end{aligned}$$

satisfies the property

$$e_S^T f_S + e_P^T f_P = -e_R^T f_R \geq 0, \tag{9.1}$$

for all $(f_S, e_S, f_P, e_P) \in \mathcal{D} \circ \mathcal{R}$.

Hence a more general viewpoint on port-Hamiltonian systems is *not* to distinguish between the Dirac structure $\mathcal{D}$ and the resistive

structure $\mathcal{R}$, but instead to start from a general (nonlinear) relation

$$\mathcal{N} := \{(f_S, e_S, f_P, e_P) \in \mathcal{F}_S \times \mathcal{E}_S \times \mathcal{F}_P \times \mathcal{E}_P\}$$

having the property that

$$e_S^T f_S + e_P^T f_P \geq 0, \tag{9.2}$$

for all $(f_S, e_S, f_P, e_P) \in \mathcal{N}$. Thus $\mathcal{N}$ *combines* the Dirac structure $\mathcal{D}$ and the resistive structure $\mathcal{R}$ into a single object.

This leads to two interesting new directions. The first one is the theory of *incrementally port-Hamiltonian systems*, as will be explored in the present chapter based on Camlibel & van der Schaft (2013). The second is the connection of port-Hamiltonian systems with pseudo-gradient systems (generalizing the Brayton-Moser equations for electrical circuits), as will be discussed in Chapter 11.

Remark 9.1. Since in the definition of $\mathcal{N}$ no explicit assumption regarding the linearity of the power-conserving interconnection structure $\mathcal{D}$ is made anymore, this also has a potential towards the modeling of systems where the assumption of linearity appears to be a stumbling block, as in thermodynamical systems, cf. Eberard et al. (2007).

9.1 Incrementally port-Hamiltonian systems

The basic idea of the definition of incrementally port-Hamiltonian systems is to replace the composition $\mathcal{N}$ of a *constant* Dirac structure $\mathcal{D}$ on a linear state space $\mathcal{X}$ and a resistive structure $\mathcal{R}$ by a *maximal monotone relation* $\mathcal{M}$, cf. Rockafellar & Wets (1998).

Definition 9.1. Let $\mathcal{F}$ be a linear space; in the sequel always assumed to be finite-dimensional. A relation $\mathcal{M} \subset \mathcal{F} \times \mathcal{E}$, with $\mathcal{E} = \mathcal{F}^*$, is said to be *monotone* if

$$(v_1 - v_2)^T (u_1 - u_2) \geq 0, \tag{9.3}$$

for all $(u_i, v_i) \in \mathcal{M}$ with $i \in \{1, 2\}$. It is called *maximal monotone* if it is monotone and the implication

$$\mathcal{M}' \text{ is monotone and } \mathcal{M} \subset \mathcal{M}' \implies \mathcal{M} = \mathcal{M}' \tag{9.4}$$

holds.

Furthermore, consider a maximal monotone relation

$$\mathcal{M} \subset \mathcal{F}_S \times \mathcal{E}_S \times \mathcal{F}_P \times \mathcal{E}_P,$$

and a Hamiltonian $H : \mathcal{X} \to \mathbb{R}$, where $\mathcal{X} = \mathcal{F}_S$ is the linear state space. Then the dynamics of the corresponding *incrementally port-Hamiltonian system* is defined as

$$\left(-\dot{x}(t), \frac{\partial H}{\partial x}(x(t)), f_P(t), e_P(t)\right) \in \mathcal{M}, \quad t \in \mathbb{R}. \tag{9.5}$$

It follows that the dynamics of incrementally port-Hamiltonian systems are characterized by the satisfaction of the inequality

$$\begin{aligned}\left(\frac{\partial^T H}{\partial x}(x_1(t)) - \frac{\partial H}{\partial x}(x_2(t)\right)^T (\dot{x}_1(t) - \dot{x}_2(t)) \\ \leq (e_P^1(t) - e_P^2(t))^T (f_P^1(t) - f_P^2(t))\end{aligned} \tag{9.6}$$

along all trajectories

$$\left(-\dot{x}_i(t), \frac{\partial H}{\partial x}(x_i(t)), f_P^i(t), e_P^i(t)\right) \in \mathcal{M}, \quad i \in \{1, 2\}.$$

What is the exact relation between port-Hamiltonian systems and incrementally port-Hamiltonian systems ? If the Dirac structure $\mathcal{D}$ is a constant Dirac structure on a linear state space, and furthermore the port-Hamiltonian system has no resistive structure (and thus is conservative), then the system is also incrementally port-Hamiltonian. This follows from the following proposition.

Proposition 9.1. Every constant Dirac structure $\mathcal{D} \subset \mathcal{F} \times \mathcal{E}$ is maximal monotone.

Proof. Let $(f_i, e_i) \in \mathcal{D}$ with $i = 1, 2$. Since $e^T f = 0$ for all $(f, e) \in \mathcal{D}$, we obtain

$$(e_1 - e_2)^T (f_1 - f_2) = 0.$$

Therefore, $\mathcal{D}$ is monotone on $\mathcal{F} \times \mathcal{E}$. Let $\mathcal{D}'$ be a monotone relation on $\mathcal{F} \times \mathcal{E}$ such that $\mathcal{D} \subset \mathcal{D}'$. Let $(f', e') \in \mathcal{D}'$ and $(f, e) \in \mathcal{D}$. Since $\mathcal{D}'$ is monotone, $\mathcal{D} \subset \mathcal{D}'$, and since $\mathcal{D}$ is a subspace, we have $0 \leq$

$(e' - \alpha e)^T (f' - \alpha f) = e'^T f' - \alpha(e'^T f + e^T f')$ for any $\alpha \in \mathbb{R}$. This means that

$$e'^T f + e^T f' = 0,$$

and hence $(f', e') \in \mathcal{D}^{\perp\!\!\!\perp} = \mathcal{D}$. Therefore, we get $\mathcal{D}' \subset \mathcal{D}$. Since the reverse inclusion already holds, we obtain $\mathcal{D}' = \mathcal{D}$. Consequently, $\mathcal{D}$ is maximal monotone. ■

On the other hand, not every maximal monotone relation is a Dirac structure, even if we restrict attention to maximal monotone relations $\mathcal{M}$ satisfying the additional property $e^T f = 0$ for all $(f, e) \in \mathcal{M}$ (no energy-dissipation), since maximal monotone relations need not be *linear*. We conclude that in the conservative case every port-Hamiltonian system is incrementally port-Hamiltonian, but not the other way around.

In the non-conservative case, the relation between port-Hamiltonian systems and incrementally port-Hamiltonian systems is less simple, as can be seen from the following examples.

Example 9.1 (Mechanical systems with friction)**.** Consider a mechanical system with standard kinetic energy and arbitrary potential energy, subject to friction. The friction characteristic corresponds to a constitutive relation between certain port variables f_R, e_R. Assume for simplicity that f_R, e_R are scalar variables, i.e., consider a single friction component with velocity f_R and friction force $-e_R$. In the case of linear friction $-e_R = df_R$ with $d > 0$, the resulting system is both port-Hamiltonian and incrementally port-Hamiltonian. In the case of a friction characteristic

$$-e_R = F(f_R),$$

the system will be port-Hamiltonian if the graph of the function F is in the first and third quadrant. On the other hand, it will be incrementally port-Hamiltonian if the relation is the function F is a monotonically non-decreasing function. For example, the *Stribeck friction* characteristic defines a port-Hamiltonian system, but *not* an incrementally port-Hamiltonian system.

Example 9.2 (Circuit with tunnel diode)**.** Consider an electrical LC-circuit (possibly with nonlinear capacitors and inductors) together

with a resistor corresponding to an electrical port $f_R = -I, e_R = V$ (current and voltage). For a linear resistor (conductor) $I = GV, G > 0$, the system is both port-Hamiltonian and incrementally port-Hamiltonian. For a nonlinear conductor $I = G(V)$ the system is *port-Hamiltonian* if the graph of the function G is in the first and third quadrant while *incrementally port-Hamiltonian* if G is monotonically non-decreasing. For example, a tunnel diode characteristic

$$I = \Phi(V - V_0) + I_0,$$

for certain positive constants V_0, I_0, and a function $\Phi(z) = \gamma z^3 - \alpha z, \alpha, \gamma > 0$, defines a system which is port-Hamiltonian but *not* incrementally port-Hamiltonian.

Example 9.3 (Sources)**.** Physical systems with *constant sources* are *not* port-Hamiltonian in the sense of a port-Hamiltonian system without external ports, but typically *are* incrementally port-Hamiltonian. Consider any nonlinear LC-circuit with passive resistors and constant voltage and/or current sources, or any arbitrary mechanical system with passive dampers and constant actuation: all are incrementally port-Hamiltonian but not a port-Hamiltonian system without external ports.

Note that physical systems with constant sources often can be also modeled as *shifted* port-Hamiltonian systems, cf. Chapter 7. Furthermore, in Maschke et al. (2000) it is discussed how systems with constant sources can be represented as the interconnection of a port-Hamiltonian system with a *port-Hamiltonian source system* (having a linear Hamiltonian which is not bounded from below, and therefore *not* a passive system).

Example 9.3 can be further generalized and formalized as follows. Consider a port-Hamiltonian system of the form

$$\begin{aligned} \dot{x} &= J(x)\frac{\partial H}{\partial x}(x) + g_R(x)f_R + g(x)f_P, \\ e_R &= g_R^T(x)\frac{\partial H}{\partial x}(x), \\ e_P &= g^T(x)\frac{\partial H}{\partial x}(x), \end{aligned} \tag{9.7}$$

where $J(x) = -J^T(x)$ and the port-variables $-f_R, e_R$ belong to a maximally monotone relation $\mathcal{M}_R$, i.e., a maximal relation with the property

$$(e_{R1} - e_{R2})^T (f_{R1} - f_{R2}) \leq 0, \tag{9.8}$$

for all $(f_{Ri}, e_{Ri}) \in \mathcal{M}_R$ with $i \in \{1, 2\}$.

Clearly for any constant vector c the subspace

$$\mathcal{C} := \{(f_P, e_P) \mid f_P = c\} \tag{9.9}$$

is also a maximal monotone relation. It follows that the product $\mathcal{M}_R \times \mathcal{C}$ is again a maximal monotone relation, and that the system (9.7) for any constant input $f_P = c$ is an incrementally port-Hamiltonian system.

9.2 Connections with incremental and differential passivity

The notion of incrementally port-Hamiltonian systems is related to *incremental passivity* and the recently introduced notion of *differential passivity*. These relations are clear if the Hamiltonian is *quadratic* and nonnegative, i.e., $H(x) = \frac{1}{2} x^T Q x$ for some symmetric positive definite matrix Q. In this case, the inequality (9.6) is equivalent to

$$\begin{aligned} \frac{d}{dt} \frac{1}{2} (x_1(t) - x_2(t))^T Q (x_1(t) - x_2(t)) \\ \leq (e_P^1(t) - e_P^2(t))^T (f_P^1(t) - f_P^2(t)). \end{aligned}$$

This property readily implies that incrementally port-Hamiltonian systems with nonnegative quadratic Hamiltonian as above are both incrementally passive and differentially passive.

Indeed, recall Desoer & Vidyasagar (1975); Pavlov et al. (2004); Angeli (2000) that a system $\dot{x} = f(x, u), y = h(x, u)$ with $x \in \mathbb{R}^n, u, y \in \mathbb{R}^m$ is called *incrementally passive* if there exists a nonnegative function $V : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ such that

$$\frac{d}{dt} V(x_1, x_2) \leq (u_1 - u_2)^T (y_1 - y_2), \tag{9.10}$$

for all $(x_i, u_i, y_i), i = 1, 2$ satisfying $\dot{x} = f(x, u), y = h(x, u)$. Taking $u = f_P, y = e_P$

$$V(x_1, x_2) := \frac{1}{2}(x_1 - x_2)^T Q(x_1 - x_2).$$

we immediately obtain the following result.

Proposition 9.2. An incrementally port-Hamiltonian system with a quadratic Hamiltonian $H(x) = \frac{1}{2}x^T Q x$ with $Q \geq 0$ is *incrementally passive*.

Recall furthermore from Forni & Sepulchre (2013); van der Schaft (2013) the following definition of differential passivity.

Definition 9.2. Consider a nonlinear control system Σ with state space $\mathcal{X}$, affine in the inputs u, and with an equal number of outputs y, given as

$$\begin{aligned} \dot{x} &= f(x) + \sum_{j=1}^{m} g_j(x) u_j, \\ y_j &= H_j(x), \quad j = 1, \ldots, m. \end{aligned} \tag{9.11}$$

The *variational system* along any input-state-output trajectory $t \in [0, T] \mapsto (x(t), u(t), y(t))$ is given by the following time-varying system,

$$\begin{aligned} \delta\dot{x}(t) = \frac{\partial f}{\partial x}(x(t))\delta x(t) + \sum_{j=1}^{m} \frac{\partial g_j}{\partial x}(x(t))u_j(t)\delta x(t) \\ + \sum_{j=1}^{m} \delta u_j g_j(x(t)), \end{aligned} \tag{9.12}$$

$$\delta y_j(t) = \frac{\partial H_j}{\partial x}(x(t))\delta x(t)\,, \quad j = 1, \ldots, m\,,$$

with state $\delta x \in \mathbb{R}^n$, where $\delta u = (\delta u_1, \ldots, \delta u_m)$ denote the inputs of the variational system and $\delta y = (\delta y_1, \ldots, \delta y_m)$ the outputs. Then the system (9.11) is called *differentially passive* if the system together with all its variational systems is passive with respect to the supply rate

$(\delta u)^T \delta y$, i.e., if there exists a function $P : T\mathcal{X} \to \mathbb{R}^+$ (called the *differential storage function*) satisfying

$$\frac{d}{dt} P \leq (\delta u)^T \delta y, \tag{9.13}$$

for all $x, u, \delta u$.

The following proposition readily follows, cf. van der Schaft (2013).

Proposition 9.3. An incrementally port-Hamiltonian system with quadratic Hamiltonian $H(x) = \frac{1}{2} x^T Q x$ with $Q \geq 0$ is differentially passive.

Proof. Consider the *infinitesimal* version of (9.6). In fact, let (f_P^1, e_P^1, x_1) and (f_P^2, e_P^2, x_2) be two triples of system trajectories arbitrarily near each other. Taking the limit we deduce from (9.6)

$$(\partial x)^T \frac{\partial^2 H}{\partial x^2}(x) \partial \dot{x} \leq (\partial e_P)^T \partial f_P, \tag{9.14}$$

where ∂x denotes the variational state, and $\partial f_P, \partial e_P$ the variational inputs and outputs). If the Hamiltonian H is a quadratic function $H(x) = \frac{1}{2} x^T Q x$ then the left-hand side of the inequality (9.14) is equal to $\frac{d}{dt} \frac{1}{2} (\partial x)^T Q \partial x$, and hence amounts to the differential dissipativity inequality

$$\frac{d}{dt} \frac{1}{2} (\partial x)^T Q \partial x \leq (\partial e_P)^T \partial f_P, \tag{9.15}$$

implying that the incrementally port-Hamiltonian system is differentially passive, with differential storage function $\frac{1}{2} (\partial x)^T Q \partial x$. ∎

9.3 Composition of maximal monotone relations

A cornerstone of port-Hamiltonian systems theory is the fact that the power-conserving interconnection of port-Hamiltonian systems defines again a port-Hamiltonian system, cf. Chapter 6. As we have seen this is based on the fact that the composition of Dirac structures is again a Dirac structure. In the present section we will show that the same property holds for incrementally port-Hamiltonian systems, but

now based on the fact that the composition of maximal monotone relations is again maximal monotone.

Consider two maximal monotonous relations $\mathcal{M}_a \subset \mathcal{F}_a \times \mathcal{F}_a^* \times \mathcal{V}_a \times \mathcal{V}_a^*$, with typical element denoted by (f_a, e_a, v_a, w_a) and $\mathcal{M}_b \subset \mathcal{F}_b \times \mathcal{F}_b^* \times \mathcal{V}_b \times \mathcal{V}_b^*$, with typical element denoted by (f_b, e_b, v_b, w_b), where $\mathcal{V}_a = \mathcal{V}_b = \mathcal{V}$, and thus $\mathcal{V}_a^* = \mathcal{V}_b^* = \mathcal{V}^*$ (shared flow and effort variables). Define as before the *composition* of $\mathcal{M}_a$ and $\mathcal{M}_b$, denoted as $\mathcal{M}_a \circ \mathcal{M}_b$, by

$$\begin{aligned}\mathcal{M}_a \circ \mathcal{M}_b := \Big\{ & (f_a, e_a, f_b, e_b) \in \mathcal{F}_a \times \mathcal{F}_a^* \times \mathcal{F}_b \times \mathcal{F}_b^* \mid \\ & \exists v \in \mathcal{V}, w \in \mathcal{V}^* \text{ s.t. } (f_a, e_a, v, w) \in \mathcal{F}_a \times \mathcal{F}_a^* \times \mathcal{V} \times \mathcal{V}^*, \\ & (f_b, e_b, -v, w) \in \mathcal{F}_b \times \mathcal{F}_b^* \times \mathcal{V} \times \mathcal{V}^* \Big\}.\end{aligned}$$

Thus the composition of $\mathcal{M}_a$ and $\mathcal{M}_b$ is obtained by imposing on the vectors $(f_a, e_a, v_a, w_a) \in \mathcal{M}_a$ and $(f_b, e_b, v_b, w_b) \in \mathcal{M}_b$ the interconnection constraints

$$v_a = -v_b, \quad w_a = w_b, \tag{9.16}$$

and looking at the resulting vectors $(f_a, e_a, f_b, e_b) \in \mathcal{F}_a \times \mathcal{F}_a^* \times \mathcal{F}_b \times \mathcal{F}_b^*$.

The main result of this section is that, whenever $\mathcal{M}_a \circ \mathcal{M}_b$ satisfies a technical condition, then the composition $\mathcal{M}_a \circ \mathcal{M}_b$ is again a maximal monotone relation. The key ingredient in the proof is the following theorem from Rockafellar & Wets (1998) [Ex. 12.46].

Theorem 9.1. Let $\mathcal{M} \subset \mathcal{F}_\alpha \times \mathcal{F}_\alpha^* \times \mathcal{F}_\beta \times \mathcal{F}_\beta^*$ be maximal monotone. Assume that the reduced relation (with $\bar{e_\beta}$ a constant vector)

$$\mathcal{M}_r := \{(f_\alpha, e_\alpha) \mid \exists f_\beta \text{ s.t. } (f_\alpha, e_\alpha, f_\beta, e_\beta = \bar{e_\beta})\} \tag{9.17}$$

is such that there exists $\bar{e_\alpha}$ for which $(\bar{e_\alpha}, \bar{e_\beta})$ is in the relative interior of the projection of $\mathcal{M}$ on the space of efforts $\{(e_\alpha, e_\beta)\}$. Then $\mathcal{M}_r$ is maximal monotone.

This theorem can be applied to the situation at hand after applying the following transformation. Define

$$\begin{aligned} y_v &:= \frac{v_a + v_b}{\sqrt{2}},\ z_v := \frac{v_a - v_b}{\sqrt{2}},\ y_v, z_v \in \mathcal{V} \\ y_w &:= \frac{w_a + w_b}{\sqrt{2}},\ z_w := \frac{w_a - w_b}{\sqrt{2}},\ y_w, z_w \in \mathcal{V}^*. \end{aligned}$$

By direct computation one obtains

$$\begin{aligned}&\langle y_{w1} - y_{w2} \mid y_{v1} - y_{v2}\rangle + \langle z_{w1} - z_{w2} \mid z_{v1} - z_{v2}\rangle \\ &\quad = \langle w_{a1} - w_{a2} \mid v_{a1} - v_{a2}\rangle + \langle w_{b1} - w_{b2} \mid v_{b1} - v_{b2}\rangle.\end{aligned} \tag{9.18}$$

Theorem 9.2. Let $\mathcal{M}_a \subset \mathcal{F}_a \times \mathcal{F}_a^* \times \mathcal{V}_a \times \mathcal{V}_a^*$ and $\mathcal{M}_b \subset \mathcal{F}_b \times \mathcal{F}_b^* \times \mathcal{V}_b \times \mathcal{V}_b^*$ be maximal monotone relations where $\mathcal{M}_a \circ \mathcal{M}_b$ is such that there exists $\bar{e}_a, \bar{e}_b$ for which $(\bar{e}_a, \bar{e}_b, y_v = 0, z_w = 0)$ is in the relative interior of the projection of the direct sum $\mathcal{M}_a \oplus \mathcal{M}_b$. Then $\mathcal{M}_a \circ \mathcal{M}_b$ is maximal monotone.

Proof. It is evident that the direct sum $\mathcal{M}_a \oplus \mathcal{M}_b \subset \mathcal{F}_a \times \mathcal{F}_a^* \times \mathcal{V}_a \times \mathcal{V}_a^* \times \mathcal{F}_b \times \mathcal{F}_b^* \times \mathcal{V}_b \times \mathcal{V}_b^*$ of $\mathcal{M}_a$ and $\mathcal{M}_b$ defined by

$$\begin{aligned}\mathcal{M}_a \oplus \mathcal{M}_b = \Big\{&(f_a, e_a, f_b, e_b, v_a, w_a, v_b, w_b) \mid \\ &(f_a, e_a, v_a, w_a) \in \mathcal{M}_a, (f_b, e_b, v_b, w_b) \in \mathcal{M}_b\Big\}\end{aligned}$$

is a maximal monotone relation. By (9.18) we deduce that $\mathcal{M}_a \oplus \mathcal{M}_b$ is also maximal monotone with respect to the coordinates $(f_a, e_a, f_b, e_b, y_v, y_w, z_v, z_w)$. It now follows from Theorem 9.1 (with $e_\beta = (y_v, z_w)$) that $\mathcal{M}_a \circ \mathcal{M}_b$ is maximal monotone. ∎

Remark 9.2. The same reasoning provides an alternative proof of the fact that the composition of two Dirac structures is a Dirac structure.

Corollary 9.3. Consider two incrementally port-Hamiltonian systems Σ_a and Σ_b with external port variables respectively $(f_P^a, e_P^a) \in \mathcal{V} \times \mathcal{V}^*$ and $(f_P^b, e_P^b) \in \mathcal{V} \times \mathcal{V}^*$, and with Hamiltonians respectively H_a and H_b, interconnected by the power-conserving interconnection constraints

$$f_P^a + f_P^b + f_P = 0, \quad e_P^a = e_P^b = e_P.$$

Then the interconnected system is incrementally port-Hamiltonian with total Hamiltonian $H_a + H_b$ and external port variables f_P, e_P.

Corollary 9.4. Let the Dirac structure $\mathcal{D}$ be constant, and the resistive structure $\mathcal{R}$ be maximal monotone. Then the composition $\mathcal{D} \circ \mathcal{R}$ is maximal monotone. Hence any port-Hamiltonian system with a constant Dirac structure and maximal monotone resistive structure $\mathcal{R}$ is also incrementally port-Hamiltonian.

10

Input-output Hamiltonian systems

In this chapter we discuss the relationships between port-Hamiltonian systems and *input-output Hamiltonian systems*, as initiated in the groundbreaking paper Brockett (1977), and further explored in e.g. van der Schaft (1984, 1982a,b); Crouch & van der Schaft (1987); Nijmeijer & van der Schaft (1990). Furthermore, we will see that in the linear case that input-output Hamiltonian systems are very close to systems with *negative-imaginary* transfer matrices Lanzon & Petersen (2008, 2010) and in the nonlinear case to systems with *counterclockwise input-output dynamics* Angeli (2006). This chapter is partly based on van der Schaft (2011)

10.1 Input-output Hamiltonian systems with dissipation

Consider a standard input-state-output port-Hamiltonian system (for simplicity without feedthrough term), cf. eqn. (4.10)

$$\begin{aligned} \dot{x} &= [J(x) - R(x)]\frac{\partial H}{\partial x}(x) + g(x)u,\ x \in \mathcal{X}, u \in \mathbb{R}^m \\ y &= g^T(x)\frac{\partial H}{\partial x}(x),\ y \in \mathbb{R}^m, \end{aligned} \tag{10.1}$$

where the $n \times n$ matrices $J(x), R(x)$ depend smoothly on x and satisfy

$$J(x) = -J^T(x),\ R(x) = R^T(x) \geq 0. \tag{10.2}$$

Now suppose that the input matrix $g(x)$ satisfies an *integrability condition*[1], in the sense that there exists a mapping $C : \mathcal{X} \to \mathbb{R}^m$ such that[2]

$$g(x) = -[J(x) - R(x)]\frac{\partial C^T}{\partial x}(x). \tag{10.3}$$

Then the system equations can be rewritten as

$$\begin{aligned}
\dot{x} &= [J(x) - R(x)] \left(\frac{\partial H}{\partial x}(x) - \frac{\partial C^T}{\partial x}(x)u \right), \\
y &= \left(\frac{\partial C^T}{\partial x}(x) \right)^T [J(x) + R(x)]\frac{\partial H}{\partial x}(x).
\end{aligned}$$

This suggests to define the *new output* vector $z = C(x) \in \mathbb{R}^m$, leading to the following system definition.

Definition 10.1. A system, with local coordinates $x = (x_1, \cdots, x_n)$ for some n-dimensional state space manifold $\mathcal{X}$, given by

$$\begin{aligned}
&\dot{x} = [J(x) - R(x)] \left(\frac{\partial H}{\partial x}(x) - \frac{\partial C^T}{\partial x}(x)u \right),\ u \in \mathbb{R}^m, \\
&z = C(x),\ y \in \mathbb{R}^m,
\end{aligned} \tag{10.4}$$

where the $n \times n$ matrices $J(x)$ and $R(x)$ depend smoothly on x and satisfy

$$J(x) = -J^T(x),\ R(x) = R^T(x) \geq 0, \tag{10.5}$$

is called an *affine input-output Hamiltonian system with dissipation* (briefly, affine IOHD system), with Hamiltonian $H : \mathcal{X} \to \mathbb{R}$, and output mapping $C : \mathcal{X} \to \mathbb{R}^m$.

[1]Notice that this amounts to assuming that each input vector field g_j is a Hamiltonian/gradient vector field with respect to the function C_j and the mixed geometric structure defined by $J(x) - R(x)$, sometimes called a Leibniz structure Morrison (1986); Ortega & Planas-Bielsa (2004).

[2]For a mapping $C : \mathbb{R}^n \to \mathbb{R}^m$ we denote by $\frac{\partial C^T}{\partial x}(x)$ the $n \times m$ matrix whose j-th column consists of the partial derivatives of the j-th component function C_j.

Remark 10.1. The definition of an IOHD system as given above is a generalization of the definition of an affine input-output Hamiltonian system as originally proposed in Brockett (1977) and studied in e.g. van der Schaft (1984, 1982a,b); Crouch & van der Schaft (1987). In fact, it reduces to this definition in case $R = 0$ and J defines a *symplectic form* (in particular, has full rank). The components of the output mapping $C : \mathcal{X} \to \mathbb{R}^m$ are called *interaction Hamiltonians*.

The new output z can be sometimes regarded as an integrated version of the old output y. In fact,

$$\dot{z} = \left(\frac{\partial C^T}{\partial x}(x)\right)^T [J(x) - R(x)] \left[\frac{\partial H}{\partial x}(x) - \frac{\partial C^T}{\partial x}(x)u\right],$$

and thus if additionally the following conditions are satisfied[3]

$$\left(\frac{\partial C^T}{\partial x}(x)\right) R(x) = 0, \ \left(\frac{\partial C^T}{\partial x}(x)\right)^T J(x) \frac{\partial C^T}{\partial x}(x) = 0, \tag{10.6}$$

then $\dot{z} = y$.

Even if the conditions (10.6) do *not* hold, and thus $\dot{z} \neq y$, then $\dot{z}$ is still an output of a *related* input-state-output port-Hamiltonian system. This follows from the following computation. Since $u^T(\frac{\partial C^T}{\partial x}(x))^T J(x) \frac{\partial C^T}{\partial x}(x)u = 0$ for all $u \in \mathbb{R}^m$ by skew-symmetry of $J(x)$, the following identity holds along trajectories of (10.4) (leaving out arguments x)

$$\frac{d}{dt}H = u^T\dot{z} - \left[\left(\frac{\partial H}{\partial x}\right)^T \ u^T\right] \begin{bmatrix} R & -R\frac{\partial C^T}{\partial x} \\ -\left(\frac{\partial C^T}{\partial x}\right)^T R & \left(\frac{\partial C^T}{\partial x}\right)^T R \frac{\partial C^T}{\partial x} \end{bmatrix} \begin{bmatrix} \frac{\partial H}{\partial x} \\ u \end{bmatrix}.$$

Noting that

$$\begin{bmatrix} R & -R\frac{\partial C^T}{\partial x} \\ -\left(\frac{\partial C^T}{\partial x}\right)^T R & \left(\frac{\partial C^T}{\partial x}\right)^T R \frac{\partial C^T}{\partial x} \end{bmatrix} = \begin{bmatrix} I_n \\ -\left(\frac{\partial C^T}{\partial x}\right)^T \end{bmatrix} R \left[I_n \ -\frac{\partial C^T}{\partial x}\right] \geq 0$$

this proves the following proposition.

[3]Note that for $m = 1$ the second condition is always satisfied by skew-symmetry of $J(x)$.

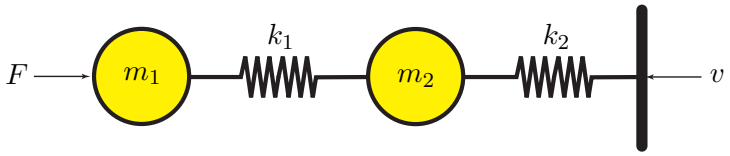


Figure 10.1: Mass-spring system connected to a moving wall.

Proposition 10.1. The affine IOHD system (10.4) with output equation

$$\tilde{y} := \dot{z} = \left(\frac{\partial C^T}{\partial x}(x)\right)^T [J(x) - R(x)] \left(\frac{\partial H}{\partial x}(x) - \frac{\partial C^T}{\partial x}(x)u\right)$$

defines an input-state-output port-Hamiltonian system with feedthrough term.

Example 10.1 (Mixed force and velocity inputs)**.** Consider a mass-spring system consisting of two masses (with masses m_1 and m_2) with an intermediate spring (with spring constant k_1). The right mass m_2 is linked by another spring (with spring constant k_2) to a movable wall with velocity v, while the left mass m_1 is directly actuated by an external force F; see Figure 10.1. The corresponding input-state-output port-Hamiltonian system is given as

$$\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{p}_1 \\ \dot{p}_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} k_1 q_1 \\ k_2 q_2 \\ \frac{p_1}{m_1} \\ \frac{p_2}{m_2} \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} v + \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} F,$$

$$y_1 = k_2 q_2,$$
$$y_2 = \frac{p_1}{m_1},$$

with q_1 and q_2 the elongations of spring k_1 and k_2 respectively, and p_1, p_2 the momenta of the two masses.

This system can be integrated to an input-output Hamiltonian system (10.4) by defining the outputs

$$z_1 = C_1(q_1, q_2, p_1, p_2) = -p_1 - p_2, \; z_2 = C_1(q_1, q_2, p_1, p_2) = q_1 + q_2.$$

Then the system with differentiated outputs $\tilde{y}_1 = \dot{z}_1, \tilde{y}_2 = \dot{z}_2$ is a port-Hamiltonian input-state-output system (with feedthrough term), but having outputs $\tilde{y}_1, \tilde{y}_2$ differing from the original outputs y_1, y_2. In fact,

$$\tilde{y}_1 = \dot{z}_1 = k_2 q_2 + F, \quad \tilde{y}_2 = \dot{z}_2 = \frac{p_1}{m_1} + v.$$

Example 10.2. Consider a linear system

$$\begin{aligned} \dot{x} &= Ax + Bu, \quad x \in \mathbb{R}^n, u \in \mathbb{R}^m, \\ y &= Cx + Du, \quad y \in \mathbb{R}^m, \end{aligned} \tag{10.7}$$

with transfer matrix $G(s) = C(Is - A)^{-1}B + D$. In Lanzon & Petersen (2008, 2010) $G(s)$ is called *negative imaginary*[4] if the transfer matrix $H(s) := s(G(s) - D)$ is *positive real* and $D = D^T$. In Angeli (2006) the same notion (mostly for the case $D = 0$) was coined as *counterclockwise input-output dynamics*.

In van der Schaft (2011) it has been shown that the system (10.7) has negative imaginary transfer matrix if and only if it can be written as

$$\begin{aligned} \dot{x} &= (J - R)(Qx - C^T u), \\ y &= Cx + Du, \ D = D^T, \end{aligned} \tag{10.8}$$

for some matrices Q, J, R of appropriate dimensions satisfying

$$Q = Q^T, \ J = -J^T, \ R = R^T \geq 0, \tag{10.9}$$

with $Q > 0$. We conclude that a linear system (10.7) has negative imaginary transfer matrix if and only it is a *linear input-output Hamiltonian system with dissipation* (10.8) satisfying $Q > 0$.

A typical instance of a linear IOHD system is a linear mechanical system with co-located position sensors and force actuators, represented in Hamiltonian state space form (with q denoting the position vector and p the momentum vector) as

$$\begin{aligned} \begin{bmatrix} \dot{q} \\ \dot{p} \end{bmatrix} &= \begin{bmatrix} 0_n & I_n \\ -I_n & 0_n \end{bmatrix} \begin{bmatrix} K & N \\ N^T & M^{-1} \end{bmatrix} \begin{bmatrix} q \\ p \end{bmatrix} + \begin{bmatrix} 0 \\ L^T \end{bmatrix} u, \\ y &= Lq. \end{aligned} \tag{10.10}$$

[4]The terminology 'negative imaginary', stems, similarly to 'positive real', from the Nyquist plot interpretation for single-input single-output systems. For the precise definition in the frequency domain we refer to Lanzon & Petersen (2008); Angeli (2006).

Clearly (10.10) defines a linear IOHD system with Hamiltonian

$$H(q,p) = \frac{1}{2}q^T K q + \frac{1}{2}p^T M^{-1} p + q^T N p, \tag{10.11}$$

where the first term is the total potential energy (with K the compliance matrix), and the second term is the kinetic energy (with M denoting the mass matrix). The term $q^T N p$ corresponds to possible 'gyroscopic forces'.

The definition of an affine IOHD system suggests the following generalization.

Definition 10.2. A (general) input-output Hamiltonian system with dissipation (IOHD system) is defined as a system of the form

$$\begin{aligned} \dot{x} &= [J(x) - R(x)\frac{\partial H}{\partial x}(x,u),\ u \in \mathbb{R}^m, \\ z &= -\frac{\partial H}{\partial u}(x,u),\ y \in \mathbb{R}^m, \end{aligned} \tag{10.12}$$

for some function $H(x,u)$, with $R(x), J(x)$ satisfying (10.5).

Obviously, this definition reduces to Definition 10.1 by taking $H(x,u) = H(x) - u^T C(x)$. For $R = 0$ and J defining a *symplectic form* the definition of a general IOHD system amounts to the definition of an input-output Hamiltonian system given in Brockett (1977), and explored in e.g. van der Schaft (1984, 1982a,b); Nijmeijer & van der Schaft (1990).

10.1.1 Dc-gain of IOHD systems

Specializing the approach of Angeli (2007) to the Hamiltonian case we can define the following notion of *dc-gain* for an IOHD system. Consider a general IOHD system (10.12) with Hamiltonian $H(x,u)$. Assume that for any constant input $\bar{u}$ there exists a unique $\bar{x}$ such that

$$\frac{\partial H}{\partial x}(\bar{x},\bar{u}) = 0. \tag{10.13}$$

It follows that $\bar{x}$ is an equilibrium of the system for $u = \bar{u}$. Define

$$\bar{y} = \frac{\partial H}{\partial u}(\bar{x},\bar{u}). \tag{10.14}$$

Then, see e.g. Wall (1977), eqns. (10.13,10.14) define a Lagrangian submanifold in the space of steady state outputs and inputs $(\bar{y}, \bar{u}) \in \mathcal{Y} \times \mathcal{U}$. Assuming additionally that this Lagrangian submanifold can be parametrized by the $\bar{u}$ variables, then there exists (locally) a generating function K such that the relation between $\bar{u}$ and $\bar{y}$ is described as

$$\bar{y} = \frac{\partial K}{\partial \bar{u}}(\bar{u}). \tag{10.15}$$

We call this relation the static input-output response or *dc-gain* of the IOHD system. Note that this dc-gain for IOHD systems enjoys an intrinsic symmetry property (reciprocity), and is solely determined by the Hamiltonian function $H(x, u)$.

Remark 10.2. In case of the linear IOHD system (10.8) the dc-gain amounts to the symmetric linear map $\bar{y} = \left(CQ^{-1}C^T + D\right)\bar{u}$.

10.2 Positive feedback interconnection and stability

We have seen before, cf. Chapter 6, that the basic interconnection property of port-Hamiltonian systems is the fact that the power-conserving interconnection of port-Hamiltonian systems again defines a port-Hamiltonian system, with Dirac structure being the composition of the Dirac structures of the composing port-Hamiltonian systems, and Hamiltonian function and resistive structure being the 'sum' of the respective Hamiltonian functions and resistive structures. A particular instance of a power-conserving interconnection is the standard *negative* feedback interconnection of two input-state-output systems given as

$$u_1 = -y_2 + e_1, \quad u_2 = y_1 + e_2,$$

where e_1, e_2 are new external inputs.

In this section we will discuss the interconnection theory for IOHD systems, and show that IOHD systems are invariant under *positive* feedback interconnection, leading to important consequences for stability analysis, cf. Section 10.2.1. First of all, consider two *affine* IOHD

systems

$$\begin{aligned}\dot{x}_i &= [J_i(x_i) - R_i(x_i)]\left(\frac{\partial H_i}{\partial x_i}(x_i) - \frac{\partial C_i^T}{\partial x_i}(x_i)u_i\right),\ u_i \in \mathbb{R}^m,\\ y_i &= C_i(x_i),\ y \in \mathbb{R}^m,\ i = 1, 2,\end{aligned} \tag{10.16}$$

interconnected by the *positive feedback interconnection*

$$u_1 = y_2 + e_1,\ u_2 = y_1 + e_2. \tag{10.17}$$

The system resulting from interconnection is the affine IOHD system

$$\begin{aligned}\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} &= \left(\begin{bmatrix}J_1(x_1) & 0\\ 0 & J_2(x_2)\end{bmatrix} - \begin{bmatrix}R_1(x_1) & 0\\ 0 & R_2(x_2)\end{bmatrix}\right)\\ &\quad \times \left(\begin{bmatrix}\dfrac{\partial H_{\rm int}}{\partial x_1}(x_1, x_2)\\ \dfrac{\partial H_{\rm int}}{\partial x_2}(x_1, x_2)\end{bmatrix} - \begin{bmatrix}\dfrac{\partial C_1^T}{\partial x_1}(x_1) & 0\\ 0 & \dfrac{\partial C_2^T}{\partial x_2}(x_2)\end{bmatrix}\begin{bmatrix}e_1\\ e_2\end{bmatrix}\right),\\ \begin{bmatrix}y_1\\ y_2\end{bmatrix} &= \begin{bmatrix}C_1(x_1)\\ C_2(x_2)\end{bmatrix},\end{aligned} \tag{10.18}$$

with *interconnected Hamiltonian* $H_{\rm int}$ given by

$$H_{\rm int}(x_1, x_2) := H_1(x_1) + H_2(x_2) - C_1^T(x_1)C_2(x_2). \tag{10.19}$$

We conclude that the interconnected Hamiltonian $H_{\rm int}$ results from addition of the individual Hamiltonians H_1 and H_2, *together with* an additional coupling term $-C_1^T(x_1)C_2(x_2)$. On the other hand, the Poisson and dissipation structures of the interconnected system are just the direct sum of the terms of the two composing subsystems. The situation is thus opposite to *negative* feedback interconnection of port-Hamiltonian systems: in this case the Hamiltonian is the sum of the Hamiltonians while the Dirac structure is determined by the Dirac structures of the two systems *together* with a coupling term.

This positive feedback interconnection property of affine IOHD systems extends to general IOHD systems as follows. For simplicity of notation take $e_1 = 0, e_2 = 0$ (no external inputs). The positive feedback interconnection of two general IOHD systems with Hamiltonians $H_i(x_i, u_i)$ results (under regularity assumptions) in a nonlinear

IOHD system, where the interconnected Hamiltonian $H_{\text{int}}(x_1, x_2)$ is constructed as follows. The functions $H_i(x_i, u_i)$ are *generating functions* for two Lagrangian submanifolds Abraham & Marsden (1994); Wall (1977); van der Schaft (1984) defined as

$$z_i = \frac{\partial H_i}{\partial x_i}(x_i, u_i),\ y_i = -\frac{\partial H_i}{\partial u_i}(x_i, u_i),\ i = 1, 2.$$

The composition of these two Lagrangian submanifolds through the positive feedback interconnection $u_1 = y_2$, $u_2 = y_1$ defines a subset in the x_1, x_2, z_1, z_2 variables, which is under a transversality condition Guillemin & Sternberg (1979) again a submanifold, and in fact Hörmander (1971); Guillemin & Sternberg (1979) is again a Lagrangian submanifold. Assuming additionally that this resulting Lagrangian submanifold can be parametrized by the x_1, x_2 variables (this corresponds to well-posedness of the interconnection), it thus possesses (at least locally) a generating function $H_{\text{int}}(x_1, x_2)$.

10.2.1 Stability of interconnected IOHD systems

Stability analysis of the positive feedback interconnection of IOHD systems is quite different from the stability analysis of the negative feedback interconnection of port-Hamiltonian systems. This directly stems from the fact that while for negative feedback interconnection of two port-Hamiltonian systems the resulting Hamiltonian is just the sum of the Hamiltonians of the two systems, the Hamiltonian of the positive feedback interconnection of two IOHD systems is more complicated, as explained above (Section 10.2). For clarity of exposition we will restrict ourselves in this subsection to *affine* input-output Hamiltonian systems with dissipation.

Proposition 10.2. Consider two affine IOHD systems with equilibria x_1^*, x_2^* satisfying $\frac{\partial H_1}{\partial x_1}(x_1^*) = 0, \frac{\partial H_2}{\partial x_2}(x_2^*) = 0$ and $C_1(x_1^*) = 0, C_2(x_2^*) = 0$. Then (x_1^*, x_2^*) is a stable equilibrium of the interconnected affine IOHD system (10.18) if the interconnected Hamiltonian H_{int} given by (10.19) has a strict minimum at the origin (x_1^*, x_2^*). A sufficient condition for this is that the Hessian matrices $\frac{\partial^2 H_1}{\partial x_1^2}(x_1^*), \frac{\partial^2 H_2}{\partial x_2^2}(x_2^*)$ are positive-definite, and furthermore the following coupling condition holds on

the linearized system

$$\lambda_{\max}\left(\frac{\partial^T C_1}{\partial x_1}(x_1^*)\left(\frac{\partial^2 H_1}{\partial x_1^2}(x_1^*)\right)^{-1}\frac{\partial C_1^T}{\partial x_1}(x_1^*)\right.$$
$$\left.\times\ \frac{\partial^T C_2}{\partial x_2}(x_2^*)\left(\frac{\partial^2 H_2}{\partial x_2^2}(x_2^*)\right)^{-1}\frac{\partial C_2^T}{\partial x_2}(x_2^*)\right) < 1. \tag{10.20}$$

Example 10.3. The dc-gain of the linear IOHD system (10.10) is given as $LK^{-1}L^T$, and thus only depends on the compliance matrix K (e.g., the spring constants) and the collocated sensor/actuator locations. Note that in this case positive feedback amounts to *positive position feedback*, while negative feedback of $z = \dot{y} = LM^{-1}p = L\dot{q}$ corresponds to negative velocity feedback; see also Lanzon & Petersen (2010).

Remark 10.3. As in (Angeli (2006), Theorem 6) the interconnected Hamiltonian $H_{\rm int}(x_1, x_2)$ can be also used for showing *boundedness* of solutions of the interconnected system; this is e.g. guaranteed if $H_{\rm int}(x_1, x_2)$ is radially unbounded. Furthermore, it leads to a *bifurcation* perspective on *multi-stability* as follows. Consider two IOHD systems with equilibria x_1^*, x_2^* corresponding to *strict global minima* of $H_1(x_1)$, respectively $H_2(x_2)$. Then the parametrized positive feedback

$$u_1 = ky_2, \quad u_2 = ky_1, \tag{10.21}$$

for $k \geq 0$ results in an interconnected Hamiltonian $H_{\rm int}^k(x_1, x_2)$, which for k small will have (by continuity) a strict minimum at (x_1^*, x_2^*), corresponding to a stable equilibrium. By increasing k the shape of $H_{\rm int}^k(x_1, x_2)$ is going to change, possibly resulting in multiple local minima, and thus multiple stable equilibria. In a general, non-Hamiltonian, setting this has been studied in Angeli (2007), where conditions were derived for *multi-stability* of the resulting interconnected system, meaning that for generic initial conditions the system trajectories will always converge to one of those stable equilibria.

11

Pseudo-gradient representations

In this chapter, which is largely based on the work presented in van der Schaft (2011), it will be demonstrated that a special subclass of port-Hamiltonian systems, namely reciprocal port-Hamiltonian systems,[1] can be naturally related with systems of the form

$$\begin{aligned} Q(z)\dot{z} &= -\frac{\partial V}{\partial z}(z,u), \\ y &= -\frac{\partial V}{\partial u}(z,u), \end{aligned} \tag{11.1}$$

where z are local coordinates for some n-dimensional state space manifold $\mathcal{Z}$, V is a potential function, and the matrix $Q(z)$ is a non-singular symmetric matrix. In case $Q(z)$ is definite, the system (11.1) defines a *gradient* system and Q can be considered a Riemannian metric. In case $Q(z)$ is indefinite, gradient systems of the form (11.1) are commonly referred to as *pseudo-gradient* systems with respect to a *pseudo*-Riemannian metric. Pseudo-gradient system establish an important class of nonlinear systems, especially for systems with nonlinear resistive relationships. Furthermore, the pair (Q, V) can be advantageously

[1] Roughly speaking, reciprocal (port-Hamiltonian) systems are systems that do not contain essential gyrators Breedveld (1984).

used as an alternative to generate a family of Lyapunov functions.

11.1 Towards the Brayton-Moser equations

Consider an input-state-output port-Hamiltonian system

$$\begin{aligned} \dot{x} &= \big[J(x) - R(x)\big]\frac{\partial H}{\partial x}(x) + g(x)u, \\ y &= g^T(x)\frac{\partial H}{\partial x}(x), \end{aligned} \tag{11.2}$$

with $x \in \mathcal{X}$, interconnection structure $J(x) = -J^T(x)$, resistive structure $R(x) = R^T(x)$, and Hamiltonian $H : \mathcal{X} \to \mathbb{R}$. Define the *co-energy* variables $z := \frac{\partial H}{\partial x}(x)$, and suppose that the mapping from the *energy* variables x to the co-energy variables z is invertible, such that

$$x = \frac{\partial H^*}{\partial z}(z),$$

where $H^*(z)$ represents the *co-Hamiltonian*, defined through the Legendre transformation of $H(x)$ given by $H^*(z) = z^T x - H(x)$, where x is solved from $z := \frac{\partial H}{\partial x}(x)$. Then, the dynamics (11.2) can also be expressed in terms of the co-energy variables z as

$$\begin{aligned} \frac{\partial^2 H^*}{\partial z^2}(z)\dot{z} &= \big[J(x) - R(x)\big]z + g(x)u, \\ y &= g^T(x)z. \end{aligned} \tag{11.3}$$

Now assume that there exists coordinates x_1 and x_2 such that

$$J(x) = \begin{bmatrix} 0 & -B(x) \\ B^T(x) & 0 \end{bmatrix},\ R(x) = \begin{bmatrix} R_1(x) & 0 \\ 0 & R_2(x) \end{bmatrix},\ g(x) = \begin{bmatrix} g_1(x) \\ 0 \end{bmatrix},$$

and that the Hamiltonian can be decomposed into

$$H(x_1, x_2) = H_1(x_1) + H_2(x_2).$$

Then, the system in co-energy variables (11.3) takes the form

$$\begin{bmatrix} \frac{\partial^2 H_1^*}{\partial z_1^2}(z) & 0 \\ 0 & \frac{\partial^2 H_2^*}{\partial z_2^2}(z) \end{bmatrix} \begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = - \begin{bmatrix} R_1(x) & B(x) \\ -B^T(x) & R_2(x) \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} g_1(x) \\ 0 \end{bmatrix} u,$$

$$y = g_1^T(x)z_1.$$

Furthermore, assuming that there exist functions $P_1(z_1)$ and $P_2(z_2)$ such that

$$R_1(x)z_1 = \frac{\partial P_1}{\partial z_1}(z_1),$$
$$-R_2(x)z_2 = \frac{\partial P_2}{\partial z_2}(z_2),$$

we can define the potential function

$$P(z) = P_1(z_1) + P_2(z_2) + P_{12}(z_1, z_2),$$

where $P_{12}(z_1, z_2) = z_1^T B(x) z_2$. Consequently, the system (11.2) is equivalent to the nonlinear pseudo-gradient system

$$\begin{bmatrix} \frac{\partial^2 H_1^*}{\partial z_1^2}(z) & 0 \\ 0 & -\frac{\partial^2 H_2^*}{\partial z_2^2}(z) \end{bmatrix} \begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = - \begin{bmatrix} \frac{\partial P}{\partial z_1}(z) \\ \frac{\partial P}{\partial z_2}(z) \end{bmatrix} + \begin{bmatrix} g_1(x) \\ 0 \end{bmatrix} u, \tag{11.4}$$
$$y = g_1^T(x) z_1,$$

which, if g_1 is constant, is equivalent to (11.1) by noting that

$$V(z, u) = P(z) - z_1^T g_1 u,$$

and defining the pseudo-Riemannian metric

$$Q(z) = \begin{bmatrix} \frac{\partial^2 H_1^*}{\partial z_1^2}(z) & 0 \\ 0 & -\frac{\partial^2 H_2^*}{\partial z_2^2}(z) \end{bmatrix}. \tag{11.5}$$

Gradient systems of the form (11.4) are known as the *Brayton-Moser equations*, which where originally derived for nonlinear electrical RLC circuits in the early sixties Brayton & Moser (1964a,b); see also Smale (1972). The function P, which, in case the Hamiltonian represent the total stored energy, has the units of power, is commonly referred to as the *mixed-potential* function due to the different nature of the potentials P_1, P_2, and P_{12}. Indeed, decomposing the dynamics of the system into two subsystems Σ_1 and Σ_2, associated to the dynamics of z_1 and z_2, the potentials P_1 and P_2 represent the resistive relationships in Σ_1 and Σ_2, respectively, whereas the potential P_{12} represents

the (instantaneous) power flow from Σ_1 to Σ_2. In the context of electrical RLC circuits, the potential associated to the current-controlled resistors is often denoted as the *content*, whereas the potential that is associated to the voltage-controlled resistors is denoted as the *co-content* as introduced by Millar in the early fifties; see Jeltsema & Scherpen (2009) and the references therein.

Example 11.1. Consider the DC motor of Example 2.5. Clearly the Hamiltonian is composed of the energy storage in the electrical and the mechanical parts of the system. Hence, application of the Legendre transformation yields the co-energy

$$H^*(I,\omega) = \frac{1}{2}LI^2 + \frac{1}{2}J\omega^2,$$

with the co-energy variables $I = \varphi/L$ and $\omega = p/J$. Furthermore, since

$$J = \begin{bmatrix} 0 & -K \\ K & 0 \end{bmatrix}, \quad R = \begin{bmatrix} R & 0 \\ 0 & b \end{bmatrix}, \quad g = \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

we readily obtain

$$P(I,\omega) = \frac{1}{2}RI^2 - \frac{1}{2}b\omega^2 + K\omega I,$$

yielding the Brayton-Moser equations

$$\begin{bmatrix} L & 0 \\ 0 & -J \end{bmatrix} \begin{bmatrix} \dot{I} \\ \dot{\omega} \end{bmatrix} = - \begin{bmatrix} RI + K\omega \\ -b\omega + KI \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} (u = V),$$
$$y = I.$$

(Compare with (2.30).)

On the other hand, if the Hamiltonian can not be decomposed as $H(x) = H_1(x_1) + H_2(x_2)$, we obtain instead of (11.4)

$$\begin{bmatrix} \dfrac{\partial^2 H^*}{\partial z_1^2}(z) & \dfrac{\partial^2 H^*}{\partial z_1 \partial z_2}(z) \\ -\dfrac{\partial^2 H^*}{\partial z_2 \partial z_1}(z) & -\dfrac{\partial^2 H^*}{\partial z_2^2}(z) \end{bmatrix} \begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = - \begin{bmatrix} \dfrac{\partial P}{\partial z_1}(z) \\ \dfrac{\partial P}{\partial z_2}(z) \end{bmatrix} + \begin{bmatrix} g_1(x) \\ 0 \end{bmatrix} u,$$
$$y = g_1^T(x) z_1.$$

However, in this case

$$Q(z) = \begin{bmatrix} \dfrac{\partial^2 H^*}{\partial z_1^2}(z) & \dfrac{\partial^2 H^*}{\partial z_1 \partial z_2}(z) \\ -\dfrac{\partial^2 H^*}{\partial z_2 \partial z_1}(z) & -\dfrac{\partial^2 H^*}{\partial z_2^2}(z) \end{bmatrix}$$

is not a symmetric matrix anymore, and therefore does not represent a pseudo-Riemannian metric.

11.2 Geometry of the Brayton-Moser equations

Instrumental for the existence of the relationship between (11.2) and (11.4) is that the pseudo-Riemannian metric Q is a Hessian with respect to the Legendre transformation of the Hamiltonian function.

Definition 11.1. A pseudo-Riemannian metric defined by the non-singular matrix $Q(z)$ is said to be Hessian if there exists a function K such that the (i, j)-th element $Q_{ij}(Z)$ of the matrix $Q(z)$ is given as

$$Q_{ij}(z) = \frac{\partial^2 K}{\partial z_i \partial z_j}(z),$$

for $i, j = 1, \ldots, n$.

A necessary and sufficient condition for the (local) existence of such function $K(z)$ is the integrability condition, cf. Duistermaat (2001),

$$\frac{\partial Q_{jk}}{\partial z_i}(z) = \frac{\partial Q_{ik}}{\partial z_j}(z),$$

for $i, j, k = 1, \ldots, n$. Note that the pseudo-Riemannian metric (11.5) is indeed Hessian with respect to the function $K(z) = H_1^*(z_1) - H_2^*(z_2)$.

As for port-Hamiltonian systems, the Brayton-Moser (BM) equations (11.4) can be also represented in the formalism of Dirac structures, using a non-canonical Dirac structure representation Blankenstein (2005). Indeed, let us for ease of presentation assume that g_1 is constant and consider the following non-canonical Dirac

structure

$$\mathcal{D}_{\text{BM}} = \left\{ (f_S, e_S, f_P, e_P) \in \mathcal{F}_S \times \mathcal{F}_S^* \times \mathcal{F}_P \times \mathcal{F}_P^* \middle| \right.$$
$$\left. Q(z) f_S = -e_S + \begin{bmatrix} g_1 \\ 0 \end{bmatrix} f_P,\ e_P = \begin{bmatrix} g_1^T & 0 \end{bmatrix} f_P \right\},$$

defined with respect to the bilinear form

$$\ll (f_{S_1}, e_{S_1}, f_{P_1}, e_{P_1}), (f_{S_2}, e_{S_2}, f_{P_2}, e_{P_2}) \gg$$
$$= e_{S_1}^T f_{S_2} + e_{S_2}^T f_{S_1} + e_{P_1}^T f_{P_2} + e_{P_2}^T f_{P_1} + f_{S_1}^T (Q(z) + Q^T(z)) f_{S_2}.$$

Then, the Brayton-Moser equations can be described as a dynamical system with respect to the non-canonical Dirac structure $\mathcal{D}_{\text{BM}}$ by setting the flow variables as the rate of change of the co-energy variables (z_1, z_2), i.e., $f_S = -(\dot{z}_1, \dot{z}_2)$, the effort variables as

$$e_S = \left(\frac{\partial P}{\partial z_1}(z), \frac{\partial P}{\partial z_2}(z) \right),$$

and the input port variables $f_P = u$. Notice that the flow and effort variables are conjugated in the sense that

$$\frac{d}{dt} P = \frac{\partial^T P}{\partial z_1}(z) \dot{z}_1 + \frac{\partial^T P}{\partial z_2}(z) \dot{z}_2.$$

Furthermore, we observe that the natural output for the Brayton-Moser equations (11.4) is given by $e_P = g_1^T z_1$, and $e_P^T f_P$ has the units of power. However, the port variables (e_P, f_P) are not conjugated with respect to $\frac{d}{dt} P$, which has the units of power per second. For, we redefine the output port variables as $e'_P = g_1^T \dot{z}_1$, so that the dynamics of the Brayton-Moser equations (11.4) can equivalently be specified by

$$\left(-\dot{z}_1, -\dot{z}_2, \frac{\partial P}{\partial z_1}(z), \frac{\partial P}{\partial z_2}(z), f_P, e'_P \right) \in \mathcal{D}_{\text{BM}}.$$

Note that $\mathcal{D}_{\text{BM}}$ satisfies a power-like balance equation

$$\frac{d}{dt} P = f_P^T e'_P - \dot{z}^T (Q(z) + Q^T(z)) \dot{z}. \tag{11.6}$$

The latter shows that, in general, since Q is indefinite, P is not conserved.

11.3 Interconnection of gradient systems

Consider two (pseudo-)gradient systems

$$Q_j(z_j)\dot{z}_j = -\frac{\partial P_j}{\partial z_j}(z_j) + \frac{\partial^T h_j}{\partial z_j}(z_j)u_j,$$
$$y_j = h_j(z_j),\ j = 1, 2,$$

and interconnect them via the standard negative feedback interconnection $u_1 = -y_2$ and $u_2 = y_1$. Then, the interconnected system is again a (pseudo-)gradient system with (pseudo-)Riemannian metric $Q_1 \oplus Q_2$ and mixed-potential function

$$P_1(z_1) - P_2(z_2) + h_1^T(z_1)h_2(z_2).$$

Example 11.2. Consider a fluid tank with capacity C_1 and pressure drop p_1. If the flow rate of the fluid flowing out of the tank is denoted by q_1, then the fluid dynamics is described by $C_1\dot{p}_1 = q_1$ and $y_1 = p_1$. Hence, the associated mixed-potential function equals $P_1 = 0$. Suppose that a long pipe, with fluid inertia L_2 and fluid resistance R_2, is connected to the tank as shown in Figure 11.1. If the flow rate in the pipe is denoted as q_2 and the pressure at the outlet is denoted as p_2, the dynamics of the pipe take the form

$$L_2\dot{q}_2 = -\frac{\partial P_2}{\partial q_2}(q_2) + p_2,$$
$$y_2 = q_2,$$

where $P_2(q_2) = \frac{1}{2}R_2q_2^2$. The overall system is described by setting $q_1 = -q_2$ and $p_2 = p_1$, yielding a mixed-potential $P(p_1, q_2) = -\frac{1}{2}R_2q_2^2 + p_1q_2$, and a pseudo-Riemannian metric $Q = \text{diag}(C_1, -L_2)$.

11.4 Generation of power-based Lyapunov functions

We have seen above that the Brayton-Moser equations (11.4) satisfy a power-like balance equation (11.6). However, we cannot establish a dissipation-like inequality since the matrix Q is, in general, indefinite. Furthermore, to obtain the passivity property an additional difficulty stems from the fact that the mixed-potential P is also not sign definite. To overcome these difficulties,

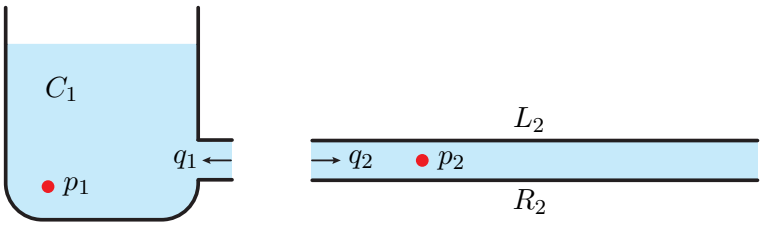


Figure 11.1: Interconnection of two gradient systems.

in Brayton & Moser (1964a,b); Ortega et al. (2003); Jeltsema et al. (2003); Jeltsema & Scherpen (2005) sufficient conditions have been given under which the equations (11.4) can be equivalently written as

$$\tilde{Q}(z)\dot{z} = -\frac{\partial \tilde{P}}{\partial z}(z) + g(z)u, \tag{11.7}$$

for some new admissible pair $(\tilde{Q}, \tilde{P})$, satisfying $\tilde{Q}(z) + \tilde{Q}^T(z) \geq 0$ and $\tilde{P} \geq 0$, for all z. Under these conditions it is clear that $\frac{d}{dt}P \leq u^T y'$, i.e., the system defines a passive system with respect to the port variables (u, y') and storage function $\tilde{P}$, with output variables $y' = g_1^T \dot{z}_1$. From the stability properties of a passive system (see Chapter 7), we know that if z^* is a local minimum of $\tilde{P}$, then is it a stable equilibrium of (11.7) when $u \equiv 0$.[2]

[2]Note that if we would start from $V(z,u) = P(z) - z_1^T g_1 u$, (constant) non-zero inputs can naturally be taken into account in the Lyapunov analysis by generating an admissible pair $(\tilde{Q}, \tilde{V})$, with $\tilde{V}$ satisfying $\tilde{V} \geq 0$ for all z.

12

Port-Hamiltonian systems on graphs

In this chapter we will see how the incidence structure of a directed graph provides a natural Poisson structure on the product of two spaces of flow and effort variables, namely those associated to the *vertices* (nodes) of the graph, and those associated to the *edges* (branches, links). In this way, many examples of network dynamics can be naturally modeled as port-Hamiltonian systems on graphs.

The Poisson structure resulting from the incidence matrix of the graph can be interpreted as a combination of two sets of *conservation* or *balance* laws. For example, in the case of a mass-spring-damper system one set of conservation laws corresponds to *momentum balance* while the other set corresponds to a continuity equation. In this sense the theory of port-Hamiltonian systems on graphs can be regarded as a discrete analog of the theory of distributed-parameter port-Hamiltonian systems with respect to a Stokes-Dirac structure as treated in Chapter 14.

This chapter is largely based on van der Schaft & Maschke (2013), to which we refer for further details and extensions.

12.1 Background on graphs

A *directed graph* $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ consists of a finite set $\mathcal{V}$ of *vertices* and a finite set $\mathcal{E}$ of directed *edges*, together with a mapping from $\mathcal{E}$ to the set of ordered pairs of $\mathcal{V}$, where no self-loops are allowed. Thus to any edge $e \in \mathcal{E}$ there corresponds an ordered pair $(v, w) \in \mathcal{V} \times \mathcal{V}$ (with $v \neq w$), representing the tail vertex v and the head vertex w of this edge.

A directed graph is completely specified by its *incidence matrix* B, which is an $N \times M$ matrix, N being the number of vertices and M being the number of edges, with (i, j)-th element equal to 1 if the j-th edge is an edge towards vertex i, equal to -1 if the j-th edge is an edge originating from vertex i, and 0 otherwise. It immediately follows that $\mathbb{1}^T B = 0$ for any incidence matrix B, where $\mathbb{1}$ is the vector consisting of all ones. A directed graph is called *connected* if between any two vertices there exists a path (a sequence of undirected edges) linking the two vertices. A directed graph is connected if and only if $\ker B^T = \operatorname{span} \mathbb{1}$; see e.g. Bollobas (1998). Since we will only consider directed graphs in the sequel 'graph' will throughout mean 'directed graph'.

Given a graph, we define its *vertex space* Λ_0 as the vector space of all functions from $\mathcal{V}$ to some linear space $\mathcal{R}$. In the examples, $\mathcal{R}$ will be mostly $\mathcal{R} = \mathbb{R}$ in which case Λ_0 can be identified with $\mathbb{R}^N$. Furthermore, we define the *edge space* Λ_1 as the vector space of all functions from $\mathcal{E}$ to $\mathcal{R}$. Again, if $\mathcal{R} = \mathbb{R}$ then Λ_1 can be identified with $\mathbb{R}^M$.

The dual spaces of Λ_0 and Λ_1 will be denoted by Λ^0, respectively by Λ^1. The duality pairing between $f \in \Lambda_0$ and $e \in \Lambda^0$ is given as

$$< f \mid e >= \sum_{v \in \mathcal{V}} < f(v) \mid e(v) > ,$$

where $< \dot{|} \dot{>}$ on the right-hand side denotes the duality pairing between $\mathcal{R}$ and $\mathcal{R}^*$, and a similar expression holds for $f \in \Lambda_1$ and $e \in \Lambda^1$ (with summation over the edges).

The incidence matrix B of the graph induces a linear map $\hat{B}$ from the edge space to the vertex space as follows. Define $\hat{B} : \Lambda_1 \to \Lambda_0$ as the linear map with matrix representation $B \otimes I$, where $I : \mathcal{R} \to \mathcal{R}$ is

the *identity map and* $\otimes$ denotes the Kronecker product. B will be called the *incidence operator*. For $\mathcal{R} = \mathbb{R}$ the incidence operator reduces to the linear map given by the matrix B itself, in which case we will throughout use B *both for the incidence matrix and for the incidence operator.* The adjoint map of $\hat{B}$ is denoted as

$$\hat{B}^* : \Lambda^0 \to \Lambda^1,$$

and is called the *co-incidence* operator. For $\mathcal{R} = \mathbb{R}^3$ the co-incidence operator is given by $B^T \otimes I_3$, while for $\mathcal{R} = \mathbb{R}$ the co-incidence operator is simply given by the transposed matrix B^T, and we will throughout use B^T *both for the co-incidence matrix and for the co-incidence operator.*

An *open graph* $\mathcal{G}$ is obtained from an ordinary graph with set of vertices $\mathcal{V}$ by identifying a subset $\mathcal{V}_b \subset \mathcal{V}$ of N_b *boundary vertices*. The interpretation of $\mathcal{V}_b$ is that these are the vertices that are open to interconnection (i.e., with other open graphs). The remaining subset $\mathcal{V}_i := \mathcal{V} - \mathcal{V}_b$ are the N_i *internal vertices* of the open graph.

The splitting of the vertices into internal and boundary vertices induces a splitting of the vertex space and its dual, given as

$$\Lambda_0 = \Lambda_{0i} \oplus \Lambda_{0b},\ \Lambda^0 = \Lambda^{0i} \oplus \Lambda^{0b},$$

where Λ_{0i} is the vertex space corresponding to the internal vertices and Λ_{0b} the vertex space corresponding to the boundary vertices. Consequently, the incidence operator $\hat{B} : \Lambda_1 \to \Lambda_0$ splits as

$$\hat{B} = \hat{B}_i \oplus \hat{B}_b,$$

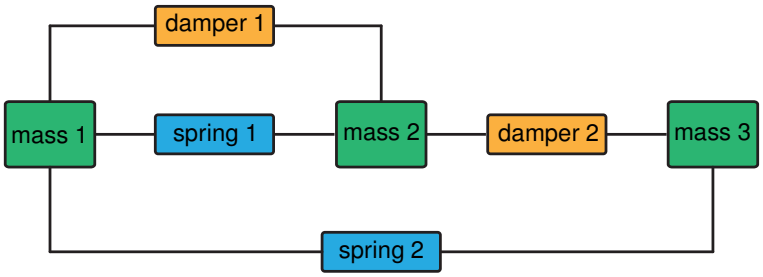
with $\hat{B}_i : \Lambda_1 \to \Lambda_{0i}$ and $\hat{B}_b : \Lambda_1 \to \Lambda_{0b}$. For $\mathcal{R} = \mathbb{R}$ we will simply write

$$B = \begin{bmatrix} B_i \\ B_b \end{bmatrix}.$$

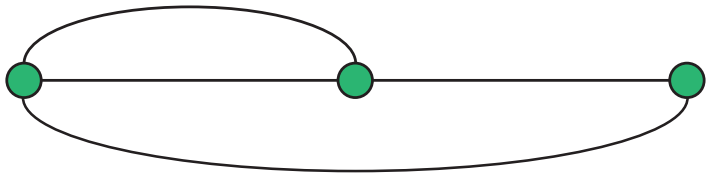
Furthermore, we will define the *boundary space* Λ_b as the linear space of all functions from the set of boundary vertices $\mathcal{V}_b$ to the linear space $\mathcal{R}$. Note that the boundary space Λ_b is equal to the linear space Λ_{0b}, and that the linear mapping $\hat{B}_b$ can be also regarded as a mapping $\hat{B}_b : \Lambda_1 \to \Lambda_b$, called the *boundary incidence operator*. The dual space of Λ_b will be denoted as Λ^b. The elements $f_b \in \Lambda_b$ are called the *boundary flows* and the elements $e^b \in \Lambda^b$ the *boundary efforts*.

12.2 Mass-spring-damper systems

The basic way of modeling a *mass-spring-damper system* as a port-Hamiltonian system on a graph is to associate the *masses* to the *vertices*, and the *springs* and *dampers* to the *edges* of the graph; see Fig. 12.1.



(a)



(b)

Figure 12.1: (a) Mass-spring-damper system; (b) the corresponding graph.

For clarity of exposition we will start with the separate treatment of mass-spring (Section 12.2.1) and mass-damper (Section 12.2.2) systems, before their merging in Section 12.2.3.

12.2.1 Mass-spring systems

Consider a graph $\mathcal{G}$ with N vertices (masses) and M edges (springs), specified by an incidence matrix B. First consider the situation that the mass-spring system is located in one-dimensional space $\mathcal{R} = \mathbb{R}$, and the springs are scalar. A vector in the vertex space Λ_0 then corresponds to the vector p of the scalar momenta of all N masses, i.e., $p \in \Lambda_0 = \mathbb{R}^N$. Furthermore, a vector in the dual edge space Λ^1 will correspond

to the total vector q of elongations of all M springs, i.e., $q \in \Lambda^1 = \mathbb{R}^M$.

Next ingredient is the definition of the Hamiltonian $H : \Lambda^1 \times \Lambda_0 \to \mathbb{R}$, which typically splits into a sum of the kinetic and potential energies of each mass and spring. In the absence of boundary vertices the dynamics of the mass-spring system is then described as the port-Hamiltonian system

$$\begin{bmatrix} \dot{q} \\ \dot{p} \end{bmatrix} = \begin{bmatrix} 0 & B^T \\ -B & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial H}{\partial q}(q,p) \\ \frac{\partial H}{\partial p}(q,p) \end{bmatrix} \tag{12.1}$$

defined with respect to the Poisson structure on the state space $\Lambda^1 \times \Lambda_0$ given by the skew-symmetric matrix

$$J := \begin{bmatrix} 0 & B^T \\ -B & 0 \end{bmatrix}. \tag{12.2}$$

The inclusion of boundary vertices, and thereby of external interaction, can be done in different ways. The first option is to associate *boundary masses* to the boundary vertices. We are then led to the port-Hamiltonian system

$$\begin{aligned} \dot{q} &= B^T \frac{\partial H}{\partial p}(q,p), \\ \dot{p} &= -B \frac{\partial H}{\partial q}(q,p) + E f_b, \\ e^b &= E^T \frac{\partial H}{\partial p}(q,p). \end{aligned} \tag{12.3}$$

Here E is a matrix with as many columns as there are boundary vertices; each column consists of zeros except for exactly one 1 in the row corresponding to the associated boundary vertex. $f_b \in \Lambda_b$ are the external *forces* exerted (by the environment) on the boundary masses, and $e_b \in \Lambda^b$ are the *velocities* of these boundary masses.

Another possibility is to regard the boundary vertices as being *massless*. In this case we obtain the port-Hamiltonian system (with p_i denoting the vector of momenta of the masses associated to the *inter-*

nal vertices)

$$\begin{aligned} \dot{q} &= B_i^T \frac{\partial H}{\partial p_i}(q, p_i) + B_b^T e^b, \\ \dot{p}_i &= -B_i \frac{\partial H}{\partial q}(q, p_i), \\ f_b &= B_b \frac{\partial H}{\partial q}(q, p_i), \end{aligned} \tag{12.4}$$

with $e^b \in \Lambda^b$ the velocities of the massless boundary vertices, and $f_b \in \Lambda_b$ the forces at the boundary vertices as *experienced* by the environment. Note that in this latter case the external velocities e^b of the boundary vertices can be considered to be *inputs* to the system and the forces f_b to be *outputs*; in contrast to the previously considered case (boundary vertices corresponding to boundary masses), where the forces f_b are inputs and the velocities e^b the outputs of the system[1].

The above formulation of mass-spring systems in $\mathcal{R} = \mathbb{R}$ directly extends to $\mathcal{R} = \mathbb{R}^3$ by using the incidence operator $\hat{B} = B \otimes I_3$ as defined before. Finally, we remark that in the above treatment we have considered springs with *arbitrary* elongation vectors $q \in \Lambda^1$. For ordinary springs the vector q of elongations is given as $q = B^T q_c$, where $q_c \in \Lambda^0$ denotes the vector of positions of the vertices. Hence in this case $q \in \operatorname{im} B^T \subset \Lambda^1$. Note that the subspace $\operatorname{im} B^T \times \Lambda_0 \subset \Lambda^1 \times \Lambda_0$ is an invariant subspace with regard to the dynamics (12.3) or (12.4). We will return to this in Section 12.6.

12.2.2 Mass-damper systems

Replacing springs by dampers leads to *mass-damper systems*. In the case of massless boundary vertices this yields the following[2] equa-

[1] One can also consider the hybrid case where *some* of the boundary vertices are associated to masses while the remaining ones are massless.

[2] Note that these equation follow from (12.4) by replacing $-\dot{q}$ by e^1 and $\frac{\partial H}{\partial q}(q,p)$ by f^1.

tions

$$\begin{aligned} B_i f_1 &= -\dot{p}_i, \\ B_b f_1 &= f_b, \\ e^1 &= -B_i^T \frac{\partial H}{\partial p_i}(p_i) - B_b^T e^b, \end{aligned} \tag{12.5}$$

where f_1, e^1 are the flows and efforts corresponding to the dampers (damping forces, respectively, velocities). For *linear* dampers $f_1 = -Re^1$, where R is the positive diagonal matrix with the damping constants on its diagonal. Substitution into (12.5) then yields the port-Hamiltonian system

$$\begin{aligned} \dot{p}_i &= -B_i R B_i^T \frac{\partial H}{\partial p_i}(p_i) - B_i R B_b^T e^b, \\ f_b &= B_b R B_i^T \frac{\partial H}{\partial p_i}(p_i) + B_b^T R B_b^T e^b, \end{aligned} \tag{12.6}$$

where, as before, the inputs e^b are the boundary velocities and f_b are the forces as experienced at the massless boundary vertices. Note that the matrix

$$\mathcal{L} := \begin{bmatrix} B_i \\ B_b \end{bmatrix} R \begin{bmatrix} B_i^T & B_b^T \end{bmatrix} = BRB^T$$

is the *weighted Laplacian matrix* of the graph $\mathcal{G}$ (with weights given by the diagonal elements of R). It is well-known Bollobas (1998) that for a connected graph the matrix $\mathcal{L}$ has exactly one eigenvalue 0, with eigenvector $\mathbb{1}$, while all other eigenvalues are positive.

12.2.3 Mass-spring-damper systems

For a mass-spring-damper system the edges will correspond partly to springs, and partly to dampers. Thus a mass-spring-damper system is described by a graph $\mathcal{G}(\mathcal{V}, \mathcal{E}_s \cup \mathcal{E}_d)$, where the vertices in $\mathcal{V}$ correspond to the *masses*, the edges in $\mathcal{E}_s$ to the *springs*, and the edges in $\mathcal{E}_d$ to the dampers of the system. This corresponds to an incidence matrix $B = \begin{bmatrix} B_s & B_d \end{bmatrix}$, where the columns of B_s reflect the spring edges and the columns of B_d the damper edges. For the case *without* boundary vertices the dynamics of such a mass-spring-damper system with

linear dampers takes the form

$$\begin{bmatrix} \dot{q} \\ \dot{p} \end{bmatrix} = \begin{bmatrix} 0 & B_s^T \\ -B_s & -B_d R B_d^T \end{bmatrix} \begin{bmatrix} \frac{\partial H}{\partial q}(q,p) \\ \frac{\partial H}{\partial p}(q,p) \end{bmatrix}. \tag{12.7}$$

In the presence of boundary vertices we may distinguish, as above, between *massless* boundary vertices, with inputs being the boundary velocities and outputs the boundary (reaction) forces, and *boundary masses*, in which case the inputs are the external forces and the outputs the velocities of the boundary masses.

Remark 12.1. The above formulation of mass-spring-damper systems with $\mathcal{R}$ equal to $\mathbb{R}$ or $\mathbb{R}^3$ can be extended to *spatial mechanisms*, that is, networks of rigid bodies in $\mathbb{R}^3$ related by joints. In this case, the linear space $\mathcal{R}$ is given by $\mathcal{R} := \mathrm{se}^*(3)$, the dual of the Lie algebra of the Lie group $SE(3)$ describing the position of a rigid body in $\mathbb{R}^3$. A spatial mechanism (or *multibody system*) is a mechanical system consisting of *rigid bodies* related by joints (defined as *kinematic pairs*) restricting the relative motion between the rigid bodies. See van der Schaft & Maschke (2013) for details.

12.2.4 Hydraulic networks

A hydraulic network can be modeled as a directed graph with edges corresponding to pipes, see e.g. Roberson & Crowe (1993); De Persis & Kallesoe (2011). The vertices may either correspond to connection points with *fluid reservoirs* (buffers), or merely to connection points of the pipes; we concentrate on the first case (the second case corresponding to a Kirchhoff-Dirac structure, cf. Section 12.8). Let x_v be the stored fluid at vertex v and let ν_e be the fluid flow through edge e. Collecting all stored fluids x_v into one vector x, and all fluid flows ν_e into one vector ν, the *mass-balance* is summarized as

$$\dot{x} = B\nu, \tag{12.8}$$

with B denoting the incidence matrix of the graph. In the absence of fluid reservoirs this reduces to Kirchhoff's current laws $B\nu = 0$.

For incompressible fluids a standard model of the fluid flow ν_e through pipe e is

$$J_e\dot{\nu}_e = P_i - P_j - \lambda_e(\nu_e), \tag{12.9}$$

where P_i and P_j are the pressures at the tail, respectively head, vertices of edge e. Note that this captures in fact *two* effects; one corresponding to energy storage and one corresponding to energy dissipation. Defining the energy variable $\varphi_e := J_e\nu_e$ the stored energy in the pipe associated with edge e is given as $\frac{1}{2J_e}\varphi_e^2 = \frac{1}{2}J_e\nu_e^2$. Secondly, $\lambda_e(\nu_e)$ is a damping force corresponding to energy dissipation.

In the case of fluid reservoirs at the vertices the pressures P_v at each vertex v are functions of x_v, and thus, being scalar functions, always derivable from an energy function $P_v = \frac{\partial H_v}{\partial x_v}(x_v), v \in \mathcal{V}$, for some Hamiltonian $H_v(x_v)$ (e.g. gravitational energy). The resulting dynamics (with state variables x_v and φ_e) is port-Hamiltonian with respect to the Poisson structure (12.2). The set-up is immediately extended to boundary vertices (either corresponding to controlled fluid reservoirs or direct in-/outflows).

12.2.5 Single species chemical reaction networks

We have already seen in the last section of Chapter 2 how isothermal detailed-balanced chemical reaction networks governed by mass action kinetics give rise to a port-Hamiltonian system defined on the graph of complexes, with respect to the Hamiltonian given by the Gibbs' free energy, and energy-dissipating relations determined by the reaction constants and the thermodynamic equilibrium.

For complexes consisting of *single species* this can be specialized to the following linear port-Hamiltonian system on a graph having the same structure as a mass-damper system. In fact, for complexes consisting of single species we have $Z = I_m$, in which case the port-Hamiltonian formulation of detailed-balanced mass action kinetics reaction networks given by (2.45) reduces to

$$\dot{x} = -B\mathcal{K}B^T\frac{x}{x^*},$$

where x^* is the assumed thermodynamic equilibrium. Defining the

diagonal matrix $M := \text{diag}(x_1^*, \cdots, x_m^*)$ this can be also written as

$$\dot{x} = -B\mathcal{K}B^T M^{-1} x, \tag{12.10}$$

which are exactly the equations of a mass-damper system on a graph, with damping constants given by the diagonal elements of $\mathcal{K}$ and Hamiltonian $H(x) = \frac{1}{2} x^T M^{-1} x$.

12.3 Swing equations for power grids

Consider a power grid consisting of n buses corresponding to the vertices of a graph. A standard model for the dynamics of the i-th bus is given by (see e.g. Machovski et al. (2008); Bürger et al. (2013))

$$\begin{aligned} \dot{\delta}_i &= \omega_i^b - \omega^r, \quad i = 1, \cdots, n, \\ M_i \dot{\omega}_i &= -a_i(\omega_i^b - \omega^r) - \sum_{j \neq i} V_i V_j S_{ij} [\sin(\delta_i - \delta_j) + u_i, \end{aligned}$$

where the summation in the last line is over all buses j which are adjacent to bus i; that is, all buses j that are directly linked to bus i by a transmission line (defining an edge of the graph). Here δ_i denotes the voltage angle, v_i the voltage amplitude, ω_i^b the frequency, $\omega_i := \omega_i^b - \omega^r$ the frequency deviation, and u_i the power generation/consumption; all at bus i. Furthermore, ω^r is the nominal (reference) frequency for the network, M_i and a_i are inertia and damping constants at bus i, and S_{ij} is the transfer susceptance of the line between bus i and j.

Define $z_k := \delta_i - \delta_j$ and $c_k := E_i E_j S_{ij}$, if the k-th edge is pointing from vertex i to vertex j. Furthermore, define the momenta $p_i = M_i \omega_i, i = 1, \cdots, n$. Then the equations can be written in the vector form

$$\begin{aligned} \dot{z} &= B^T M^{-1} p, \\ \dot{p} &= -AM^{-1} p - BC \,\text{Sin}\, z + u, \end{aligned}$$

where z is the m-vector with components z_k, M is the diagonal matrix with diagonal elements M_i, A is the diagonal matrix with diagonal elements a_i, and C is the diagonal matrix with elements c_k. Furthermore, $\text{Sin} : \mathbb{R}^m \to \mathbb{R}^m$ denotes the elementwise $\sin$ function, and z^* is the m-vector with k-th component δ_{ij}^*.

Defining the Hamiltonian $H(z,p)$ as

$$H(z,p) = \frac{1}{2}p^T M^{-1} p - \mathbb{1}^T C \operatorname{Cos} z, \tag{12.11}$$

the equations take the port-Hamiltonian form

$$\begin{bmatrix} \dot{z} \\ \dot{p} \end{bmatrix} = \begin{bmatrix} 0 & B^T \\ -B & -A \end{bmatrix} \begin{bmatrix} \dfrac{\partial H}{\partial z}(z,p) \\ \dfrac{\partial H}{\partial p}(z,p) \end{bmatrix} + \begin{bmatrix} 0 \\ I \end{bmatrix} u. \tag{12.12}$$

Note that the Hamiltonian $H(z,p)$ is of the standard 'kinetic energy plus potential energy' form, with potential energy $-\mathbb{1}^T C \operatorname{Cos} z = -\sum c_k \cos z_k$ similar to the gravitational energy of a pendulum; whence the name 'swing equations'. Note that, as in the mass-spring system example, the potential energy is associated to the edges of the graph, while the kinetic energy is associated to its vertices. A difference with the mass-spring-damper system example is that in the current example the 'damping torques' $A\frac{\partial H}{\partial p}(z,p)$ are associated to the vertices, instead of to the edges.

12.4 Available storage

Consider the simplest port-Hamiltonian system on a graph, given as

$$\begin{aligned} \dot{x} &= Bu, \quad u \in \mathbb{R}^m,\ x \in \mathbb{R}^n, \\ y &= B^T \frac{\partial H}{\partial x}(x), \quad y \in \mathbb{R}^m, \end{aligned} \tag{12.13}$$

where B is the incidence matrix of the graph, and $H(x) = \frac{1}{2}\|x\|^2$ is the Hamiltonian function. Clearly, since H is non-negative it defines a storage function, and the system is passive. On the other hand it will turn out that the *minimal* storage function for the system, called the *available storage* (see Chapter 7), is *different* from H.

Throughout this section we will assume[3] that the graph is *connected*, or equivalently Bollobas (1998) $\ker B^T = \operatorname{span} \mathbb{1}$. Based on

[3]Without loss of generality, since otherwise the analysis can be repeated for every connected component of the graph.

Chapter 7 we know that the available storage S_a is given as

$$S_a(x) = \sup - \int_0^\tau u^T(t)y(t)dt, \tag{12.14}$$

where we consider the supremum over all $\tau \geq 0$ and all input functions $u : [0, \tau] \to \mathbb{R}^m$, and where $y : [0, \tau] \to \mathbb{R}^m$ is the output resulting from the input function $u : [0, \tau] \to \mathbb{R}^m$ and initial condition $x(0) = x$. Noting that

$$\begin{aligned}\int_0^\tau u^T(t)y(t)dt &= \int_0^\tau u^T(t)B^T x(t)dt \\ &= \int_0^\tau \dot{x}^T(t)x(t)dt = \frac{1}{2}\|x(\tau)\|^2 - \frac{1}{2}\|x(0)\|^2,\end{aligned}$$

we see that the available storage is equivalently given as

$$S_a(x) = \sup \left(\frac{1}{2}\|x\|^2 - \frac{1}{2}\|x(\tau)\|^2 \right), \tag{12.15}$$

where we take the supremum over all $\tau \geq 0$ and all possible states $x(\tau)$ resulting from input functions $u : [0, \tau] \to \mathbb{R}^m$. By connectedness of the graph, we know that from $x(0) = x$ we can reach, by choosing the input function suitably, any state $x(\tau)$ satisfying

$$\mathbb{1}^T x(\tau) = \mathbb{1}^T x. \tag{12.16}$$

Hence the available storage $S_a(x)$ is given by (12.15) where we take the supremum over all states $x(\tau)$ satisfying (12.16). This corresponds to minimizing $\frac{1}{2}\|x(\tau)\|^2$ over all $x(\tau)$ satisfying (12.16), having the solution

$$x(\tau) = \frac{1}{n}\mathbb{1}^T x \mathbb{1}, \tag{12.17}$$

Thus the available storage S_a is given by the explicit expression

$$S_a(x) = \frac{1}{2}\|x\|^2 - \frac{1}{2}\left(\frac{1}{n}\mathbb{1}^T x\right)^2 \|\mathbb{1}\|^2 = \frac{1}{2}x^T\left(I_n - \frac{1}{n}\mathbb{1}\mathbb{1}^T\right)x. \tag{12.18}$$

We conclude that for all initial conditions $x(0) = x$ which are such that $\mathbb{1}^T x \neq 0$ the available storage $S_a(x)$ is strictly smaller than the Hamiltonian $\frac{1}{2}\|x\|^2$. The reason is that, since the system $\dot{x} = Bu$ is not controllable, it is not possible to drive every initial state to the

origin; the position of zero energy. Instead, by extracting the maximal amount of energy the system is brought from state x to a state x^* with $x_1^* = \cdots = x_n^*$, satisfying $x_1^* + \cdots + x_n^* = x_1 + \cdots + x_n$.

Note that the matrix $I_n - \frac{1}{n}\mathbb{1}\mathbb{1}^T$ defines a symmetric weighted Laplacian matrix for an extended graph; namely the *complete graph* for the vertices of the original graph[4].

The above analysis can be extended to any port-Hamiltonian system (12.13) for which the Hamiltonian H is non-negative (and thus the system (12.13) is passive). Indeed, in this case the available storage can be seen to be

$$S_a(x) = H(x) - H(v^*(x)),$$

where $H(v^*(x))$ is the solution of minimizing $H(v)$ over all $v \in \mathbb{R}^n$ satisfying $\mathbb{1}^T v = \mathbb{1}^T x$. Equivalently, this amounts to the minimization of

$$H(v) + \lambda(\mathbb{1}^T v - \mathbb{1}^T x)$$

over v and the Lagrangian multiplier $\lambda \in \mathbb{R}$, yielding the minimizer $v^*(x)$ as the solution of the equations

$$\begin{aligned} \frac{\partial H}{\partial v_1}(v^*(x)) = \frac{\partial H}{\partial v_2}(v^*(x)) = \cdots = \frac{\partial H}{\partial v_n}(v^*(x)), \\ v_1 + v_2 + \cdots + v_n = x_1 + x_2 + \cdots + x_n. \end{aligned} \tag{12.19}$$

The first equation of (12.19) can be interpreted as a consensus condition on the co-energy variables $\frac{\partial H}{\partial v_1}, \cdots, \frac{\partial H}{\partial v_n}$. Note that, as in the case $H(x) = \frac{1}{2}\|x\|^2$, the expression for the available storage is independent of the graph (as long as it is connected).

Example 12.1. Consider a system of n point masses $M_1, \cdots, M_n$ in $\mathbb{R}$ with state variables being the momenta $p_1, \cdots, p_n$, and with Hamiltonian equal to the *kinetic energy*

$$H(p) = \sum_{i=1}^{n} \frac{p_i^2}{2M_i}.$$

The available storage can be computed

$$S_a(p) = \frac{1}{2} \sum_{i<j} \frac{M_i M_j}{M_1 + \cdots + M_n} \left(\frac{p_i}{M_i} - \frac{p_j}{M_j} \right)^2.$$

[4] A graph is called *complete* if there is an edge between every pair of vertices.

This quantity was called the *motion energy* in Willems (2013). It amounts to the maximal energy which can be extracted from the system by applying forces $F_1, \cdots, F_n$ satisfying $\sum_{j=1}^{n} F_j = 0$, or equivalently (since $\mathbb{1}^T B = 0$)

$$\dot{p} = F = Bu,$$

where F is the vector with components $F_1, \cdots, F_n$ and B is the incidence matrix of the complete graph with vertices corresponding to the masses $M_1, \cdots, M_n$. Note that as a result of extracting the maximal energy the system will end up in a consensus state $v_1 = \cdots = v_n$, with $v_i = \frac{p_i}{M_i}$ the velocities of the point masses.

The above expression for the available storage can be readily extended to point masses in $\mathbb{R}^3$; replacing the expression

$$\left(\frac{p_i}{M_i} - \frac{p_j}{M_j} \right)^2 \text{ with } \left\| \frac{p_i}{M_i} - \frac{p_j}{M_j} \right\|^2 .$$

Note that contrary to the Hamiltonian function, the available storage is not *additive*: the available storage of an interconnected port-Hamiltonian system on a graph is not necessarily the sum of the available storages of the individual subsystems, as was already noted in Willems (2013). A simple example is provided by the juxtaposition of two systems each consisting of two masses. The sum of the energies which can be extracted from the two systems separately by applying for each system two external forces whose sum is zero, is strictly smaller than the amount of energy which can be extracted from the four masses by applying four external forces whose sum is zero.

12.5 Analysis of port-Hamiltonian systems on graphs

In this section we will investigate the dynamical properties of the mass-spring-damper system as discussed in Section 12.2.3. As we have seen, many other examples share the same mathematical structure, and their analysis will follow the same lines.

Thus we will consider a mass-spring-damper system as described by a graph $\mathcal{G}(\mathcal{V}, \mathcal{E}_s \cup \mathcal{E}_d)$, where the vertices in $\mathcal{V}$ correspond to the *masses*, the edges in $\mathcal{E}_s$ to the *springs*, and the edges in $\mathcal{E}_d$ to the

dampers of the system, with incidence matrix $B = \begin{bmatrix} B_s & B_d \end{bmatrix}$, where the columns of B_s reflect the spring edges and the columns of B_d the damper edges. Without boundary vertices the dynamics takes the form (see equation (12.7) in Section 12.2.3)

$$\begin{bmatrix} \dot{q} \\ \dot{p} \end{bmatrix} = \begin{bmatrix} 0 & B_s^T \\ -B_s & -B_d R B_d^T \end{bmatrix} \begin{bmatrix} \dfrac{\partial H}{\partial q}(q,p) \\ \dfrac{\partial H}{\partial p}(q,p) \end{bmatrix}. \tag{12.20}$$

Throughout this section we make the following simplifying assumption[5]. The graph $\mathcal{G}(\mathcal{V}, \mathcal{E}_s \cup \mathcal{E}_d)$ is connected, or equivalently $\ker B_s^T \cap \ker B_d^T = \operatorname{span} \mathbb{1}$.

12.5.1 Equilibria and Casimirs

We start with the following proposition regarding the equilibria.

Proposition 12.1. The set of equilibria $\mathcal{E}$ of (12.20) is given as

$$\mathcal{E} = \left\{ (q,p) \in \Lambda^1 \times \Lambda_0 \,\middle|\, \frac{\partial H}{\partial q}(q,p) \in \ker B_s, \frac{\partial H}{\partial p}(q,p) \in \operatorname{span} \mathbb{1} \right\}.$$

Proof. The state (q,p) is an equilibrium whenever

$$B_s^T \frac{\partial H}{\partial p}(q,p) = 0, \; B_s \frac{\partial H}{\partial q}(q,p) + B_d R B_d^T \frac{\partial H}{\partial p}(q,p) = 0.$$

Premultiplication of the second equation by the row-vector $\frac{\partial^T H}{\partial p}(q,p)$, making use of the first equation, yields

$$\frac{\partial^T H}{\partial p}(q,p) B_d R B_d^T \frac{\partial H}{\partial p}(q,p) = 0,$$

or equivalently $B_d^T \frac{\partial H}{\partial p}(q,p) = 0$, which implies $B_s \frac{\partial H}{\partial q}(q,p) = 0$. ∎

In other words, for (q,p) to be an equilibrium, $\frac{\partial H}{\partial p}(q,p)$ should satisfy the consensus conditions corresponding to the mass-damper graph $\mathcal{G}(\mathcal{V}, \mathcal{E}_s \cup \mathcal{E}_d)$, whereas $\frac{\partial H}{\partial q}(q,p)$ should be in the space of *cycles*

[5] Again, this assumption can be made without loss of generality, since otherwise the same analysis can be performed for each connected component.

of the mass-spring graph $\mathcal{G}(\mathcal{V}, \mathcal{E}_s)$ (corresponding to zero net spring forces applied to the masses at the vertices).

Similarly, the *Casimirs* (conserved quantities independent of the Hamiltonian H, cf. Chapter 8), are computed as follows.

Proposition 12.2. The Casimir functions are all functions $C(q,p)$ satisfying

$$\frac{\partial C}{\partial p}(q,p) \in \operatorname{span} \mathbb{1}, \ \frac{\partial C}{\partial q}(q,p) \in \ker B_s. \tag{12.21}$$

Proof. The function $C(q,p)$ is a Casimir if

$$\begin{bmatrix} \frac{\partial C}{\partial q}(q,p) & \frac{\partial C}{\partial p}(q,p) \end{bmatrix} \begin{bmatrix} 0 & B_s^T \\ -B_s & -B_d R B_d^T \end{bmatrix} = 0,$$

or equivalently

$$\frac{\partial^T C}{\partial p}(q,p) B_s = 0, \ \frac{\partial^T C}{\partial q}(q,p) B_s^T + \frac{\partial^T C}{\partial p}(q,p) B_d R B_d^T = 0.$$

Postmultiplication of the second equation by $\frac{\partial C}{\partial p}(q,p)$, making use of the first equation, gives the result. ∎

Therefore all Casimir functions can be expressed as functions of the *linear* Casimir functions

$$C(q,p) = \mathbb{1}^T p, \quad C(q,p) = k^T q, k \in \ker B_s \tag{12.22}$$

This implies that starting from an arbitrary initial position $(q_0, p_0) \in \Lambda^1 \times \Lambda_0$ the solution of the mass-spring-damper system (12.20) will be contained in the affine space

$$\mathcal{A}_{(q_0,p_0)} := \begin{bmatrix} q_0 \\ p_0 \end{bmatrix} + \begin{bmatrix} 0 \\ \ker \mathbb{1}^T \end{bmatrix} + \begin{bmatrix} \operatorname{im} B_s^T \\ 0 \end{bmatrix} \tag{12.23}$$

i.e., for all t the difference $q(t) - q_0$ remains in the space of *co-cycles* of the spring graph, while $\mathbb{1}^T p(t) = \mathbb{1}^T p_0$.

12.5.2 Stability analysis

Under generic conditions on the Hamiltonian $H(q,p)$, each affine space $\mathcal{A}_{(q_0,p_0)}$ will intersect the set of equilibria $\mathcal{E}$ in a *single* point

(q_∞, p_∞), which will qualify as the point of asymptotic convergence starting from (q_0, p_0) (provided there is enough damping present). In order to simplify the statement of the results we will throughout this subsection consider *linear* mass-spring systems, corresponding to a quadratic and decoupled Hamiltonian function

$$H(q,p) = \frac{1}{2}q^T K q + \frac{1}{2}p^T G p, \tag{12.24}$$

where K is the positive diagonal matrix of spring constants, and G is the positive diagonal matrix of reciprocals of the masses. It follows that the set of equilibria is given as $\mathcal{E} = \{(q,p) \in \Lambda^1 \times \Lambda_0 \mid Kq \in \ker B_s, Gp \in \operatorname{span} \mathbb{1}\}$, while for each (q_0, p_0) there exists a *unique* point $(q_\infty, p_\infty) \in \mathcal{E} \cap \mathcal{A}_{(q_0,p_0)}$. In fact, q_∞ is given by the spring graph co-cycle/cycle decomposition

$$q_0 = v_0 + q_\infty, \quad v_0 \in \operatorname{im} B_s^T \subset \Lambda^1, Kq_\infty \in \ker B_s \subset \Lambda_1, \tag{12.25}$$

while p_∞ is uniquely determined by

$$Gp_\infty \in \operatorname{span} \mathbb{1}, \quad \mathbb{1}^T p_\infty = \mathbb{1}^T p_0. \tag{12.26}$$

This leads to the following asymptotic stability theorem. First note that the energy $H(q,p) = \frac{1}{2}q^T K q + \frac{1}{2}p^T G p$ satisfies

$$\begin{aligned} \frac{d}{dt}H(q,p) &= -\frac{\partial^T H}{\partial p}(q,p) B_d R B_d^T \frac{\partial H}{\partial p}(q,p) \\ &= -p^T B_d G R B_d^T G p \leq 0, \end{aligned} \tag{12.27}$$

and thus qualifies as a Lyapunov function; showing at least stability.

Theorem 12.1. Consider a linear mass-spring-damper system with $H(q,p) = \frac{1}{2}q^T K q + \frac{1}{2}p^T G p$, where K and G are diagonal positive matrices. Then for every (q_0, p_0) there exists a unique equilibrium point $(q_\infty, p_\infty) \in \mathcal{E} \cap \mathcal{A}_{(q_0,p_0)}$, determined by (12.25, 12.26). Define the spring Laplacian matrix $L_s := B_s K B_s^T$. Then for every (q_0, p_0) the following holds: the trajectory starting from (q_0, p_0) converges asymptotically to (q_∞, p_∞) if and only if the largest GL_s-invariant subspace contained in $\ker B_d^T$ is equal to $\operatorname{span} \mathbb{1}$.

The condition that the largest GL_s-invariant subspace contained in $\ker B_d^T$ is equal to $\operatorname{span} \mathbb{1}$ amounts to *pervasive damping*: the influence of the dampers spreads through the whole system. Another feature of the dynamics of the mass-spring-damper system (12.20) is its *robustness* with regard to constant external (disturbance) forces. Indeed, consider a mass-spring-damper system with boundary masses (see Section 12.2) and general Hamiltonian $H(q,p)$, subject to *constant* forces $\bar{f}_b$

$$\begin{bmatrix} \dot{q} \\ \dot{p} \end{bmatrix} = \begin{bmatrix} 0 & B_s^T \\ -B_s & -B_d R B_d^T \end{bmatrix} \begin{bmatrix} \dfrac{\partial H}{\partial q}(q,p) \\ \dfrac{\partial H}{\partial p}(q,p) \end{bmatrix} + \begin{bmatrix} 0 \\ E \end{bmatrix} \bar{f}_b, \tag{12.28}$$

where we *assume*[6] the existence of a $\bar{q}$ such that

$$B_s \frac{\partial H}{\partial q}(\bar{q},0) = E\bar{f}_b. \tag{12.29}$$

Then, the shifted Hamiltonian $\bar{H}(q,p) := H(q,p) - (q-\bar{q})^T \frac{\partial H}{\partial q}(\bar{q},0) - H(\bar{q},0)$ introduced before, satisfies

$$\frac{d}{dt}\bar{H}(q,p) = -\frac{\partial^T H}{\partial p}(q,p) B_d R B_d^T \frac{\partial H}{\partial p}(q,p) \leq 0. \tag{12.30}$$

Specializing to $H(q,p) = \frac{1}{2}q^T K q + \frac{1}{2}p^T G p$, in which case $\bar{H}(q,p) = \frac{1}{2}(q-\bar{q})^T K (q-\bar{q}) + \frac{1}{2}p^T G p$, we obtain the following analog of Theorem 12.1.

Proposition 12.3. Consider a linear mass-spring-damper system (12.28) with constant external disturbance $\bar{f}_b$ and Hamiltonian $H(q,p) = \frac{1}{2}q^T K q + \frac{1}{2}p^T G p$, where K and G are diagonal positive matrices. and with $\operatorname{im} E \subset \operatorname{im} B_s$. The set of controlled equilibria is given by $\bar{\mathcal{E}} = \{(q,p) \in \Lambda^1 \times \Lambda_0 \mid B_s K q = E\bar{f}_b, Gp \in \operatorname{span} \mathbb{1}\}$. For every (q_0,p_0) there exists a unique equilibrium point $(\bar{q}_\infty, p_\infty) \in \bar{\mathcal{E}} \cap \mathcal{A}_{(q_0,p_0)}$. Here p_∞ is determined by (12.26), while $\bar{q}_\infty = \bar{q} + q_\infty$, with $\bar{q}$ such that $B_s K \bar{q} = E\bar{f}_b$ and q_∞ the unique solution of (12.25) with q_0 replaced by $q_0 - \bar{q}$. Furthermore, for each (q_0,p_0) the trajectory starting from

[6] If the mapping $q \to \frac{\partial H}{\partial q}(q,0)$ is surjective, then there exists for every $\bar{f}_b$ such a $\bar{q}$ if and only if $\operatorname{im} E \subset \operatorname{im} B_s$.

(q_0, p_0) converges asymptotically to $(\bar{q}_\infty, p_\infty)$ if and only if the largest GL_s-invariant subspace contained in $\ker B_d^T$ is equal to $\operatorname{span} \mathbb{1}$.

Note that the above proposition has a classical interpretation in terms of the robustness of *integral control* with regard to constant disturbances: the springs act as integral controllers which counteract the influence of the unknown external force $\bar{f}_b$ so that the vector of momenta p will still converge to consensus.

Thanks to the systematic use of the port-Hamiltonian structure, the stability analysis given above is readily extendable to the nonlinear case.

12.6 Symmetry reduction

In this subsection we will show how port-Hamiltonian systems on graphs, such as the mass-spring-damper systems, can be alternatively obtained by *symmetry reduction* from a *symplectic* formulation, exploiting the invariance of the Hamiltonian function (in particular, of the spring potential energies).

Let us return to the formulation of a mass-spring system in Section 12.2, where the vertices correspond to the masses, and the edges to the springs in between them. An alternative is to consider the configuration vector $q_c \in \Lambda^0 =: Q_c$, describing the *positions* of all the masses. In fact, this is the classical starting point for Lagrangian mechanics, where we do *not* start with the *energy variables* q and p, but instead we start with the configuration vector q_c and the corresponding velocity vector $\dot{q}_c$. The classical Hamiltonian formulation is then obtained by *defining* the vector of momenta $p \in \Lambda_0 = Q_c^*$ as $p = M\dot{q}_c$ (with M the diagonal mass matrix), resulting in the *symplectic phase space* $Q_c \times Q_c^* = \Lambda^0 \times \Lambda_0$. For ordinary springs the relation between $q_c \in \Lambda^0$ and the vector $q \in \Lambda^1$ describing the elongations of the springs is given as $q = B^T q_c$. Hence in this case the Hamiltonian can be also expressed as a function H_c of (q_c, p) by defining

$$H_c(q_c, p) := H(B^T q_c, p). \tag{12.31}$$

It follows that the equations of motion of the mass-spring system (with

boundary masses) are given by the canonical Hamiltonian equations

$$\begin{aligned} \dot{q}_c &= \frac{\partial H_c}{\partial p}(q_c, p), \\ \dot{p} &= -\frac{\partial H_c}{\partial q_c}(q_c, p) + E f_b, \\ e^b &= E^T \frac{\partial H_c}{\partial p}(q_c, p), \end{aligned} \tag{12.32}$$

where, as before, f_b are the external forces exerted on the boundary masses and e^b are their velocities.

What is the relation with the port-Hamiltonian formulation given in Section 12.2 ? It turns out that this relation is precisely given by the standard procedure of *symmetry reduction* of a Hamiltonian system[7]. Indeed, since $\mathbb{1}^T B = 0$ the Hamiltonian function $H_c(q_c, p)$ given in (12.31) is *invariant* under the action of the group $\mathfrak{G} = \mathbb{R}$ acting on the phase space $\Lambda^0 \times \Lambda_0 \simeq \mathbb{R}^{2N}$ by the symplectic group action

$$(q_c, p) \mapsto (q_c + \alpha \mathbb{1}, p), \quad \alpha \in \mathfrak{G} = \mathbb{R}. \tag{12.33}$$

From standard reduction theory, see e.g. Marsden & Ratiu (1999); Libermann & Marle (1987) and the references quoted therein, it follows that we may factor out the configuration space $Q_c := \Lambda^0$ to the *reduced configuration space*

$$Q := \Lambda^0 / \mathfrak{G} \tag{12.34}$$

Let us assume that the graph is *connected*, or equivalently $\ker B^T = \operatorname{span} \mathbb{1}$. Then we have the following identification

$$Q := \Lambda^0 / \mathfrak{G} \simeq B^T \Lambda^0 \subset \Lambda^1. \tag{12.35}$$

Hence the *reduced state space* of the mass-spring system is given by $\operatorname{im} B^T \times \Lambda_0$, where $\operatorname{im} B^T \subset \Lambda^1$. Furthermore, under the symmetry action the canonical Hamiltonian equations (12.32) on the symplectic space $\Lambda^0 \times \Lambda_0$ reduce to the port-Hamiltonian equations (12.3) on

[7] This relation can be regarded as the discrete, graph-theoretic, version, of the correspondence between the port-Hamiltonian formulation of the Maxwell equations (using the Stokes-Dirac structure) and its symplectic formulation using the vector potential of the magnetic field, cf. Marsden & Ratiu (1999); Vankerschaver et al. (2010).

$\operatorname{im} B^T \times \Lambda_0 \subset \Lambda^1 \times \Lambda_0$ obtained before:

$$\begin{aligned} \dot{q} &= B^T \dot{q}_c = B^T \frac{\partial H_c}{\partial p}(q_c, p) = B^T \frac{\partial H}{\partial p}(q, p), \\ \dot{p} &= -\frac{\partial H_c}{\partial q_c}(q_c, p) + E f_b = -B \frac{\partial H}{\partial q}(q, p) + E f_b, \\ e^b &= E^T \frac{\partial H}{\partial p}(q, p). \end{aligned} \tag{12.36}$$

In case the graph is not connected, then the above symmetry reduction can be performed for each component of the graph (i.e., the symmetry group is $\mathbb{R}^{c_\mathcal{G}}$, with $c_\mathcal{G}$ denoting the number of components of the graph $\mathcal{G}$), yielding again the reduced state space[8] $\operatorname{im} B^T \times \Lambda_0$.

For a mass-spring-*damper* system, although in the standard symmetry reduction framework not considered as a Hamiltonian system, the same reduction procedure can still be applied. A mass-spring-damper system in coordinates (q_c, p) takes the form

$$\begin{aligned} \dot{q}_c &= \frac{\partial H_c}{\partial p}(q_c, p), \\ \dot{p} &= -\frac{\partial H_c}{\partial q_c}(q_c, p) - B_d R B_d^T \frac{\partial H_c}{\partial p}(q_c, p) + E f_b, \\ e^b &= E^T \frac{\partial H_c}{\partial p}(q_c, p), \end{aligned} \tag{12.37}$$

where $H_c(q_c, p) = H(B_s^T q, p)$ with $q = B_s^T q_c$ the spring elongations. Here B_s and B_d denote, as before, the incidence matrices of the spring, respectively, damper graph. Under the same symmetry action as above this reduces to the equations (12.20) on the reduced state space $\operatorname{im} B_s^T \times \Lambda_0$.

Furthermore we obtain the following corollary to Theorem 12.1 regarding to 'second-order consensus' (see also Goldin et al. (2010); Camlibel & Zhang (2012)):

Corollary 12.2. Consider the mass-spring-damper system (12.37) in coordinates (q_c, p) where we assume the spring graph to be connected.

[8] Note that in fact the subspace $\operatorname{im} B^T \subset \Lambda^1$ is determined by the Casimirs $k^T q, Bk = 0$ in the sense that $\operatorname{im} B^T = \{q \in \Lambda^0 \mid k^T q = 0, \text{ for all } k \in \ker B\}$. Furthermore, $\operatorname{im} B^T = \Lambda^1$ if and only if the graph does not contain cycles.

Then for all initial conditions $q_c(t) \to \operatorname{span} \mathbb{1}, p(t) \to \operatorname{span} \mathbb{1}$ if and only the largest GL_s-invariant subspace contained in $\ker B_d^T$ is equal to $\operatorname{span} \mathbb{1}$, and moreover $\ker B_s = 0$.

12.7 The graph Dirac structures and interconnection

Starting point for the definition of port-Hamiltonian systems on graphs in this chapter is the identification of the Poisson structure (12.2) corresponding to the incidence matrix of the graph. Inclusion of the boundary vertices leads to the definition of the following two 'canonical' Dirac structures.

Definition 12.1. Consider an open graph $\mathcal{G}$ with vertex, edge and boundary spaces, incidence matrix B and boundary incidence matrix B_b. The *flow-continuous graph Dirac structure* $\mathcal{D}_f(\mathcal{G})$ is defined as

$$\mathcal{D}_f(\mathcal{G}) := \Big\{(f_1, e^1, f_{0i}, e^{0i}, f_b, e^b) \in \Lambda_1 \times \Lambda^1 \times \Lambda_{0i} \times \Lambda^{0i} \times \Lambda_b \times \Lambda^b \mid B_i f_1 = f_{0i}, B_b f_1 = f_b, e^1 = -B_i^T e^{0i} - B_b^T e^b\Big\}. \tag{12.38}$$

The *effort-continuous graph Dirac structure* $\mathcal{D}_e(\mathcal{G})$ is defined as

$$\mathcal{D}_e(\mathcal{G}) := \Big\{(f_1, e^1, f_0, e^0, f_b, e^b) \in \Lambda_1 \times \Lambda^1 \times \Lambda_0 \times \Lambda^0 \times \Lambda_b \times \Lambda^b \mid B_i f_1 = f_{0i}, B_b f_1 = f_{0b} + f_b, e^1 = -B^T e^0, e^b = e^{0b}\}. \tag{12.39}$$

By Proposition 2.3 both $\mathcal{D}_f(\mathcal{G})$ and $\mathcal{D}_e(\mathcal{G})$ are separable Dirac structures. Note that $\mathcal{D}_f(\mathcal{G})$ and $\mathcal{D}_e(\mathcal{G})$ only differ in the role of the boundary flows and efforts, and that $\mathcal{D}_f(\mathcal{G}) = \mathcal{D}_e(\mathcal{G})$ if there are no boundary vertices.

Interconnection of two open graphs $\mathcal{G}^\alpha$ and $\mathcal{G}^\beta$ is done by identifying some of their boundary vertices, and equating (up to a minus sign) the boundary efforts and flows corresponding to these boundary vertices, resulting in a new graph. For simplicity of exposition consider the case that the open graphs have *all* their boundary vertices in common, resulting in a (closed) graph with set of vertices $\mathcal{V}_i^\alpha \cup \mathcal{V}_i^\beta \cup \mathcal{V}$, where $\mathcal{V} := \mathcal{V}_b^\alpha = \mathcal{V}_b^\beta$ denotes the set of boundary vertices of both graphs. The incidence operator of the interconnected (closed) graph is

obtained as follows. For simplicity of notation consider the case that $\mathcal{R} = \mathbb{R}$. Let $\mathcal{G}^j$ have incidence matrices

$$B^j = \begin{bmatrix} B_i^j \\ B_b^j \end{bmatrix}, \quad j = \alpha, \beta.$$

The incidence operator B of the interconnected graph is then given as

$$B = \begin{bmatrix} B_i^\alpha & 0 \\ 0 & B_i^\beta \\ B_b^\alpha & B_b^\beta \end{bmatrix}, \tag{12.40}$$

corresponding to the interconnection constraints on the boundary potentials and currents given by

$$e^{b\alpha} = e^{b\beta}, \quad f_b^\alpha + f_b^\beta = 0. \tag{12.41}$$

Comparing the *interconnection* of open graphs with the *composition* of their graph Dirac structures (see e.g. Proposition 6.1) it is readily seen that the flow/effort-continuous graph Dirac structure of an interconnected graph equals the composition of the flow/effort-continuous graph Dirac structures of $\mathcal{G}^\alpha$ and $\mathcal{G}^\beta$.

12.8 The Kirchhoff-Dirac structure

In this section we consider a third canonical graph Dirac structure, which results from *constraining* the flows at the internal vertices to *zero* (and thus there is no energy-storage or dissipation associated with the vertices for the corresponding port-Hamiltonian system).

The *Kirchhoff-Dirac structure* is defined as

$$\begin{aligned} \mathcal{D}_K(\mathcal{G}) := \Big\{ & (f_1, e^1, f_b, e^b) \in \Lambda_1 \times \Lambda^1 \times \Lambda_b \times \Lambda^b \mid \\ & B_i f_1 = 0, B_b f_1 = f_b, \exists e^{0i} \in \Lambda^{0i} \text{ s.t. } e^1 = -B_i^T e^{0i} - B_b^T e^b \Big\}. \end{aligned} \tag{12.42}$$

Note that, in contrast to the flow/effort-continuous graph Dirac structures, the Kirchhoff-Dirac structure only involves the flow and effort variables of the *edge* and *boundary* vertex spaces (not of the internal vertex spaces).

Proposition 12.4. $\mathcal{D}_K(\mathcal{G})$ is a separable Dirac structure.

Proof. The Kirchhoff-Dirac structure is equal to the composition of the flow-continuous[9] graph Dirac structure $\mathcal{D}_f(\mathcal{G})$ with the trivial separable Dirac structure defined as

$$\{(f_{0i}, e^{0i}) \in \Lambda_{0i} \times \Lambda^{0i} \mid f_{0i} = 0\}.$$

The result then follows from Proposition 6.1. ∎

Port-Hamiltonian systems with respect to the Kirchhoff-Dirac structure are defined completely similar to the case of the flow/effort-continuous graph Dirac structure; the difference being that energy-storing or dissipative relations are now only defined for the flow and effort variables corresponding to the edges.

12.8.1 Electrical circuits

The prime example of a port-Hamiltonian system[10] with respect to a Kirchhoff-Dirac structure is an electrical RLC-circuit, with circuit graph $\mathcal{G}$. In this case the elements of Λ_1 and Λ^1 denote the vectors of currents through, respectively the voltages across, the edges, and the Kirchhoff-Dirac structure amounts to Kirchhoff's current and voltage laws (whence its name). Furthermore, the effort variables e^0 are the *potentials* at the vertices, while the boundary flows and efforts f_b, e^b are the *boundary currents*, respectively *boundary potentials* at the boundary vertices (the *terminals* of the electrical circuit).

On top of Kirchhoff's laws, the dynamics is defined by the energy-storage relations corresponding to either capacitors or inductors, and dissipative relations corresponding to resistors. The energy-storing re-

[9] Or the composition of the effort-continuous graph Dirac structure with $\{(f_0, e^0) \in \Lambda_0 \times \Lambda^0 \mid f_0 = 0\}$.

[10] The terminology 'port-Hamiltonian' may be confusing in this context, because 'ports' in electrical circuits are usually defined by *pairs of terminals*, that is *pairs* of boundary vertices with external variables being the currents through and the voltages across an edge corresponding to each such port. See also the discussion in Willems (2007, 2010); van der Schaft & Maschke (2009).

lations for a capacitor at edge e are given by

$$\dot{Q}_e = -I_e, \quad V_e = \frac{dH_{Ce}}{dQ_e}(Q_e), \tag{12.43}$$

with Q_e the charge, and $H_{Ce}(Q_e)$ denoting the electric energy stored in the capacitor. Alternatively, in the case of an inductor one specifies the magnetic energy $H_{Le}(\Phi_e)$, where Φ_e is the magnetic flux linkage, together with the dynamic relations

$$\dot{\Phi}_e = V_e, \quad -I_e = \frac{dH_{Le}}{d\Phi_e}(\Phi_e). \tag{12.44}$$

Finally, a resistor at edge e corresponds to a static relation between the current I_e through and the voltage V_e across this edge, such that $V_e I_e \leq 0$. In particular, a linear (ohmic) resistor at edge e is specified by a relation $V_e = -R_e I_e$, with $R_e \geq 0$.

Alternatively, we can decompose the circuit graph $\mathcal{G}$ as the interconnection of a graph corresponding to the capacitors, a graph corresponding to the inductors, and a graph corresponding to the resistors. For simplicity let us restrict ourselves to the case of an LC-circuit without boundary vertices. Define $\hat{\mathcal{V}}$ as the set of all vertices that are adjacent to at least one capacitor *as well as* to at least one inductor. Then split the circuit graph into an open circuit graph $\mathcal{G}^C$ corresponding to the capacitors and an open circuit graph $\mathcal{G}^L$ corresponding to the inductors, both with set of boundary vertices $\hat{\mathcal{V}}$. Denote the incidence matrices of these two circuit graphs by

$$B^C := \begin{bmatrix} B_i^C \\ B_b^C \end{bmatrix}, \; B^L := \begin{bmatrix} B_i^L \\ B_b^L \end{bmatrix}$$

Assuming for simplicity that all capacitors and inductors are linear we arrive at the following equations for the C-circuit

$$\begin{aligned} B_b^C \dot{Q} &= I_b^C, \\ B_i^C \dot{Q} &= 0, \\ B_b^{CT} \psi_b^C &= C^{-1} Q - B_i^{CT} \psi_i^C. \end{aligned}$$

with Q the vector of charges of the capacitors and C the diagonal matrix with diagonal elements given by the capacitances of the capaci-

tors. Similarly for the L-circuit we obtain the equations

$$\begin{aligned}\dot{\Phi} &= B_b^{LT}\psi_b^L + B_i^{LT}\psi_i^L,\\ 0 &= B_i^L L^{-1}\Phi,\\ I_b^L &= -B_b^L L^{-1}\Phi,\end{aligned}$$

with Φ the vector of fluxes and L the diagonal matrix of inductances of all the inductors.

The equations of the LC-circuit are obtained by imposing the interconnection constraints $\psi_b^C = \psi_b^L =: \psi_i$ and $I_b^C + I_b^L = 0$. By eliminating the boundary currents I_b^C, I_b^L one thus arrives at the differential-algebraic port-Hamiltonian equations[11]

$$\begin{bmatrix} B_i^C & 0\\ 0 & B_i^L\\ B_b^C & B_b^L\end{bmatrix}\begin{bmatrix}-\dot{Q}\\ L^{-1}\Phi\end{bmatrix} = 0,\ \begin{bmatrix}C^{-1}Q\\ -\dot{\Phi}\end{bmatrix} = \begin{bmatrix} B_i^{CT} & 0 & B_b^{CT}\\ 0 & B_i^{LT} & B_b^{LT}\end{bmatrix}\begin{bmatrix}\psi_i^C\\ \psi_i^L\\ \psi_i\end{bmatrix}.$$

12.8.2 Boundary flows and efforts of the Kirchhoff-Dirac structure

The fact that the internal vertex flows in the definition of the Kirchhoff-Dirac structure are all zero (and consequently no storage or dissipation at the vertices takes place) has a number of specific consequences for the behavior of the boundary flows and efforts (see Willems (2010) for closely related considerations).

Assume (for simplicity of exposition) that $\mathcal{R} = \mathbb{R}$. From the definition of the Kirchhoff-Dirac structure and $\mathbb{1}^T B = 0$ it follows that

$$0 = \mathbb{1}^T B f_1 = \mathbb{1}_b^T B_b f_1 = -\mathbb{1}_b^T f_b, \tag{12.45}$$

with $\mathbb{1}_b$ denoting the vector with all ones of dimension equal to the number of boundary vertices. Hence the boundary part of the Kirchhoff-Dirac structure of an open graph is constrained by the fact that the boundary flows add up to zero. Dually, we may always add to the vector of vertex efforts e^0 the vector $\mathbb{1}$ leaving invariant the edge

[11] For a formulation of pure R, L or C circuits, and their weighted Laplacian matrices, we refer to van der Schaft (2010).

efforts $e^1 = B^T e^0$. Hence, to the vector of boundary efforts e^b we may always add the vector $\mathbb{1}_b$.

Proposition 12.5. Consider an open graph $\mathcal{G}$ with Kirchhoff-Dirac structure $\mathcal{D}_K(\mathcal{G})$. Then for each $(f_1, e^1, f_b, e^b) \in \mathcal{D}_K(\mathcal{G})$ it holds that

$$\mathbb{1}_b^T f_b = 0\,,$$

while for any constant $c \in \mathbb{R}$

$$(f_1, e^1, f_b, e^b + c\mathbb{1}_b) \in \mathcal{D}_K(\mathcal{G}).$$

12.9 Topological analogies

From the above formulation of an RLC-circuit in Section 12.8.1 we conclude that the structure of the dynamical equations of an inductor are structurally *different* from that of a capacitor. In order to elucidate this basic difference we zoom in on the description of an inductor and a capacitor as two-terminal elements. To this end consider the elementary open graph consisting of one edge with two boundary vertices α, β, described by the incidence matrix $b = \begin{bmatrix} 1 & -1 \end{bmatrix}^T$. It follows that an inductor with magnetic energy $H(\Phi)$ is described by the equations

$$\begin{aligned} \dot{\Phi} &= b^T \begin{bmatrix} \psi_\alpha \\ \psi_\beta \end{bmatrix}, \\ \begin{bmatrix} I_\alpha \\ I_\beta \end{bmatrix} &= b \frac{dH}{d\Phi}(\Phi), \end{aligned} \tag{12.46}$$

whereas a capacitor with electric energy $H(Q)$ is described as

$$\begin{aligned} b\dot{Q} &= \begin{bmatrix} I_\alpha \\ I_\beta \end{bmatrix}, \\ \frac{dH}{dQ}(Q) &= b^T \begin{bmatrix} \psi_\alpha \\ \psi_\beta \end{bmatrix}. \end{aligned} \tag{12.47}$$

This difference stems from the fact that the energy variable Q of a capacitor, as well as the current I, takes values in the linear space Λ_1, while the state variable Φ of an inductor, as well as the voltage V,

takes values in the *dual* space Λ^1. Recalling from Section 12.2.1 the description of a spring system

$$\begin{aligned} \dot{q} &= b^T \begin{bmatrix} v_\alpha \\ v_\beta \end{bmatrix}, \\ \begin{bmatrix} F_\alpha \\ F_\beta \end{bmatrix} &= b\frac{dH}{dq}(q), \end{aligned} \tag{12.48}$$

with q the elongation of the spring, and $H(q)$ its potential energy, we conclude that there is a strict analogy between a *spring* and an *inductor*[12]. On the other hand, a moving *mass* is *not* a strict analog of a *capacitor*. Instead, it can be considered to be the analog of a *grounded* capacitor, while the strict analog of a capacitor (12.47) is the so-called *inerter* Smith (2002)

$$b\dot{p} = \begin{bmatrix} F_\alpha \\ F_\beta \end{bmatrix}, \quad \frac{dH}{dp}(p) = b^T \begin{bmatrix} v_\alpha \\ v_\beta \end{bmatrix},$$

where p is the momentum of the inerter and $H(p)$ its kinetic energy, while F_α, F_β and v_1, v_2 denote the forces, respectively, velocities, at the two terminals of the inerter. For a further discussion on analogies, see Appendix B.

[12]Thus we favor the so-called *force-current analogy* instead of the *force-voltage analogy*.

13

Switching port-Hamiltonian systems

In quite a few application areas (e.g., power converters, robotics, hydraulic networks) systems arise which operate in different modes. Such systems are commonly called *switching* or *multi-modal* physical systems. In many cases of interest it is appropriate to model the fast transitions between the different modes of operation of these systems by *ideal switches*. In this chapter we will investigate how multi-modal physical systems can be approached from the port-Hamiltonian point of view. It will turn out that the varying topology of the system corresponds to a varying Dirac structure, while the energy-storage and energy-dissipation is the same for all the modes.

Another aspect of multi-modal physical systems is that often the modes of the system may involve algebraic constraints on the state variables, while at the moment of switching the current state does not satisfy the algebraic constraints of the next mode. This problem needs to be resolved by the formulation of a *jump rule*, stating how the present state should be changed instantaneously in order that it satisfies the algebraic constraints of the next mode.

A classical example of such a jump rule arises in electrical circuit theory, and concerns the characterization of the discontinuous change

in the charges of the capacitors and/or in the magnetic fluxes of the inductors whenever switches are instantaneously closed or opened. This is sometimes referred to as the *charge and flux conservation principle*, and is usually only discussed on the basis of examples; see e.g. Seshu & Balabanian (1964). In this chapter we will state a jump rule for general switching port-Hamiltonian systems, which will include the classical charge and flux conservation principle as a special case. The discontinuous change of the state involved in the jump rule amounts to an impulsive motion satisfying a set of conservation laws derived from the general conservation laws of the port-Hamiltonian system. Furthermore, if the Hamiltonian function is convex and non-negative it follows that the switching port-Hamiltonian system with this jump rule is *passive*.

This chapter is largely based on van der Schaft & Camlibel (2009), to which we refer for some of the proofs and for further extensions.

13.1 Switching port-Hamiltonian systems

For the definition of a *switching port-Hamiltonian system* we need the following ingredients. We start with an overall Dirac structure $\mathcal{D}$ on the space of all flow and effort variables involved:

$$\mathcal{D} \subset \mathcal{F}_x \times \mathcal{E}_x \times \mathcal{F}_R \times \mathcal{E}_R \times \mathcal{F}_P \times \mathcal{E}_P \times \mathcal{F}_S \times \mathcal{E}_S. \tag{13.1}$$

The space $\mathcal{F}_x \times \mathcal{E}_x$ is the space of flow and effort variables corresponding to the *energy-storing* elements[1], the space $\mathcal{F}_R \times \mathcal{E}_R$ denotes the space of flow and effort variables of the *energy-dissipating* elements, while $\mathcal{F}_P \times \mathcal{E}_P$ is the space of flow and effort variables corresponding to the *external ports* (or sources). Finally, the linear spaces $\mathcal{F}_S$, respectively $\mathcal{E}_S := \mathcal{F}_S^*$, denote the flow and effort spaces of the ideal *switches*. Let s be the number of switches, then every subset $\pi \subset \{1, 2, \ldots, s\}$ defines a *switch configuration*, according to

$$e_S^i = 0, \quad i \in \pi, \qquad f_S^j = 0, \quad j \notin \pi. \tag{13.2}$$

[1]Note the slight change in notation with respect to other chapters: we have reserved the notation $\mathcal{F}_S$ and $\mathcal{E}_S$ for the flow and effort spaces of the *switches*, and $\mathcal{F}_x$ and $\mathcal{E}_x$ for the flow and effort spaces of the *energy-storing* elements.

We will say that in switch configuration π, for all $i \in \pi$ the i-th switch is *closed*, while for $j \notin \pi$ the j-th switch is *open*.

For each fixed switch configuration π this leads to the following subspace $\mathcal{D}_\pi$ of the restricted space of flows and efforts $\mathcal{F}_x \times \mathcal{E}_x \times \mathcal{F}_R \times \mathcal{E}_R \times \mathcal{F}_P \times \mathcal{E}_P$:

$$\begin{aligned}\mathcal{D}_\pi = \Big\{ (f_x, e_x, f_R, e_R, f_P, e_P) \mid \exists f_S \in \mathcal{F}_S, e_S \in \mathcal{E}_S \\ \text{s.t. } e_S^i = 0, i \in \pi,\ f_S^j = 0, j \notin \pi, \text{ and} \\ (f_x, e_x, f_R, e_R, f_P, e_P, f_S, e_S) \in \mathcal{D} \Big\}.\end{aligned} \tag{13.3}$$

For every π the subspace $\mathcal{D}_\pi$ defines a Dirac structure. Indeed, every switch configuration π given by (13.2) defines a Dirac structure on the space of flow and effort variables f_S, e_S of the switches, and $\mathcal{D}_\pi$ equals the composition of this Dirac structure with the overall Dirac structure $\mathcal{D}$. Since the composition of any two Dirac structures is again a Dirac structure (cf. Chapter 6) it thus follows that $\mathcal{D}_\pi$ is a Dirac structure for every switch configuration π.

The dynamics of the switching port-Hamiltonian system is defined by specifying as before, next to its Dirac structure $\mathcal{D}$, the constitutive relations of the energy-storing and energy-dissipating (resistive) elements. We will restrict to linear resistive structures given by the graph of a resistive mapping (see Chapter 4).

Definition 13.1. Consider a linear state space $\mathcal{X} = \mathcal{F}_x$, a Dirac structure $\mathcal{D}$ given by (13.1), a Hamiltonian $H : \mathcal{X} \to \mathbb{R}$, and a linear resistive structure $f_R = -Re_R$ with $R = R^T \geq 0$. Then the dynamics of the corresponding switching port-Hamiltonian system is given as

$$(-\dot{x}(t), \frac{\partial H}{\partial x}(x(t)), -Re_R(t), e_R(t), f_P(t), e_P(t)) \in \mathcal{D}_\pi \tag{13.4}$$

at all time instants t during which the system is in switch configuration π.

It follows from the power-conservation property of Dirac structures that during the time-interval in which the system is in a fixed switch configuration

$$\frac{d}{dt}H = -e_R^T R e_R + e_P^T f_P \leq e_P^T f_P, \tag{13.5}$$

thus showing *passivity for each fixed switch configuration* if the Hamiltonian H is non-negative.

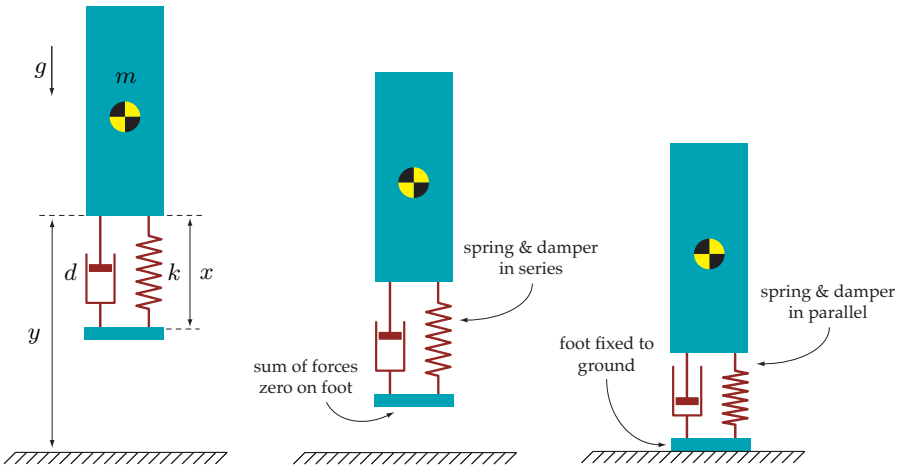


Figure 13.1: Bouncing pogo-stick: definition of the variables (left), flying phase (middle), contact phase (right).

Example 13.1. Consider a pogo-stick that bounces on a horizontal plate of variable height (see Figure 13.1). It consists of a mass m and a mass-less foot, interconnected by a linear spring (with stiffness k and rest length x_0) and a linear damper d. The states of the system are x (length of the spring), y (height of the bottom of the mass), and $z = m\dot{y}$ (momentum of the mass). The total energy is

$$H(x, y, z) = \frac{1}{2}k(x - x_0)^2 + mg(y + y_0) + \frac{1}{2m}p^2,$$

where y_0 is the distance from the bottom of the mass to the center of mass. This leads to the constitutive relations given as

$$\begin{aligned} &f_x = -\dot{x},\ f_y = -\dot{y},\ f_p = -\dot{p},\ f_R = -\tfrac{1}{d}e_R, \\ &e_x = k(x - x_0),\ e_y = mg,\ e_p = \tfrac{p}{m}, \end{aligned}$$

where f_R, e_R are the flow and effort variables associated to the damper (energy dissipation), and d is the damping constant. Furthermore, the overall Dirac structure of the system is described by the linear equa-

tions

$$f_y = f_x - f_S,\ f_R = f_x,\ f_p = e_x + e_y + e_R,$$
$$e_S + e_x + e_R = 0,\ e_p = -f_y.$$

Here the third equation $f_p = e_x + e_y + e_R$ represents the total force balance on the mass m. In the switch configuration $e_S = 0$ (no external force on the foot) the pogo-stick is in its flying mode, while for $f_S = 0$ the foot is in contact with the horizontal plate. Hence the equation $e_S + e_x + e_R = 0$ expresses that for $e_S = 0$ (flying mode) the spring force on the mass-less foot balances the damping force, while for $f_S = 0$ (contact mode) $f_y = f_x$ and e_S represents the constraint force exerted by the ground.

The conditions (13.4) for a particular switch configuration π may entail algebraic constraints on the state variables x. These are characterized by the effort *constraint subspace* defined for each switch configuration π as follows (see also Chapter 8):

$$\begin{aligned}\mathcal{C}_\pi := \Big\{ e_x \in \mathcal{E}_x \mid \exists f_x, f_R, e_R, f_P, e_P, \text{ s.t.} \\ (f_x, e_x, f_R, e_R, f_P, e_P) \in \mathcal{D}_\pi,\ f_R = -R e_R \Big\}.\end{aligned} \tag{13.6}$$

The subspace[2] $\mathcal{C}_\pi$ determines, together with H, the *algebraic constraints* in each switch configuration π. Indeed, from (13.4) it follows that

$$\frac{\partial H}{\partial x}(x(t)) \in \mathcal{C}_\pi, \tag{13.7}$$

for all time instants t during which the system is in switch configuration π. Hence if $\mathcal{C}_\pi \neq \mathcal{E}_x$ then in general (depending on the Hamiltonian H) this imposes algebraic constraints on the state vector $x(t)$.

Example 13.2 (Pogo-stick continued)**.** In the example of the pogo-stick the subspace $\mathcal{C}_\pi$ is *equal* to $\mathcal{E}_x$ for any of the two switch configurations, and hence there are no algebraic constraints. This changes, however, if the mass of the foot is taken into account. Indeed, by assuming a mass $M > 0$ of the foot, there is an additional state variable

[2]Note that $\mathcal{C}_\pi$ may depend on the linear resistive structure, but *not* on the energy storage constitutive relation.

p_M (momentum of the foot) with kinetic energy $\frac{1}{2}p_M^2$, and the equation $e_S + e_x + e_R = 0$ changes into $f_M = e_S + e_x + e_R$ with $f_M = -\dot{p}_M$, while furthermore, an extra equation $e_{p_M} == f_y - f_x$ with $e_M = \frac{p_M}{m}$ is added to the equations of the overall Dirac structure.

In the contact mode π given by $f_S = 0$, this means that $e_{p_M} = f_y - f_x = 0$, so that

$$\mathcal{C}_\pi = \{(e_x, e_y, e_p, e_{p_M}) \mid e_{p_M} = 0\},$$

implying the obvious algebraic constraint $\frac{p_M}{m} = e_{p_M} = 0$.

13.2 Jump rule for switching port-Hamiltonian systems

Next, we define for each π the *jump space*

$$\mathcal{J}_\pi := \{f_x \mid (f_x, 0, 0, 0, 0, 0) \in D_\pi\}. \tag{13.8}$$

The following crucial relation between the jump space $\mathcal{J}_\pi$ and the effort constraint subspace $\mathcal{C}_\pi$ holds true. Recall that $\mathcal{J}_\pi \subset \mathcal{F}_x$ while $\mathcal{C}_\pi \subset \mathcal{E}_x$, where $\mathcal{E}_x = \mathcal{F}_x^*$.

Theorem 13.1.

$$\mathcal{J}_\pi = \mathcal{C}_\pi^\perp, \tag{13.9}$$

where $^\perp$ denotes the orthogonal complement with respect to the duality product between the dual spaces $\mathcal{F}_x$ and $\mathcal{E}_x$.

The jump rule for a switch configuration π is now formulated as follows.

Definition 13.2 (Jump rule)**.** Consider the state x^- of a switching port-Hamiltonian system at a switching time where the switch configuration of the system changes into π. Suppose x^- is *not* satisfying the algebraic constraints corresponding to π, that is

$$\frac{\partial H}{\partial x}(x^-) \notin \mathcal{C}_\pi. \tag{13.10}$$

Then, the new state x^+ just after the switching time satisfies

$$x^+ - x^- \in \mathcal{J}_\pi, \quad \frac{\partial H}{\partial x}(x^+) \in \mathcal{C}_\pi. \tag{13.11}$$

This means that at this switching time an instantaneous jump ('*state transfer*') from x^- to x^+ with $x_{\text{transfer}} := x^+ - x^- \in \mathcal{J}_\pi$ will take place, in such a manner that $\frac{\partial H}{\partial x}(x^+) \in \mathcal{C}_\pi$.

The jump space $\mathcal{J}_\pi$ is the space of flows in the state space $\mathcal{X} = \mathcal{F}_x$ that is compatible with zero effort e_x at the energy-storing elements and zero flows f_R, f_P and efforts e_R, e_P at the resistive elements and external ports. Said otherwise, the jump space consists of all flow vectors f_x that may be *added* to the present flow vector corresponding to a certain effort vector at the energy storage and certain flow and effort vectors at the resistive elements and external ports, while remaining in the Dirac structure $\mathcal{D}_\pi$, *without changing* these other effort and flow vectors. Since $\mathcal{D}_\pi$ captures the full power-conserving interconnection structure of the system while in switch configuration π, reflecting the underlying *conservation laws* of the system, the jump space J_π thus corresponds to a particular subset of conservation laws, and the jump rule formulated above proclaims that the discontinuous change in the state vector is an impulsive motion satisfying this particular set of conservation laws.

For physical systems one would expect that the value of the Hamiltonian $H(x^+)$ immediately *after* the switching time is less than or equal to the value $H(x^-)$ just before. This property is ensured whenever the Hamiltonian is a *convex* function.

Theorem 13.2. Consider a switching port-Hamiltonian system with H a *convex* function. Then for any x^- and x^+ satisfying the jump rule (13.11)

$$H(x^+) \leq H(x^-). \tag{13.12}$$

The proof is based on the fact that a function $f : \mathbb{R}^n \to \mathbb{R}$ is convex if and only if [Rockafellar & Wets (1998)]

$$f(y) \geq f(x) + < \frac{\partial f}{\partial x}(x) \mid y - x >,$$

for all x, y. Application to H with $x = x^+$ and $y = x^-$ yields

$$H(x^-) \geq H(x^+) + < \frac{\partial H}{\partial x}(x^+) \mid x^- - x^+ > .$$

However, by (13.11) $< \frac{\partial H}{\partial x}(x^+) \mid x^- - x^+ >= 0$, since $\mathcal{J}_\pi = \mathcal{C}_\pi^\perp$.

Corollary 13.3. Consider a switching port-Hamiltonian system satisfying the jump rule, with its Hamiltonian H being a convex function. Then for all $t_2 \geq t_1$

$$H(x(t_2)) \leq H(x(t_1)) + \int_{t_1}^{t_2} e_P^T(t) f_P(t) dt,$$

and thus the system is *passive* if H is nonnegative (see Chapter 7).

If the Hamiltonian H is a quadratic function $H(x) = \frac{1}{2} x^T K x$ (and thus the port-Hamiltonian system is linear), then the jump rule reduces to

$$x_{\text{transfer}} = x^+ - x^- \in \mathcal{J}_\pi, \quad Kx^+ \in \mathcal{C}_\pi. \tag{13.13}$$

If $K \geq 0$ then it follows from Theorem 13.2 and Corollary 13.3 that the switching port-Hamiltonian system is passive. Furthermore, for each x^- there exists a x^+ satisfying (13.13), and moreover if $K > 0$ this x^+ (and the jump x_{transfer}) is *unique*. Indeed, the property $\mathcal{J}_\pi = \mathcal{C}_\pi^\perp$ implies $\lambda^T K x = 0$ for all $\lambda \in \mathcal{J}_\pi$ and all $x \in \mathcal{X}$ with $Kx \in \mathcal{C}_\pi$, or equivalently $\lambda^T K x = 0$ for all $x \in \mathcal{C}_\pi^K := \{x \in \mathcal{X} \mid Kx \in \mathcal{C}_\pi\}$ and all $\lambda \in J_\pi$. Thus, J_π is the orthogonal complement of the subspace $\mathcal{C}_\pi^K$ where the inner product on $\mathcal{X}$ is defined by the positive definite matrix K. Hence it follows that the vector x^+ satisfying (13.13) is unique.

The jump rule in the linear case also allows for a *variational characterization* (see also Camlibel & Zhang (2012); Gerritsen et al. (2002)).

Theorem 13.4. Let $K \geq 0$. A state x^+ satisfying (13.13) is a solution of the minimization problem (for given x^-)

$$\min_{x, Kx \in \mathcal{C}_\pi} \frac{1}{2}(x - x^-)^T K (x - x^-), \tag{13.14}$$

and conversely if $K > 0$ then the unique solution of (13.14) is the unique solution to (13.13).

Furthermore, an application of *Dorn's duality* Mangasarian (1969); Camlibel (2001) yields (see also Camlibel (2001); Gerritsen et al. (2002))

Theorem 13.5. Let $K > 0$. Then the jump $\lambda = x^+ - x^-$ is the unique minimum of

$$\min_{\lambda \in \mathcal{J}_\pi} \frac{1}{2}(x^- + \lambda)^T K (x^- + \lambda). \tag{13.15}$$

13.3 Charge and flux transfer in switched RLC circuits

In this subsection we will show how the jump rule for switching port-Hamiltonian systems, as formulated above, includes the classical *charge and flux conservation principles* for RLC-circuits with switches as a special case, and in fact formalizes these principles in an insightful way.

Consider an RLC-circuit with switches with an *arbitrary topology*. It can be described as a switched port-Hamiltonian system as follows (see also Escobar et al. (1999)). First consider the directed graph associated with the circuit. Identify every capacitor, every inductor, every resistor and every switch with an edge. Furthermore, associate with every external port an edge (between the terminals of the port). Denote the incidence matrix of this directed graph by B; cf. Section 12.1. The incidence matrix has as many columns as there are edges, and as many rows as there are vertices in the graph. By reordering the edges we partition the incidence matrix as $B = \begin{bmatrix} B_C & B_L & B_R & B_S & B_P \end{bmatrix}$, where the sub-matrices B_C, B_L, B_R, B_S correspond, respectively, to the capacitor, inductor, resistor, and switch edges, and B_P corresponds to the external ports. Then Kirchhoff's current laws are given as

$$B_C I_C + B_L I_L + B_R I_R + B_S I_S + B_P I_P = 0, \tag{13.16}$$

with I_C, I_L, I_R, I_S, I_P denoting the currents through, respectively, the capacitors, inductors, resistors, switches, and the external ports. Correspondingly, Kirchhoff's voltage laws are given as

$$\begin{aligned} V_C &= B_C^T \psi, \\ V_L &= B_L^T \psi, \\ V_R &= B_R^T \psi, \\ V_S &= B_S^T \psi, \\ V_P &= B_P^T \psi, \end{aligned} \tag{13.17}$$

with V_C, V_L, V_R, V_S, V_P denoting the voltages across the capacitors, inductors, resistors, switches, and ports, respectively, and ψ being the vector of potentials at the vertices.

Kirchhoff's current and voltage laws define a separable Dirac structure $\mathcal{D}$ on the space of flow and effort variables given as

$$\begin{aligned} f_x &= (I_C, V_L), \\ e_x &= (V_C, I_L), \\ f_R &= V_R, \\ e_R &= I_R, \\ f_S &= V_S, \\ e_S &= I_S, \\ f_P &= V_P, \\ e_P &= I_P. \end{aligned} \tag{13.18}$$

The constitutive relations for the energy storage are given as

$$\begin{aligned} (\dot{Q}, \dot{\Phi}) &= -(I_C, V_L), \\ (V_C, I_L) &= \left(\frac{\partial H}{\partial Q}, \frac{\partial H}{\partial \Phi} \right), \end{aligned} \tag{13.19}$$

where Q is the vector of charges at the capacitors, and Φ the vector of fluxes of the inductors. For a linear RLC-circuit

$$H(Q, \Phi) = \frac{1}{2} Q^T C^{-1} Q + \frac{1}{2} \Phi^T L^{-1} \Phi, \tag{13.20}$$

where the diagonal elements of the diagonal matrices C and L are the capacitances, respectively, inductances, of the capacitors and inductors.

Similarly, the constitutive relations for the linear (Ohmic) resistors are given as

$$V_R = -R I_R, \tag{13.21}$$

with R denoting a diagonal matrix with diagonal elements being the resistances of the resistors.

For every subset $\pi \subset \{1, \cdots, s\}$ (where s is the number of

switches) the Dirac structure $\mathcal{D}_\pi$ is defined by the equations

$$\begin{aligned} 0 &= B_C I_C + B_L I_L + B_R I_R + B_S I_S + B_P I_P, \\ V_C &= B_C^T \psi, \\ V_L &= B_L^T \psi, \\ V_R &= B_R^T \psi, \\ V_S &= B_S^T \psi, \\ V_P &= B_P^T \psi, \\ V_S^i &= 0,\ i \in \pi, \quad I_S^j = 0,\ j \notin \pi. \end{aligned} \tag{13.22}$$

Thus, all switches corresponding to the subset π are closed, while the remaining ones are open.

The constraint subspace $\mathcal{C}_\pi$ for each switch configuration π is given as

$$\begin{aligned} \mathcal{C}_\pi = \{(V_C, I_L) \mid \exists I_C, V_L, V_R, I_R, V_S, I_S, V_P, I_P \\ \text{such that (13.21) and (13.22) is satisfied } \}. \end{aligned} \tag{13.23}$$

Furthermore, the jump space $\mathcal{J}_\pi$ is given as the set of all (I_C, V_L) satisfying for some ψ the equations

$$\begin{aligned} 0 &= B_C I_C + B_S I_S, \\ 0 &= B_C^T \psi, \\ V_L &= B_L^T \psi, \\ 0 &= B_R^T \psi, \\ V_S &= B_S^T \psi, \\ 0 &= B_P^T \psi, \\ V_S^i &= 0,\ i \in \pi, \quad I_S^j = 0,\ j \notin \pi. \end{aligned} \tag{13.24}$$

Hence the jump space can be written as the product of the space

$$\{I_C \mid \exists I_S,\ I_S^j = 0, j \notin \pi,\ B_C I_C + B_S I_S = 0\},$$

with the space

$$\begin{aligned} \{V_L \mid \exists \psi \text{ such that } V_L = B_L^T \psi, 0 = B_C^T \psi, 0 = B_R^T \psi, \\ 0 = B_P^T \psi, V_S = B_S^T \psi, V_S^i = 0, i \in \pi\}. \end{aligned}$$

It follows that for RLC circuits with switches the jump (state transfer) splits into a *charge transfer* $Q^+ - Q^- = Q_{\text{transfer}}$ and a *flux transfer* $\Phi^+ - \Phi^- = \Phi_{\text{transfer}}$. The charge transfer Q_{transfer} corresponding to the switch configuration π is specified as follows. The *direction* of the charge transfer Q_{transfer} is determined by

$$B_C Q_{\text{transfer}} + B_S I_S = 0, \quad I_S^j = 0,\ j \notin \pi, \tag{13.25}$$

corresponding to *Kirchhoff's current laws* for the circuit with switch configuration π, where the inductors and resistors have been *open-circuited*, and the currents through the external ports are all zero. This recovers and formalizes Frasca et al. (2010) the classical *charge conservation principle*. On the other hand, the *amount* of charge transfer Q_{transfer} is determined by

$$\begin{aligned} &C^{-1}(Q^- + Q_{\text{transfer}}) = B_C^T \psi \\ &\text{for some } \psi \text{ satisfying } V_S = B_S^T \psi,\ V_S^i = 0, i \in \pi. \end{aligned}$$

Furthermore, the direction of the flux transfer Φ_{transfer} is determined by the equations

$$\begin{aligned} 0 &= B_C^T \psi, \\ \Phi_{\text{transfer}} &= B_L^T \psi, \\ 0 &= B_R^T \psi, \\ V_S &= B_S^T \psi, \quad V_S^i = 0,\ i \in \pi, \\ 0 &= B_P^T \psi, \end{aligned} \tag{13.26}$$

corresponding to *Kirchhoff's voltage laws* for the circuit corresponding to the switch configuration π, where the capacitors and the resistors have been *short-circuited*, and the voltages across the external ports are all zero. This formalizes the classical *flux conservation principle*. On the other hand, the *amount* of flux transfer is uniquely determined by the condition

$$\begin{aligned} &B_C I_C + B_L L^{-1}(\Phi^- + \Phi_{\text{transfer}}) \\ &+ B_R I_R + B_S I_S + B_P I_P = 0, \\ &\text{for some } I_C, I_R, I_P, I_S \text{ with } I_S^j = 0,\ j \notin \pi. \end{aligned}$$

Since in the case of a linear circuit the Hamiltonian $H(Q,\Phi) = \frac{1}{2}Q^T C^{-1} Q + \frac{1}{2}\Phi^T L^{-1}\Phi$ splits as the sum of a quadratic function of the charge Q and the flux Φ, the variational characterization of the jump (state transfer) rule also splits into the variational characterization of the charge transfer principle, given as the minimization of

$$\min_{Q, C^{-1}Q \in C_\pi^V} \frac{1}{2}(Q - Q^-)^T C^{-1}(Q - Q^-), \tag{13.27}$$

where C_π^V denotes the projection of the subspace C_π on the space of voltages V_C, and the variational characterization of the flux transfer principle, given as the minimization of

$$\min_{\Phi, L^{-1}\Phi \in C_\pi^I} \frac{1}{2}(\Phi - \Phi^-)^T L^{-1}(\Phi - \Phi^-), \tag{13.28}$$

where C_π^I denotes the projection of the subspace $\mathcal{C}_\pi$ on the space of currents I_L.

13.4 The jump rule for switched mechanical systems

As a second example consider mechanical systems subject to linear damping and kinematic constraints, written in port-Hamiltonian form as (see e.g. Chapter 2)

$$\begin{aligned}
\dot{q} &= \frac{\partial H}{\partial p}(q,p), \\
\dot{p} &= -\frac{\partial H}{\partial q}(q,p) - \bar{R}(q,p)\frac{\partial H}{\partial p}(q,p) + A(q)\lambda + B(q)F, \\
0 &= A^T(q)\frac{\partial H}{\partial p}(q,p), \\
v &= B^T(q)\frac{\partial H}{\partial p}(q,p),
\end{aligned} \tag{13.29}$$

with $q = (q_1, \cdots, q_n)$ the vector of generalized position coordinates, $p = (p_1, \cdots, p_n)$ the vector of generalized momenta, $F \in \mathbb{R}^m$ the vector of external generalized forces, $v \in \mathbb{R}^m$ the vector of conjugated generalized velocities, and $H(q,p)$ the total energy of the system (which usually can be split into a kinetic and a potential energy contribution). Furthermore, $0 = A^T(q)\frac{\partial H}{\partial p}(q,p) = A^T(q)\dot{q}$ denotes the

kinematic constraints (such as rolling without slipping) with corresponding constraint forces $\lambda \in \mathbb{R}^s$, where s is the number of kinematic constraints (equal to the number of rows of the matrix $A^T(q)$).

The damping is characterized by the $n \times n$ matrix $\bar{R}(q,p)$ which is assumed to be symmetric and positive semi-definite, that is, $\bar{R}^T = \bar{R} \geq 0$. This implies the usual energy-balance

$$\frac{dH}{dt}(q,p) = -\frac{\partial H}{\partial p}^T(q,p)\bar{R}(q,p)\frac{\partial H}{\partial p}(q,p) + v^T F \leq v^T F.$$

We throughout assume that the matrix $\bar{R}(q,p)$ admits a factorization

$$\bar{R}(q,p) = P^T(q,p)RP(q,p), \quad R = R^T > 0,$$

for some $r \times n$ matrix $P(q,p)$ and *constant* $r \times r$ matrix R.

A *switching mechanical system* arises if the kinematic constraints can be turned on and off. Denoting $f_S := \lambda$ and replacing the kinematic constraints in (13.29) by

$$e_S := A^T(q)\frac{\partial H}{\partial p}(q,p), \tag{13.30}$$

this defines a switching port-Hamiltonian system as before, where any subset $\pi \subset \{1, \cdots, r\}$ defines as before the switch configuration $e_S^i = 0, i \in \pi$, $f_S^j = 0, j \notin \pi$. Thus in switch configuration π each i-th kinematic constraint, with $i \in \pi$, is active, while the other kinematic constraints (corresponding to indices not in π) are inactive.

It follows that the effort constraint subspace $\mathcal{C}_\pi$ in this case is given as

$$\begin{aligned}
&\mathcal{C}_\pi = \{e_x \mid \exists f_x, f_R, e_R, F, f_S, \text{ with} \\
&f_S^j = 0, j \notin \pi, \quad f_R = -Re_R,\, e_R = P^T(q,p)\tfrac{\partial H}{\partial p}(q,p) \\
&-f_x = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix} e_x + P(q,p)f_R + B(q)F + A(q)f_S \\
&e_S = A^T(q)\tfrac{\partial H}{\partial p}(q,p), \quad e_S^i = 0, i \in \pi\}.
\end{aligned}$$

Furthermore, the jump space $\mathcal{J}_\pi$ is given as

$$\mathcal{J}_\pi = \{f_x \mid f_x \in \operatorname{im} \begin{bmatrix} 0 \\ A_\pi(q) \end{bmatrix}\}$$

where the matrix $A_\pi(q)$ is obtained from the matrix $A(q)$ by leaving out every j-th column with $j \notin \pi$.

Thus jump rule in this case amounts to a jump in the momentum variables p given as

$$p_{\text{transfer}} = p^+ - p^- \in A_\pi(q), \quad A_\pi^T(q)\frac{\partial H}{\partial p}(q, p^+) = 0$$

If H can be written as the sum of a kinetic and a potential energy $H(q,p) = \frac{1}{2}p^T M^{-1}(q)p + V(q)$, with $M(q) > 0$ denoting the generalized mass matrix, then a variational characterization of the jump rule is given by defining p^+ to be the unique minimum of

$$\min_{p, A_\Pi^T(q)M^{-1}(q)p=0} \frac{1}{2}(p-p^-)^T M^{-1}(q)(p-p^-) \tag{13.31}$$

Furthermore, since in this case the kinetic energy is a convex function of the momenta, it follows from Theorem 13.2 and Corollary 13.3 that the switching mechanical system is passive if the potential energy is non-negative.

14

Distributed-parameter systems

The aim of this chapter is to introduce the main concepts behind the extension of finite-dimensional port-Hamiltonian systems of the previous chapters to distributed-parameter systems. Dynamic models of distributed-parameter systems are defined by considering not only the time but also the space as independent parameters on which the physical quantities are defined. They allow to model objects such as vibrating strings or plates, transmission lines, or electromagnetic fields and mass and heat transfer phenomena. A port-Hamiltonian formulation of classes of distributed-parameter systems is presented, which incorporates the energy flow through the boundary of the spatial domain of the system. Instrumental for its contraction is the notion of an infinite-dimensional Dirac structure associated with the exterior derivative and based on Stokes' theorem. The theory is exemplified using the telegrapher's equations for an ideal transmission line, Maxwell's equations on a bounded domain with non-zero Poynting vector at its boundary, and a vibrating string with traction forces at its ends. Finally, some properties of the Stokes-Dirac structure are reviewed, including the analysis of conservation laws. For further details we refer to van der Schaft & Maschke (2002), on which this

chapter is largely based. A detailed treatment of linear distributed-parameter port-Hamiltonian systems on a one-dimensional spatial domain, including *well-posedness* and *stability* theory, can be found in Jacob & Zwart (2012).

14.1 The Stokes-Dirac structure

We start by introducing the underlying geometric framework for the port-Hamiltonian formulation of distributed-parameter systems on a bounded spatial domain, with non-zero energy flow through the boundary. The key concept is the introduction of a special type of Dirac structure on suitable spaces of differential forms on the spatial domain and its boundary, making use of Stokes' theorem. Throughout, let Z be an n-dimensional smooth manifold with smooth $(n-1)$-dimensional boundary ∂Z, representing the space of *spatial variables*.

Denote by $\Omega^k(Z)$, for $k = 0, 1, \cdots, n$, the space of differential k-forms on Z, and by $\Omega^k(\partial Z)$, for $k = 0, 1, \cdots, n-1$, the space of k-forms on ∂Z.[1] Clearly, $\Omega^k(Z)$ and $\Omega^k(\partial Z)$ are (infinite-dimensional) linear spaces (over $\mathbb{R}$). Furthermore, there is a natural pairing between $\Omega^k(Z)$ and $\Omega^{n-k}(Z)$ given by

$$< \beta | \alpha >:= \int_Z \beta \wedge \alpha \quad (\in \mathbb{R}), \tag{14.1}$$

with $\alpha \in \Omega^k(Z), \beta \in \Omega^{n-k}(Z)$, where $\wedge$ is the usual wedge product of differential forms yielding the n-form $\beta \wedge \alpha$. In fact, the pairing (14.1) is *non-degenerate* in the sense that if $< \beta | \alpha >= 0$ for all α, respectively for all β, then $\beta = 0$, respectively $\alpha = 0$.

Similarly, there is a pairing between $\Omega^k(\partial Z)$ and $\Omega^{n-1-k}(\partial Z)$ given by

$$< \beta | \alpha >:= \int_{\partial Z} \beta \wedge \alpha, \tag{14.2}$$

with $\alpha \in \Omega^k(\partial Z), \beta \in \Omega^{n-1-k}(\partial Z)$. Now, let us define the linear space

$$\mathcal{F}_{p,q} := \Omega^p(Z) \times \Omega^q(Z) \times \Omega^{n-p}(\partial Z), \tag{14.3}$$

[1] Note that $\Omega^0(Z)$ and $\Omega^0(\partial Z)$ are the spaces of smooth functions on Z and ∂Z, respectively.

for any pair p, q of positive integers satisfying

$$p + q = n + 1, \tag{14.4}$$

and, correspondingly, let us define

$$\mathcal{E}_{p,q} := \Omega^{n-p}(Z) \times \Omega^{n-q}(Z) \times \Omega^{n-q}(\partial Z). \tag{14.5}$$

Then, the pairing (14.1) and (14.2) yields a (non-degenerate) pairing between $\mathcal{F}_{p,q}$ and $\mathcal{E}_{p,q}$. As for finite-dimensional systems, symmetrization of this pairing yields the bilinear form on $\mathcal{F}_{p,q} \times \mathcal{E}_{p,q}$, with values in $\mathbb{R}$

$$\begin{aligned} \ll (f_p^1, f_q^1, f_b^1, e_p^1, e_q^1, e_b^1), (f_p^2, f_q^2, f_b^2, e_p^2, e_q^2, e_b^2) \gg \\ := \int_Z \left[e_p^1 \wedge f_p^2 + e_q^1 \wedge f_q^2 + e_p^2 \wedge f_p^1 + e_q^2 \wedge f_q^1 \right] \\ + \int_{\partial Z} \left[e_b^1 \wedge f_b^2 + e_b^2 \wedge f_b^1 \right], \end{aligned} \tag{14.6}$$

where for $i = 1, 2$,

$$f_p^i \in \Omega^p(Z),\ f_q^i \in \Omega^q(Z),\ e_p^i \in \Omega^{n-p}(Z),\ e_p^i \in \Omega^{n-q}(Z),$$

$$f_b^i \in \Omega^{n-p}(\partial Z),\ e_b^i \in \Omega^{n-q}(\partial Z).$$

The spaces of differential forms $\Omega^p(Z)$ and $\Omega^q(Z)$ represent the energy variables of two different physical energy domains interacting with each other, while $\Omega^{n-p}(\partial Z)$ and $\Omega^{n-q}(\partial Z)$ will denote the boundary variables whose (wedge) product represents the boundary energy flow. The following theorem is proved in van der Schaft & Maschke (2002), based on 'integration by parts' and Stokes' theorem. Recall that d denotes the exterior derivative, mapping k-forms into $k+1$-forms (and generalizing in $\mathbb{R}^3$ the vector calculus operations $\mathrm{grad}, \mathrm{curl}, \mathrm{div}$).

Theorem 14.1. Consider $\mathcal{F}_{p,q}$ and $\mathcal{E}_{p,q}$ given in (14.3) and (14.5), respectively, with p, q satisfying (14.4), and bilinear form $\ll, \gg$ given by (14.6). Let $(\cdot)|_{\partial Z}$ denote the restriction to the boundary ∂Z, then the

linear subspace

$$\mathcal{D} = \left\{ (f_p, f_q, f_b, e_p, e_q, e_b) \in \mathcal{F}_{p,q} \times \mathcal{E}_{p,q} \middle| \begin{array}{l} \left[\begin{array}{c} f_p \\ f_q \end{array} \right] = \left[\begin{array}{cc} 0 & (-1)^{pq+1} d \\ d & 0 \end{array} \right] \left[\begin{array}{c} e_p \\ e_q \end{array} \right], \\ \left[\begin{array}{c} f_b \\ e_b \end{array} \right] = \left[\begin{array}{cc} 1 & 0 \\ 0 & -(-1)^{n-q} \end{array} \right] \left[\begin{array}{c} e_{p|\partial Z} \\ e_{q|\partial Z} \end{array} \right] \end{array} \right\}, \tag{14.7}$$

is a Dirac structure.

The subspace (14.7) is called a *Stokes-Dirac* structure.

14.2 Distributed-parameter port-Hamiltonian systems

The definition of a distributed-parameter Hamiltonian system with respect to a Stokes-Dirac structure can now be stated as follows. Let Z be an n-dimensional manifold with boundary ∂Z, and let $\mathcal{D}$ be a Stokes-Dirac structure as in Theorem 14.1. Consider furthermore a *Hamiltonian density* (energy per volume element)

$$\mathcal{H} : \Omega^p(Z) \times \Omega^q(Z) \times Z \to \Omega^n(Z),$$

resulting in the total energy

$$H := \int_Z \mathcal{H} \in \mathbb{R}.$$

From (14.1), we know that there exists a non-degenerate pairing between $\Omega^p(Z)$ and $\Omega^{n-p}(Z)$, respectively between $\Omega^q(Z)$ and $\Omega^{n-q}(Z)$. This means that $\Omega^{n-p}(Z)$ and $\Omega^{n-q}(Z)$ can be regarded as *dual spaces* to $\Omega^p(Z)$, respectively $\Omega^q(Z)$ (although strictly contained in their functional analytic duals). Let $\alpha_p, \partial\alpha_p \in \Omega^p(Z)$ and $\alpha_q, \partial\alpha_q \in \Omega^q(Z)$. Then, under weak smoothness conditions on $\mathcal{H}$, we have

$$\begin{aligned} H(\alpha_p + \partial\alpha_p, \alpha_q + \partial\alpha_q) &= \int_Z \mathcal{H}\left(\alpha_p + \partial\alpha_p, \alpha_q + \partial\alpha_q, z\right) \\ &= \int_Z \mathcal{H}\left(\alpha_p, \alpha_q, z\right) + \int_Z \left[\delta_p H \wedge \partial\alpha_p + \delta_q H \wedge \partial\alpha_q\right] \\ &\quad + \text{ higher order terms in } \partial\alpha_p, \partial\alpha_q, \end{aligned} \tag{14.8}$$

for certain differential forms

$$\delta_p H \in \Omega^{n-p}(Z),$$
$$\delta_q H \in \Omega^{n-q}(Z).$$

Furthermore, from the non-degeneracity of the pairing between $\Omega^p(Z)$ and $\Omega^{n-p}(Z)$, respectively between $\Omega^q(Z)$ and $\Omega^{n-q}(Z)$, it immediately follows that these differential forms are uniquely determined. This means that $(\delta_p H, \delta_q H) \in \Omega^{n-p}(Z) \times \Omega^{n-q}(Z)$ can be regarded as the (partial) *variational derivatives* (see e.g. Olver (1993)) of H at $(\alpha_p, \alpha_q) \in \Omega^p(Z) \times \Omega^q(Z)$. Throughout this chapter we assume that the Hamiltonian H admits variational derivatives satisfying (14.8).

Now, consider a time-function

$$(\alpha_p(t), \alpha_q(t)) \in \Omega^p(Z) \times \Omega^q(Z), \quad t \in \mathbb{R},$$

and the Hamiltonian $H(\alpha_p(t), \alpha_q(t))$ evaluated along this trajectory. It follows that at any time t

$$\frac{dH}{dt} = \int_Z \left[\delta_p H \wedge \frac{\partial \alpha_p}{\partial t} + \delta_q H \wedge \frac{\partial \alpha_q}{\partial t} \right]. \tag{14.9}$$

The differential forms $\frac{\partial \alpha_p}{\partial t}, \frac{\partial \alpha_q}{\partial t}$ represent the generalized velocities of the energy variables α_p, α_q. In a similar fashion as for finite-dimensional systems, we set

$$\begin{aligned} f_p &= -\frac{\partial \alpha_p}{\partial t}, \quad e_p = \delta_p H, \\ f_q &= -\frac{\partial \alpha_q}{\partial t}, \quad e_q = \delta_q H, \end{aligned} \tag{14.10}$$

where, as before, the minus sign is included to have a consistent energy flow description.

Definition 14.1. The distributed-parameter port-Hamiltonian system with n-dimensional manifold of spatial variables Z, state-space $\Omega^p(Z) \times \Omega^q(Z)$ (with $p+q = n+1$), Stokes-Dirac structure $\mathcal{D}$ given by (14.7), and Hamiltonian H, is given as

$$\begin{aligned} -\frac{\partial}{\partial t} \begin{bmatrix} \alpha_p \\ \alpha_q \end{bmatrix} &= \begin{bmatrix} 0 & (-1)^r d \\ d & 0 \end{bmatrix} \begin{bmatrix} \delta_p H \\ \delta_q H \end{bmatrix}, \\ \begin{bmatrix} f_b \\ e_b \end{bmatrix} &= \begin{bmatrix} 1 & 0 \\ 0 & -(-1)^{n-q} \end{bmatrix} \begin{bmatrix} \delta_p H|_{\partial Z} \\ \delta_q H|_{\partial Z} \end{bmatrix}, \end{aligned} \tag{14.11}$$

with $r = pq + 1$.

By the power-preserving property of any Dirac structure, it immediately follows that for any $(f_p, f_q, f_b, e_p, e_q, e_b)$ in the Stokes-Dirac structure $\mathcal{D}$

$$\int_Z [e_p \wedge f_p + e_q \wedge f_q] + \int_{\partial Z} e_b \wedge f_b = 0.$$

Hence, by substitution of (14.10) and using (14.9), we obtain the following proposition.

Proposition 14.1. Consider the port-Hamiltonian system (14.11). Then, the associated power-balance satisfies

$$\frac{dH}{dt} = \int_{\partial Z} e_b \wedge f_b, \tag{14.12}$$

expressing that the rate of change of energy on the domain Z is equal to the power supplied to the system through the boundary ∂Z, i.e.,

The system (14.11) represents a (nonlinear) boundary control system in the sense of e.g. Fattorini (1968). Indeed, we could interpret f_b as the boundary control inputs to the system, and e_b as the measured outputs (or the other way around). This feature is illustrated in the following examples.

Example 14.1 (Telegrapher's equations)**.** Consider a lossless transmission line with spacial domain $Z = [0, 1] \subset \mathbb{R}$. The energy variables are the charge density 1-form $Q = Q(t, z)dz \in \Omega^1([0, 1])$, and the flux density 1-form $\varphi = \varphi(t, z)dz \in \Omega^1([0, 1])$; thus $p = q = n = 1$. The total energy stored at time t in the transmission line is given as

$$H(Q, \varphi) = \int_0^1 \frac{1}{2} \left[\frac{Q^2(t, z)}{C(z)} + \frac{\varphi^2(t, z)}{L(z)} \right] dz,$$

with

$$\begin{bmatrix} \delta_Q H \\ \delta_\varphi H \end{bmatrix} = \begin{bmatrix} \dfrac{Q(t, z)}{C(z)} \\ \dfrac{\varphi(t, z)}{L(z)} \end{bmatrix} = \begin{bmatrix} V(t, z) \\ I(t, z) \end{bmatrix},$$

where $C(z)$ and $L(z)$ are the distributed capacitance and distributed inductance of the line, respectively, whereas $V(t,z)$ and $I(t,z)$ represent the corresponding voltage and current. The resulting port-Hamiltonian system is given by

$$-\frac{\partial}{\partial t}\begin{bmatrix} Q(t,z) \\ \varphi(t,z) \end{bmatrix} = \begin{bmatrix} 0 & \frac{\partial}{\partial z} \\ \frac{\partial}{\partial z} & 0 \end{bmatrix} \begin{bmatrix} V(t,z) \\ I(t,z) \end{bmatrix},$$

which represents the well-known telegrapher's equations, together with the boundary variables

$$\begin{bmatrix} f_b^0(t) \\ f_b^1(t) \end{bmatrix} = \begin{bmatrix} V(t,0) \\ V(t,1) \end{bmatrix}, \quad \begin{bmatrix} e_b^0(t) \\ e_b^1(t) \end{bmatrix} = -\begin{bmatrix} I(t,0) \\ I(t,1) \end{bmatrix}.$$

The associated power-balance reads

$$\frac{dH}{dt} = \int_0^1 e_b f_b dz = I(t,0)V(t,0) - I(t,1)V(t,1), \tag{14.13}$$

which is in accordance with (14.12).

Similar equations as the telegrapher's equations hold for a vibrating string van der Schaft & Maschke (2002), or for a compressible gas/fluid in a one-dimensional pipe.

Example 14.2 (Shallow water equations)**.** The dynamical behavior of water in an open canal with spatial domain $Z = [a,b] \subset \mathbb{R}$ can be described by

$$\frac{\partial}{\partial t}\begin{bmatrix} h \\ v \end{bmatrix} + \begin{bmatrix} v & h \\ g & v \end{bmatrix} \frac{\partial}{\partial z}\begin{bmatrix} h \\ v \end{bmatrix},$$

with $h(t,z)$ the height of the water at position z at time t, $v(z,t)$ the corresponding velocity, and g the gravitational constant. These *shallow water equations* can be written as a port-Hamiltonian system by recognizing the internally stored energy

$$H(h,v) = \int_a^b \frac{1}{2}\left[hv^2 + gh^2\right] dz,$$

yielding

$$e_h = \delta_h H = \frac{1}{2}v^2 + gh \quad \text{(Bernoulli function)},$$
$$e_v = \delta_v H = hv \quad \text{(mass flow)}.$$

Hence, in a similar fashion as the telegrapher's equations, we obtain

$$-\frac{\partial}{\partial t}\begin{bmatrix} h(t,z) \\ v(t,z) \end{bmatrix} = \begin{bmatrix} 0 & \frac{\partial}{\partial z} \\ \frac{\partial}{\partial z} & 0 \end{bmatrix} \begin{bmatrix} \delta_h H \\ \delta_v H \end{bmatrix},$$

with boundary variables $-hv|_{[a,b]}$ and $(\frac{1}{2}v^2 + gh)|_{[a,b]}$. The associated power-balance is obtained by taking the time-derivative of H, i.e.,

$$\begin{aligned} \frac{d}{dt}\int_a^b \frac{1}{2}\left[hv^2 + gh^2\right] dz &= -hv\left(\frac{1}{2}v^2 + gh\right)\Big|_{[a,b]} \\ &= -v\left(\frac{1}{2}gh^2\right)\Big|_{[a,b]} - v\left(\frac{1}{2}hv^2 + \frac{1}{2}gh^2\right)\Big|_{[a,b]}, \end{aligned}$$

which expresses that the power flow through the boundary of the channel equals velocity $\times$ pressure $+$ energy flux through the boundary.

14.3 Presence of sources and dissipation

Energy exchange through the boundary is not the only possible way a distributed-parameter system may interact with its environment. An example of this is provided by Maxwell's equations (Example 14.3), where interaction may also take place via the current density $\mathfrak{J}$, which directly affects the electric charge distribution in the domain Z. In order to cope with this situation, we augment the spaces $\mathcal{F}_{p,q}$ and $\mathcal{E}_{p,q}$ as defined in (14.3) and (14.5), respectively, to

$$\begin{aligned} \mathcal{F}^s_{q,p} &:= \mathcal{F}_{p,q} \times \Omega^s(S), \\ \mathcal{E}^s_{q,p} &:= \mathcal{E}_{p,q} \times \Omega^{n-s}(S), \end{aligned} \tag{14.14}$$

for some m-dimensional manifold S and some $s \in \{0, 1, \cdots, m\}$, with $f_s \in \Omega^s(S)$ denoting the externally supplied distributed control flow,

and $e_s \in \Omega^{n-s}(S)$ the conjugate distributed quantity, corresponding to an energy exchange

$$\int_S e_s \wedge f_s. \tag{14.15}$$

The Stokes-Dirac structure (14.7) is then extended to

$$\begin{aligned}
\begin{bmatrix} f_p \\ f_q \end{bmatrix} &= \begin{bmatrix} 0 & (-1)^r d \\ d & 0 \end{bmatrix} \begin{bmatrix} e_p \\ e_q \end{bmatrix} + G(f_s), \\
\begin{bmatrix} f_b \\ e_b \end{bmatrix} &= \begin{bmatrix} 1 & 0 \\ 0 & -(-1)^{n-q} \end{bmatrix} \begin{bmatrix} e_{p|\partial Z} \\ e_{q|\partial Z} \end{bmatrix}, \\
e_s &= -G^* \begin{bmatrix} e_p \\ e_q \end{bmatrix},
\end{aligned} \tag{14.16}$$

with G denoting a linear map

$$G = \begin{bmatrix} G_p \\ G_q \end{bmatrix} : \Omega^s(S) \to \Omega^p(Z) \times \Omega^q(Z), \tag{14.17}$$

with dual map (again we consider $\Omega^{n-p}(Z)$ and $\Omega^{n-q}(Z)$ as dual spaces to $\Omega^p(Z)$ and $\Omega^{n-q}(Z)$, respectively)

$$G^* = (G_p^*, G_q^*) : \Omega^{n-p}(Z) \times \Omega^{n-q}(Z) \to \Omega^{n-s}(S),$$

satisfying

$$\int_Z \left[e_p \wedge G_p(f_s) + e_q \wedge G_q(f_s) \right] = \int_S \left[G_p^*(e_p) + G_q^*(e_q) \right] \wedge f_s,$$

for all $e_p \in \Omega^{n-p}(Z)$, $e_q \in \Omega^{n-q}(Z)$, and $f_s \in \Omega^s(S)$.

Proposition 14.2. Equations (14.16) determine a Dirac structure

$$\mathcal{D}^s \subset \mathcal{F}^s_{p,q} \times \mathcal{E}^s_{p,q},$$

with respect to the augmented bilinear form on $\mathcal{F}^s_{p,q} \times \mathcal{E}^s_{p,q}$, which is obtained by adding to the bilinear form (14.6) the term

$$\int_S \left[e_s^1 \wedge f_s^2 + e_s^2 \wedge f_s^1 \right]. \tag{14.18}$$

Then, substitution of (14.10) into $\mathcal{D}^s$ given by (14.16) yields a port-Hamiltonian system with external variables (f_b, f_s, e_b, e_s), with (f_b, e_b) the *boundary* external variables and (f_s, e_s) the *distributed* external variables. Furthermore, the power-balance (14.12) extends to

$$\frac{dH}{dt} = \int_{\partial Z} e_b \wedge f_b + \int_S e_s \wedge f_s, \tag{14.19}$$

with the first term on the right-hand side denoting the power flow through the boundary, and the second term denoting the distributed power flow. We conclude this section with the following example.

Example 14.3 (Maxwell's equations)**.** Let $Z \subset \mathbb{R}^3$ be a 3-dimensional manifold with boundary ∂Z, defining the spatial domain, and consider an electromagnetic field in Z. The energy variables are the magnetic field induction 2-form $\alpha_q = \mathfrak{B} \in \Omega^2(Z)$:

$$\mathfrak{B} = \frac{1}{2} B_{ij}(t,z) dz^i \wedge dz^j,$$

and the electric field induction 2-form $\alpha_p = \mathfrak{D} \in \Omega^2(Z)$:

$$\mathfrak{D} = \frac{1}{2} D_{ij}(t,z) dz^i \wedge dz^j.$$

Furthermore, the associated magnetic and electric field intensities are given by $\mathfrak{H} = H_i(t,z)dz^i \in \Omega^1(Z)$ and $\mathfrak{E} = E_i(t,z)dz^i \in \Omega^1(Z)$, respectively. These 1-forms are related to the energy variables through the constitutive relations of the medium (or material equations) $\star\mathcal{B} = \mu H$ and $\star\mathfrak{D} = \epsilon\mathfrak{E}$, with the scalar functions $\mu(t,z)$ and $\epsilon(t,z)$ denoting the magnetic permeability and electric permittivity, respectively, and $\star$ denoting the Hodge star operator (corresponding to a Riemannian metric on Z), converting 2-forms into 1-forms.

Then, the Hamiltonian H is defined as

$$H = \int_Z \frac{1}{2}(\mathfrak{E} \wedge \mathfrak{D} + \mathfrak{H} \wedge \mathfrak{B}),$$

where one readily verifies that $\delta_p H = \mathfrak{E}, \delta_q H = \mathfrak{H}$, and the corresponding Stokes-Dirac structure $(n = 3, p = 2, q = 2)$ takes the form

$$\begin{bmatrix} f_p \\ f_q \end{bmatrix} = \begin{bmatrix} 0 & -d \\ d & 0 \end{bmatrix} \begin{bmatrix} e_p \\ e_q \end{bmatrix}, \quad \begin{bmatrix} f_b \\ e_b \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} e_{p|\partial Z} \\ e_{q|\partial Z} \end{bmatrix}. \tag{14.20}$$

Under the assumption that the current density $\mathfrak{J}$ in the medium is zero, and explicitly taking into account the behavior at the boundary, Maxwell's equations are then represented as the port-Hamiltonian system with respect to the Stokes-Dirac structure given by (14.20), as

$$-\frac{\partial}{\partial t}\begin{bmatrix}\mathfrak{D}\\ \mathfrak{B}\end{bmatrix} = \begin{bmatrix}0 & -d\\ d & 0\end{bmatrix}\begin{bmatrix}\delta_{\mathfrak{D}}H\\ \delta_{\mathfrak{B}}H\end{bmatrix}, \tag{14.21}$$

$$\begin{bmatrix}f_b\\ e_b\end{bmatrix} = \begin{bmatrix}\delta_{\mathfrak{D}}H|_{\partial Z}\\ \delta_{\mathfrak{B}}H|_{\partial Z}\end{bmatrix}, \tag{14.22}$$

and the power-balance (14.19) takes the form

$$\frac{dH}{dt} = \int_{\partial Z}\delta_{\mathfrak{B}}H\wedge\delta_{\mathfrak{D}}H = \int_{\partial Z}\mathfrak{H}\wedge\mathfrak{E} = -\int_{\partial Z}\mathfrak{E}\wedge\mathfrak{H},$$

with $\mathfrak{E}\wedge\mathfrak{H}$ a 2-form corresponding to the Poynting vector (see Ingarden & Jamiolkowski (1985)).

In the case of a non-zero current density, we have to modify (14.21) to

$$-\frac{\partial}{\partial t}\begin{bmatrix}\mathfrak{D}\\ \mathfrak{B}\end{bmatrix} = \begin{bmatrix}0 & -d\\ d & 0\end{bmatrix}\begin{bmatrix}\delta_{\mathfrak{D}}H\\ \delta_{\mathfrak{B}}H\end{bmatrix} + \begin{bmatrix}I\\ 0\end{bmatrix}\mathfrak{J}, \tag{14.23}$$

with I denoting the identity operator from $\mathfrak{J}\in\Omega^2(Z)$ to $\Omega^2(Z)$. (Thus, in the notation of (14.17), $f_s=\mathfrak{J}$, $S=Z$, and $\Omega^s(S)=\Omega^2(Z)$.) Furthermore, we add the equation

$$e_s = -\begin{bmatrix}0 & I\end{bmatrix}\begin{bmatrix}\delta_{\mathfrak{D}}H\\ \delta_{\mathfrak{B}}H\end{bmatrix} = -\mathfrak{E}, \tag{14.24}$$

yielding the augmented power-balance

$$\frac{dH}{dt} = -\int_{\partial Z}\mathfrak{E}\wedge\mathfrak{H} - \int_{Z}\mathfrak{E}\wedge\mathfrak{J},$$

which is known as Poynting's theorem.

Energy dissipation can be incorporated in the framework of distributed-parameter port-Hamiltonian systems by terminating some of the ports (boundary or distributed) with a resistive relation. For example, for distributed dissipation, let $R:\Omega^{n-s}(S)\to\Omega^s(S)$ be a map satisfying

$$\int_S e_s\wedge R(e_s)\geq 0,\quad \forall e_s\in\Omega^{n-s}(S).$$

Then, by adding the relation

$$f_s = -R(e_s)$$

to the port-Hamiltonian system defined with respect to the Dirac structure $\mathcal{D}^s$, we obtain a port-Hamiltonian system with dissipation, satisfying the differential dissipation inequality

$$\frac{dH}{dt} = \int_{\partial Z} e_b \wedge f_b - \int_S e_s \wedge R(e_s) \leq \int_{\partial Z} e_b \wedge f_b.$$

Example 14.4 (Maxwell's equations (cont'd))**.** In order to incorporate energy dissipation into the Maxwell equations (14.23), we decompose the current density into $\mathfrak{J} = \mathfrak{J}_s + \bar{\mathfrak{J}}$, and impose Ohm's law

$$\star\mathfrak{J}_s = \sigma\mathfrak{E},$$

with $\sigma(t, z)$ the specific conductivity of the medium.

14.4 Conservation laws

In Chapter 8, we introduced the notion of a conserved quantity—independent of the Hamiltonian—called a Casimir function. In this section, this notion is extended to distributed-parameter systems, and for non-zero boundary conditions gives rise to certain conservation laws. For, consider the distributed-parameter port-Hamiltonian system as defined by (14.11). Conserved quantities that are independent from the Hamiltonian H are obtained as follows. Let

$$C : \Omega^p(Z) \times \Omega^q(Z) \times Z \to \mathbb{R} \tag{14.25}$$

be a function satisfying

$$\begin{aligned} d(\delta_p C) &= 0, \\ d(\delta_q C) &= 0. \end{aligned} \tag{14.26}$$

Then, the time-derivative of C along the trajectories of (14.11) is given by

$$\begin{aligned}\frac{dC}{dt} &= \int_Z \delta_p C \wedge \frac{\partial \alpha_p}{\partial t} + \int_Z \delta_q C \wedge \frac{\partial \alpha_q}{\partial t} \\ &= -\int_Z \delta_p C \wedge (-1)^r d(\delta_q H) - \int_Z \delta_q C \wedge d(\delta_p H) \\ &= -(-1)^{n-q} \int_Z d(\delta_q H \wedge \delta_p C) - (-1)^{n-q} \int_Z d(\delta_q C \wedge \delta_p H) \\ &= \int_{\partial Z} e_b \wedge f_b^C + \int_{\partial Z} e_b^C \wedge f_b,\end{aligned}$$

where we have denoted, in analogy with (14.7),

$$f_b^C := \delta_p C|_{\partial Z}, \quad e_b^C := -(-1)^{n-q} \delta_q C|_{\partial Z}.$$

In particular, if in addition to (14.26), the function C satisfies

$$\begin{aligned}\delta_p C|_{\partial Z} &= 0, \\ \delta_q C|_{\partial Z} &= 0,\end{aligned} \tag{14.27}$$

then $\frac{dC}{dt} = 0$ along the system trajectories of (14.11) for any Hamiltonian H. Therefore, a function C satisfying (14.26) and (14.27) is called a Casimir function. If C satisfies (14.26), but not (14.27), then the time-derivative of C is determined by the boundary conditions of (14.11), and therefore is called a *conservation law* for (14.11).

Example 14.5 (Telegrapher's equations (cont'd))**.** In the case of the telegrapher's equations, the total charge

$$C_Q = \int_0^1 Q(t,z)dz,$$

is a conservation law. Indeed

$$\frac{dC_Q}{dt} = -\int_0^1 \frac{\partial I}{\partial z}(t,z)dz = I(t,0) - I(t,1).$$

Similarly, differentiating the total magnetic flux

$$C_\varphi = \int_0^1 \varphi(t,z)dz$$

with respect to time yields

$$\frac{dC_\varphi}{dt} = -\int_0^1 \frac{\partial V}{\partial z}(t,z)dz = V(t,0) - V(t,1).$$

For a further discussion on Casimir functions and conservation laws, also for the extended Dirac structure (14.16), we refer to van der Schaft & Maschke (2002).

14.5 Covariant formulation of port-Hamiltonian systems

A covariant formulation that is well-known for Maxwell's equations (see Ingarden & Jamiolkowski (1985)) can be generalized to general distributed-parameter port-Hamiltonian systems (14.11), defined with respect to a general Stokes-Dirac structure $\mathcal{D}$. Define on $Z \times \mathbb{R}$, with space-time coordinates (z,t), the following p-form and q-form

$$\begin{aligned}\gamma_p &:= \alpha_p + (-1)^r \delta_q H \wedge dt,\\ \gamma_q &:= \alpha_q + \delta_p H \wedge dt,\end{aligned}$$

respectively. Then the first part of the equations (14.11) can be equivalently stated as

$$\begin{aligned}L_{\frac{\partial}{\partial t}} \bar{d}\gamma_p = 0,\\ L_{\frac{\partial}{\partial t}} \bar{d}\gamma_q = 0,\end{aligned} \tag{14.28}$$

with $\bar{d}$ denoting the exterior derivative with respect to space-time (z,t). Indeed, (14.28) means that $\bar{d}\gamma_p$ and $\bar{d}\gamma_q$ do not depend on t, i.e.,

$$\begin{aligned}\bar{d}\gamma_p = \beta_p,\\ \bar{d}\gamma_q = \beta_q,\end{aligned} \tag{14.29}$$

for certain $(p+1)-$ and $(q+1)-$forms β_p, respectively β_q, not depending on t. Writing out (14.29) yields

$$\begin{aligned}d\alpha_p + \frac{\partial \alpha_p}{\partial t} \wedge dt + (-1)^r d(\delta_q H) \wedge dt = \beta_p,\\ d\alpha_q + \frac{\partial \alpha_q}{\partial t} \wedge dt + d(\delta_p H) \wedge dt = \beta_q,\end{aligned} \tag{14.30}$$

with d denoting the exterior derivative with respect to the spatial variables z, resulting in the equations of a port-Hamiltonian system (14.11)

$$\begin{aligned} -\frac{\partial \alpha_p}{\partial t} &= (-1)^r d(\delta_q H), \\ -\frac{\partial \alpha_q}{\partial t} &= d(\delta_p H), \end{aligned} \tag{14.31}$$

together with the conserved quantities (cf. Chapter 8) $d\alpha_p = \beta_p$ and $d\alpha_q = \beta_q$. Furthermore, the boundary variables of the port-Hamiltonian system (14.11) can be re-formulated as

$$\begin{aligned} \left(i_{\frac{\partial}{\partial t}} \gamma_q\right)\Big|_{\partial Z} &= f_b, \\ \left(i_{\frac{\partial}{\partial t}} \gamma_p\right)\Big|_{\partial Z} &= (-1)^q e_b. \end{aligned}$$

15

Control of port-Hamiltonian systems

In the previous chapters, we have witnessed that one of the advantages of the port-Hamiltonian (pH) framework is that the Hamiltonian can be used as a basis to construct a candidate Lyapunov function, thus providing insight into various system properties like stability, passivity, finite L_2 gain, etc.. Another key feature of pH systems is that a power-preserving interconnection of pH systems results in another pH system, with total Hamiltonian being the sum of the Hamiltonian functions and with a Dirac structure defined by the composition of the Dirac structures of the subsystems. These features have led to a research focus on the control of port-Hamiltonian systems.

15.1 Control by interconnection

Consider a *plant* pH system, with state space $\mathcal{X}_p$, Hamiltonian $H_p : \mathcal{X}_p \to \mathbb{R}$, resistive (energy-dissipating) port (f_R, e_R), and external port (f_P, e_P), and a *controller* pH system, with state space $\mathcal{X}_c$, Hamiltonian $H_c : \mathcal{X}_c \to \mathbb{R}$, resistive port $(\bar{f}_R, \bar{e}_R)$, and external port $(\bar{f}_P, \bar{e}_P)$. From Chapter 6, we know that the composition of Dirac structures via a power-preserving interconnection is again a Dirac structure. Indeed,

any interconnection between the plant and the controller through their respective external ports satisfying the power-preserving property

$$e_P^T f_P + \bar{e}_P^T \bar{f}_P = 0, \tag{15.1}$$

results in a closed-loop pH system, with state space $\mathcal{X}_p \times \mathcal{X}_c$, Hamiltonian $H_p + H_c$, resistive port $((f_R, e_R), (\bar{f}_R, \bar{e}_R))$, satisfying the power-balance $\dot{H}_p + \dot{H}_c = e_R^T f_R + \bar{e}_R^T \bar{f}_R$. Using standard Lyapunov stability theory, we immediately infer that since both $e_R^T f_R \leq 0$ and $\bar{e}_R^T \bar{f}_R \leq 0$, and $(x_p^*, x_c^*) \in \mathcal{X}_p \times \mathcal{X}_c$ is a minimum of $H_p + H_c$, then (x_p^*, x_c^*) will be a stable equilibrium of the closed-loop system. Moreover, the equilibrium (x_p^*, x_c^*) is asymptotically stable under an additional zero-state detectability condition on the plant and/or controller system.

Example 15.1. Consider a controlled version of the spinning rigid body of Example 3.1:

$$\begin{bmatrix} \dot{p}_x \\ \dot{p}_y \\ \dot{p}_z \end{bmatrix} = \begin{bmatrix} 0 & -p_z & p_y \\ p_z & 0 & -p_x \\ -p_y & p_x & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial H}{\partial p_x} \\ \frac{\partial H}{\partial p_y} \\ \frac{\partial H}{\partial p_z} \end{bmatrix} + \begin{bmatrix} g_x \\ g_y \\ g_z \end{bmatrix} u,$$

with control u and natural output

$$y = g_x \frac{\partial H}{\partial p_x} + g_y \frac{\partial H}{\partial p_y} + g_z \frac{\partial H}{\partial p_z}.$$

Since for $u = 0$, $\dot{H}(p) = 0$ and $H(p)$ has its minimum at $p = 0$. Thus, the equilibrium point $p_k^* = 0$, for $k \in \{x, y, z\}$, is stable. To render the closed-loop asymptotically stable, we apply the feedback

$$u = -y = -\sum_k g_k \frac{p_k}{I_k},$$

yielding convergence to the largest invariant set contained in

$$\mathbb{O} := \left\{ p \in \mathbb{R}^3 \mid \dot{H}(p) = 0 \right\} = \left\{ p \in \mathbb{R}^3 \,\middle|\, \sum_k g_k \frac{p_k}{I_k} = 0 \right\},$$

which actually is $p = 0$ if, and only if, $g_k \neq 0$ for all $k \in \{x, y, z\}$.

The above example shows that a pH system having its energetic minimum at the origin can be asymptotically stabilized by interconnecting it with a static controller pH system (i.e., $H_c = 0$), which, in general terms, has the form $\bar{e}_P = K_d \bar{f}_P$, with $K_d = K_d^T$ a positive definite damping injection matrix, through a power-conserving interconnection $f_P = -\bar{e}_P$ and $e_P = \bar{f}_P$. Note that in the example, we set $K_d = 1$, and $f_P = u$, $e_P = y$, $\bar{f}_P = \bar{u}$, and $\bar{e}_P = \bar{y}$. Asymptotic stability of the origin can be inferred provided a detectability condition is satisfied.

Another application of control by interconnection is energy transfer control.

15.2 Energy transfer control

Consider two pH systems Σ_i (without internal dissipation) in input-state-output form

$$\Sigma_i : \begin{cases} \dot{x}_i = J_i(x_i)\dfrac{\partial H_i}{\partial x_i}(x_i) + g_i(x_i)u_i, \\ y_i = g_i^T(x_i)\dfrac{\partial H_i}{\partial x_i}(x_i), \quad i = 1, 2, \end{cases}$$

both satisfying the power-balance $\dot{H}_i(x_i) = y_i^T u_i$. Suppose now that we want to transfer the energy from the port-Hamiltonian system Σ_1 to the port-Hamiltonian system Σ_2, while keeping the total energy $H_1 + H_2$ constant. This can be done by using the following output feedback

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0 & -y_1 y_2^T \\ y_2 y_1^T & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}, \tag{15.2}$$

which, due to the skew-symmetry property of the interconnection matrix, clearly satisfies (15.1). Hence, the closed-loop system composed of Σ_1 and Σ_2 is energy-preserving, that is $\dot{H}_1 + \dot{H}_2 = 0$. However, if we consider the individual energies then we notice that

$$\dot{H}_1(x) = -y_1^T y_1 y_2^T y_2 = -||y_1||^2 ||y_2||^2 \leq 0, \tag{15.3}$$

implying that H_1 is decreasing as long as $||y_1||$ and $||y_2||$ are different from 0. Conversely, as expected since the total energy is constant,

$$\dot{H}_2(x) = y_2^T y_2 y_1^T y_1 = ||y_2||^2 ||y_1||^2 \geq 0, \tag{15.4}$$

implying that H_2 is increasing at the same rate. In particular, if H_1 has a minimum at the zero equilibrium, and Σ_1 is zero-state detectable, then all the energy H_1 of Σ_1 will be transferred to Σ_2, provided that $||y_2||$ is not identically zero.

If there is internal energy dissipation, then this energy transfer mechanism still works. However, the fact that H_2 grows or not will depend on the balance between the energy delivered by Σ_1 to Σ_2 and the internal loss of energy in Σ_2 due to dissipation. We conclude that this particular scheme of power-conserving energy transfer is accomplished by a skew-symmetric output feedback, which is *modulated* by the values of the output vectors of both systems. Of course this raises, among others, the question of the efficiency of the proposed energy-transfer scheme, and the need for a systematic quest of similar power-conserving energy-transfer schemes. We refer to Duindam et al. (2004) for a similar energy-transfer scheme directly motivated by the structure of the example (control of a snakeboard).

15.3 Stabilization by Casimir generation

Stabilizing a system at the origin, which often coincides with the open-loop minimum energy, is generally not an enticing control problem. Of wider practical interest is to stabilize the system at a non-zero set-point. Indeed, suppose that we want to stabilize a plant pH system around a set-point x_p^*. We know that for any controller pH system, the closed-loop power-balance satisfies

$$\dot{H}_p + \dot{H}_c = e_R^T f_R + \bar{e}_R^T \bar{f}_R \leq 0. \tag{15.5}$$

If x_p^* is not a minimum of H_p, a possible strategy is to consider Casimir functions $C : \mathcal{X}_p \times \mathcal{X}_c \to \mathbb{R}$ to generate a candidate Lyapunov function for the closed-loop system of the form $V := H_p + H_c + C$. The design is then completed by selecting the controller in such a way that V has a minimum at (x_p^*, x_c^*). As discussed in Chapter 8, Casimirs are conserved quantities that are completely characterized by the Dirac structure of the system. Since we are interested in the Casimir functions that are based on the closed-loop Dirac structure, this strategy

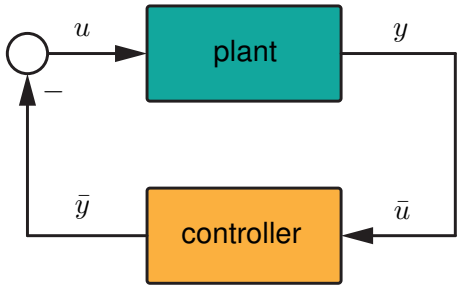


Figure 15.1: Feedback interconnection of a plant and a controller port-Hamiltonian system.

reduces to finding all the achievable closed-loop Dirac structures. A comprehensive analysis is given in Cervera et al. (2007).

Another way to interpret the generation of Casimir functions for the closed-loop system is the following. Let us, for ease of presentation, assume that the plant can be represented by an input-state-output pH system of the form

$$\begin{aligned} \dot{x}_p &= [J_p(x_p) - R(x_p)] \frac{\partial H_p}{\partial x_p}(x_p) + g_p(x_p)u, \\ y &= g_p^T(x_p) \frac{\partial H_p}{\partial x_p}(x_p), \end{aligned} \tag{15.6}$$

with $x_p \in \mathcal{X}_p$ and $u, y \in \mathbb{R}^m$. Furthermore, if the controller is also an input-state-output pH system of the form

$$\begin{aligned} \dot{x}_c &= [J_c(x_c) - R_c(x_c)] \frac{\partial H_c}{\partial x_c}(x_c) + g_c(x_c)\bar{u}, \\ \bar{y} &= g_c^T(x_c) \frac{\partial H_c}{\partial x_c}(x_c), \end{aligned} \tag{15.7}$$

with $x_c \in \mathcal{X}_c$ and $\bar{u}, \bar{y} \in \mathbb{R}^m$, then the interconnection of the plant system (15.6) with (15.7) via the standard power-preserving feedback interconnection $u = -\bar{y}$ and $\bar{u} = y$, as shown in Fig. 15.1, yields the

closed-loop system

$$\begin{aligned}\begin{bmatrix}\dot{x}_p\\ \dot{x}_c\end{bmatrix} &= \begin{bmatrix}J_p(x_p)-R_p(x_p) & -g_p(x_p)g_c^T(x_c)\\ g_c(x_c)g_p^T(x_p) & J_c(x_c)-R_c(x_c)\end{bmatrix}\begin{bmatrix}\dfrac{\partial H_p}{\partial x_p}(x_p)\\ \dfrac{\partial H_c}{\partial x_c}(x_c)\end{bmatrix},\\ \begin{bmatrix}y\\ \bar{y}\end{bmatrix} &= \begin{bmatrix}g_p^T(x_p) & 0\\ 0 & g_c^T(x_c)\end{bmatrix}\begin{bmatrix}\dfrac{\partial H_p}{\partial x_p}(x_p)\\ \dfrac{\partial H_c}{\partial x_c}(x_c)\end{bmatrix},\end{aligned} \tag{15.8}$$

which is again a pH system with Hamiltonian $H_p(x_p) + H_c(x_c)$.

The main idea then is to design the controller system such that the closed-loop system (15.8) has useful Casimirs. If both the plant and the controller are lossless, i.e., $R_p(x_p) = 0$ and $R_c(x_c) = 0$, we thus look for functions $C(x_d, x_c)$ satisfying

$$\begin{aligned}&\begin{bmatrix}\dfrac{\partial^T C}{\partial x_p}(x_p,x_c) & \dfrac{\partial^T C}{\partial x_c}(x_p,x_c)\end{bmatrix}\\ &\qquad\times\begin{bmatrix}J_p(x_p) & -g_p(x_p)g_c^T(x_c)\\ g_c(x_c)g_p^T(x_p) & J_c(x_c)\end{bmatrix} = 0,\end{aligned} \tag{15.9}$$

such that the candidate Lyapunov function

$$V(x_p, x_c) = H_p(x_p) + H_c(x_c) + C(x_p, x_c) \tag{15.10}$$

has a minimum at (x_p^*, x_c^*) for a certain x_c^*. Subsequently, one may add extra damping, directly or in the dynamics of the controller, to achieve asymptotic stability.

Example 15.2. Consider a pendulum with normalized Hamiltonian

$$H(q,p) = \frac{1}{2}p^2 + 1 - \cos q$$

actuated by a torque u, with output $y = p$ (angular velocity). Suppose we wish to stabilize the pendulum at $q^* \neq 0$ and $p^* = 0$. Apply the nonlinear integral control

$$\begin{aligned}\dot{x}_c &= \bar{u}\\ \bar{y} &= \frac{\partial H_c}{\partial x_c}(x_c),\end{aligned}$$

which is a port-Hamiltonian controller system with $J_c = 0$. After interconnecting the pendulum with the controller by setting $u = -\bar{y}$ and $\bar{u} = y$, we proceed by searching for Casimirs $C(q, p, \xi)$ which are found by solving

$$\begin{bmatrix} \dfrac{\partial C}{\partial q} & \dfrac{\partial C}{\partial p} & \dfrac{\partial C}{\partial \xi} \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} = [0 \quad 0 \quad 0],$$

leading to $C(q, p, \xi) = K(q - \xi)$, and candidate Lyapunov functions

$$V(q, p, \xi) = \frac{1}{2}p^2 + 1 - \cos q + H_c(\xi) + K(q - \xi),$$

with the functions H_c and K to be determined. For a local minimum, determine K and H_c such that

- Equilibrium assignment:

$$\begin{aligned} &\frac{\partial V}{\partial q}(q^*, 0, \xi^*) = \sin q^* + \frac{\partial K}{\partial q}(q^* - \xi^*) = 0, \\ &\frac{\partial V}{\partial p}(q^*, 0, \xi^*) = 0, \\ &\frac{\partial V}{\partial \xi}(q^*, 0, \xi^*) = \frac{\partial H_c}{\partial \xi}(\xi^*) - \frac{\partial K}{\partial \xi}(q^* - \xi^*) = 0. \end{aligned}$$

- Minimum condition:

$$\begin{bmatrix} \cos q^* + \dfrac{\partial^2 K}{\partial q^2}(q^* - \xi^*) & 0 & -\dfrac{\partial^2 K}{\partial q \partial \xi}(q^* - \xi^*) \\ 0 & 1 & 0 \\ -\dfrac{\partial^2 K}{\partial \xi \partial q}(q^* - \xi^*) & 0 & \dfrac{\partial^2 K}{\partial \xi^2}(q^* - \xi^*) + \dfrac{\partial^2 H_c}{\partial \xi^2}(\xi^*) \end{bmatrix} > 0,$$

which provides many possibilities to accomplish the stabilization task.

Example 15.3. A similar approach can be applied to distributed-parameter systems. Consider for instance the shallow water equations of Example 14.2:

$$-\frac{\partial}{\partial t}\begin{bmatrix} h(t,z) \\ v(t,z) \end{bmatrix} = \begin{bmatrix} 0 & \dfrac{\partial}{\partial z} \\ \dfrac{\partial}{\partial z} & 0 \end{bmatrix} \begin{bmatrix} \delta_h H \\ \delta_v H \end{bmatrix},$$

with boundary variables $hv|_{[a,b]}$ and $-(\frac{1}{2}v^2+gh)|_{[a,b]}$ and Hamiltonian

$$H(h,v)=\frac{1}{2}\int_a^b \left[hv^2+gh^2\right]dz.$$

An obvious 'physical' controller is to add to one side of the canal, say the right-end b, an infinite water reservoir of height h^*, corresponding to the port-Hamiltonian 'source' system

$$\begin{aligned}\dot{x}_c &= \bar{u}\\ \bar{y} &= \frac{\partial H_c}{\partial x_c}(x_c),\end{aligned}$$

with Hamiltonian $H_c(x_c)=gh^*x_c$, via the feedback interconnection

$$\bar{u}=y=h(b)v(b),\ \bar{y}=-u=\frac{1}{2}v^2(b)+gh(b).$$

By mass-balance, we find that

$$\int_a^b h(z,t)dz+x_c+\kappa$$

is a Casimir for the closed-loop system. Thus, we may take as a candidate Lyapunov function

$$\begin{aligned}V(h,v,x_c) &= \frac{1}{2}\int_a^b \left[hv^2+gh^2\right]dz+gh^*x_c\\ &\quad -gh^*\left(\int_a^b h(z,t)dz+x_c\right)+\frac{1}{2}g(h^*)^2(b-a)\\ &= \frac{1}{2}\int_a^b \left[hv^2+g(h-h^*)^2\right]dz,\end{aligned}$$

which has a minimum at the desired set-point (h^*,v^*,x_c^*), with $v^*=0$ and x_c^* arbitrary.

Note that if we restrict the motion of (15.8) to the subset

$$\Omega=\{(x_p,x_c)\in\mathcal{X}_p\times\mathcal{X}_c|C(x_p,x_c)=\kappa\},\tag{15.11}$$

with $C(x_p,x_c)=x_c-S(x_p)$, where $S(x_p)$ is differentiable and κ is some constant, the candidate Lyapunov function (15.10) reduces to a *shaped*

closed-loop Hamiltonian $H_p(x_p)+H_c(S(x_p)+\kappa)$. This is accomplished if, along the trajectories of (15.8), the functions $S(x_p)$ are such that

$$\dot{C}(x_p, x_c)\big|_\Omega = 0.$$

Hence the dynamic feedback reduces to a state feedback scheme, and we are thus looking for solutions $S(x)$ of the partial differential equations (PDE's)

$$\begin{bmatrix} -\dfrac{\partial^T S}{\partial x_p}(x_p) & I_{n_c} \end{bmatrix} \begin{bmatrix} J_p(x_p) - R_p(x_p) & -g_p(x_p)g_c^T(x_c) \\ g_c(x_c)g_p^T(x_p) & J_c(x_c) - R_c(x_c) \end{bmatrix} = 0, \quad (15.12)$$

which, under the assumption that $R(x_p) \geq 0$ and $R_c(x_c) \geq 0$, are characterized by the following chain of equalities van der Schaft (2000).

Proposition 15.1. The system of PDE's (15.12) has a solution if and only if

$$\begin{aligned} \frac{\partial^T S}{\partial x_p}(x_p) J_p(x_p) \frac{\partial S}{\partial x_p}(x_p) &= J_c(x_c), \\ R_p(x_p) \frac{\partial S}{\partial x_p}(x_p) &= 0, \\ R_c(x_c) &= 0, \\ J_p(x_p) \frac{\partial S}{\partial x_p}(x_p) &= -g_p(x_p) g_c^T(x_c). \end{aligned} \quad (15.13)$$

15.4 The dissipation obstacle and beyond

Surprisingly, the presence of dissipation may pose a problem. Indeed, if $R_p(x_p) \geq 0$ and $R_c(x_c) \geq 0$, the set of PDE's (15.9) extends to

$$\begin{aligned} &\begin{bmatrix} \dfrac{\partial^T C}{\partial x_p}(x_p, x_c) & \dfrac{\partial^T C}{\partial x_c}(x_p, x_c) \end{bmatrix} \\ &\qquad \times \begin{bmatrix} J_p(x_p) - R_p(x_p) & -g_p(x_p)g_c^T(x_c) \\ g_c(x_c)g_p^T(x_p) & J_c(x_c) - R_c(x_c) \end{bmatrix} = 0, \end{aligned} \quad (15.14)$$

which implies

$$-\frac{\partial^T C}{\partial x_p}(x_p, x_c) R_p(x_p) \frac{\partial C}{\partial x_p}(x_p, x_c) = \frac{\partial^T C}{\partial x_c}(x_p, x_c) R_c(x_c) \frac{\partial C}{\partial x_c}(x_p, x_c).$$

However, since $R_p(x_p)$ and $R_c(x_c)$ are semi-positive definite, we have

$$R_p(x_p)\frac{\partial C}{\partial x_p}(x_p, x_c) = 0, \quad R_c(x_c)\frac{\partial C}{\partial x_c}(x_p, x_c) = 0.$$

The same condition is also appearing in (15.13) when we consider Casimirs of the form (15.11). This restriction is known as the *dissipation obstacle*, which, roughly speaking, dictates that the Casimir functions cannot depend on the coordinates that are subject to dissipation. This means that dissipation is admissible only on the coordinates of the closed-loop system that do not require shaping of the Hamiltonian.

Example 15.4. Consider the levitated ball system of Example 2.4 and assume that the inductance of the coil is given by $L(q) = k/(1-q)$, with k some real parameter depending on the geometry of the coil. Then, the desired equilibrium is given by

$$(q^*, p^*, \varphi^*) = (q^*, 0, \pm\sqrt{2kmg}),$$

which suggests that both q and φ need to be shaped in order to stabilize the system at the desired equilibrium. However, we can not find any useful Casimir since

$$\begin{bmatrix}\frac{\partial C}{\partial q} & \frac{\partial C}{\partial p} & \frac{\partial C}{\partial \varphi}\end{bmatrix}\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & R\end{bmatrix} = [0 \quad 0 \quad 0].$$

The dissipation obstacle stems from the assumption that *both* the plant and controller dissipation structures satisfy $R_p(x_p) \geq 0$ and $R_c(x_c) \geq 0$. Although these properties are necessary to ensure that the plant and the controller are both passive systems, they are merely sufficient for passivity of the closed-loop system. Indeed, in Koopman & Jeltsema (2012) it is shown that removing the passivity constraint on the controller naturally resolves the dissipation obstacle. Furthermore, using the Casimir relation between the plant and controller states as in (15.11) guaranties stability of the closed loop. Another method to circumvent the dissipation obstacle is discussed in Section 15.7.

15.5 Passivity-based control

In the previous sections, we have considered the control of pH systems from the perspective of interconnecting a plant pH system with a controller pH system. The aim is then to design a controller system that generates a suitable candidate Lyapunov function to ensure stability of the closed-loop system. In the remaining sections, the control of pH systems is considered from the notion of passivity. In the past two decades, passivity-based control (PBC) has emerged as a model-based non-linear control design method that respects, and successfully exploits, the physical structure of a system. In essence, the PBC methodology exploits the property that physical systems satisfy the energy-balance, i.e., stored energy equals the difference between the supplied energy and the dissipated energy, to arrive, via control, at a modified energy balance satisfying: desired stored energy equals the difference between a new supplied energy and a desired dissipated energy. Hence, PBC aims at energy shaping plus damping injection by rendering the closed-loop passive with respect to some desired storage function.

For ease of presentation, we will assume that the plant system can be represented as an input-state-output pH system of the form

$$\begin{aligned} \dot{x} &= [J(x) - R(x)] \frac{\partial H}{\partial x}(x) + g(x)u, \\ y &= g^T(x) \frac{\partial H}{\partial x}(x), \end{aligned} \tag{15.15}$$

where $x \in \mathcal{X}$ and $u, y \in \mathbb{R}^m$.

15.6 Energy-shaping and damping injection

For a plant system (15.15), the energy-shaping and damping injection (ES-DI) objective is to obtain a target closed-loop system

$$\dot{x} = [J(x) - R_d(x)] \frac{\partial H_d}{\partial x}(x), \tag{15.16}$$

where $R_d(x)$ is the desired dissipation matrix given by

$$R_d(x) = R(x) + g(x)K_d(x)g^T(x),$$

in terms of a damping injection matrix $K_d(x)$. The desired closed-loop Hamiltonian $H_d(x)$ is obtained by augmenting the open-loop Hamiltonian with an additional energy $H_a(x)$ such that $H_d(x) = H(x) + H_a(x)$ has a minimum at the desired equilibrium, i.e.,

$$x^* = \arg\min\{H_d(x)\}. \tag{15.17}$$

The target closed-loop dynamics (15.16) can then be achieved by the static state feedback control

$$u(x) = u_{\text{ES}}(x) + u_{\text{DI}}(x),$$

where

$$u_{\text{ES}}(x) = \left(g^T(x)g(x)\right)^{-1} g^T(x)\left[J(x) - R(x)\right]\frac{\partial H_a}{\partial x}(x),$$
$$u_{\text{DI}}(x) = -K_d(x)g^T(x)\frac{\partial H_d}{\partial x}(x),$$

represent the energy-shaping and damping injection components of the control, respectively. The added energy $H_a(x)$ us a solution to the set of PDE's

$$\begin{bmatrix} g^\perp(x)\left[J(x) - R(x)\right] \\ g^T(x) \end{bmatrix}\frac{\partial H_a}{\partial x}(x) = 0, \tag{15.18}$$

where $g^\perp(x)$ is a left-annihilator of $g(x)$, i.e., $g^\perp(x)g(x) = 0$, of maximal rank. Among all possible solutions (15.18), the one satisfying (15.17) is chosen.

Example 15.5. Consider a (fully actuated) mechanical system with total energy

$$H(q,p) = \tfrac{1}{2}p^T M^{-1}(q)p + P(q),$$

where $M(q) = M^T(q) > 0$ represents the generalized mass matrix. Assume that the potential energy $P(q)$ is bounded from below. The associated pH equations are given by

$$\begin{bmatrix}\dot{q}\\ \dot{p}\end{bmatrix} = \underbrace{\begin{bmatrix} 0 & I_k \\ -I_k & 0\end{bmatrix}}_{J=-J^T}\begin{bmatrix}\frac{\partial H}{\partial q}\\ \frac{\partial H}{\partial p}\end{bmatrix} + \underbrace{\begin{bmatrix}0\\ B(q)\end{bmatrix}}_{g(q)}u,$$
$$y = \begin{bmatrix}0 & B^T(q)\end{bmatrix}\begin{bmatrix}\frac{\partial H}{\partial q}\\ \frac{\partial H}{\partial p}\end{bmatrix}.$$

Clearly, the system has as passive outputs the generalized velocities:

$$\dot{H}(q,p) = u^T y = u^T B^T(q)\frac{\partial H}{\partial p}(q,p) = u^T M^{-1}(q)p = u^T \dot{q}.$$

Considering (15.18), the simplest way to ensure that the closed-loop energy has a minimum at $(q,p) = (q^*,0)$ is to select

$$H_a(q) = -P(q) + \frac{1}{2}(q-q^*)^T K_p(q-q^*) + \kappa,$$

where $K_p = K_p^T > 0$ is the energy-shaping gain and κ is an arbitrary constant. This, in turn, provides the energy-shaping part of the control

$$u_{\text{ES}}(q) = \frac{\partial P}{\partial q}(q) - K_p(q-q^*),$$

and the resulting closed-loop energy

$$H_d(q,p) = \frac{1}{2}p^T M^{-1}(q)p + \frac{1}{2}(q-q^*)^T K_p(q-q^*).$$

To ensure that the trajectories actually converge to $(q^*,0)$, we need to render the closed-loop asymptotically stable by adding some damping

$$u_{\text{DI}}(p) = -K_d\frac{\partial H}{\partial p}(q,p) = -K_d\dot{q},$$

where $K_d = K_d^T > 0$. Note that the energy-balance of the system is now

$$H_d[q(t),p(t)] - H_d[q(0),p(0)] = -\int_0^t \dot{q}^T(\tau)K_d\dot{q}(\tau)\mathrm{d}\tau.$$

Observe that the controller obtained in Example 15.5 is just the classical PD plus gravity compensation controller. However, the design via energy-shaping and damping injection provides a new interpretation of the controller, namely, that the closed-loop energy is (up to a constant) equal to

$$H_d(x) = H(x) - \int_0^t u^T(\tau)y(\tau)\mathrm{d}\tau,$$

i.e., the difference between the plant and the controller energy. For that reason, the ES-DI methodology is often referred to as energy-balancing (EB) control.

15.7 Interconnection and damping assignment

As with the conventional Casimir-based control method, a major drawback of the ES-DI approach is the solvability of the PDE's (15.18), which is also stymied by dissipation obstacle. A generalization of the ES-DI method that circumvents the dissipation obstacle is provided by the interconnection and damping assignment passivity-based control (IDA-PBC) method.

For a plant system (15.15), the interconnection and damping assignment passivity-based control (IDA-PBC) design objective is to obtain a closed-loop system of the form

$$\dot{x} = [J_d(x) - R_d(x)] \frac{\partial H_d}{\partial x}(x), \tag{15.19}$$

where the desired interconnection and the dissipation matrices satisfy skew-symmetry and symmetric positive semi-definiteness respectively, i.e., $J_d(x) = -J_d^T(x)$ and $R_d(x) = R_d^T(x)$, with $R_d(x) \geq 0$, respectively. As before, the desired closed-loop Hamiltonian is defined by $H_d(x) = H(x) + H_a(x)$, where the added energy $H_a(x)$ satisfies

$$\begin{aligned}[J(x) + J_a(x) - R(x) \ -R_a(x)] \frac{\partial H_a}{\partial x}(x) = \\ [J_a(x) - R_a(x)] \frac{\partial H}{\partial x}(x) + g(x)u(x),\end{aligned} \tag{15.20}$$

where $J_a(x) := J_d(x) - J(x)$ and $R_a(x) := R_d(x) - R(x)$.

Proposition 15.2 (Ortega et al. (2001b))**.** Consider the system (15.15) and a desired equilibrium x^* to be stabilized. Assume that we can find functions $u(x)$ and $H_a(x)$, and matrices $J_a(x)$ and $R_a(x)$ satisfying (15.20) and such that the following conditions occur.

- Equilibrium assignment: at x^* the gradient of $H_a(x)$ verifies

$$\frac{\partial H_a}{\partial x}(x^*) + \frac{\partial H}{\partial x}(x^*) = 0. \tag{15.21}$$

- Minimum condition: the Hessian of $H_a(x)$ at x^* satisfies

$$\frac{\partial^2 H_a}{\partial x^2}(x^*) + \frac{\partial^2 H}{\partial x^2}(x^*) > 0. \tag{15.22}$$

Then, x^* will be a (locally) stable equilibrium of the closed-loop system (15.19). It will be (locally) assymptotically stable if, in addition, the largest invariant set under the closed-loop dynamics contained in

$$\left\{ x \in \mathcal{X} \middle| \frac{\partial^T H_d}{\partial x}(x) R_d(x) \frac{\partial H_d}{\partial x}(x) = 0 \right\}$$

equals $\{x^*\}$.

Remark 15.1. Note that if the interconnection and damping matrices of the open-loop system are not changed, i.e., $J_d(x) = J(x)$ and $R_d(x) = R(x)$, or equivalently, if $J_a(x) = 0$ and $R_a(x) = 0$, then the IDA-PBC methodology reduces to the ES-DI scheme outlined in the previous section.

Example 15.6. Consider again the levitated ball system of Example 2.4. Suppose that we first try to stabilize the system by only shaping the Hamiltonian without altering the interconnection and damping matrices. Then, the PDE (15.20) reduces to

$$\frac{\partial H_a}{\partial q} = 0, \ \frac{\partial H_a}{\partial p} = 0, \ -R\frac{\partial H_a}{\partial \varphi} = u(x),$$

which means that H_a can only depend on φ. Thus, the resulting closed-loop Hamiltonian will be of the form

$$H_d(q, p, \varphi) = mgq + \frac{p^2}{2m} + \frac{(1-q)}{2k}\varphi^2 + H_a(\varphi).$$

Even though, with a suitable selection of H_a, we can satisfy the equilibrium assignment condition, the Hessian of H_d at $(q^*, 0, \pm\sqrt{2kmg})$ will never be positive definite. The source of the problem is the lack of an effective coupling between the electrical and mechanical subsystems. Indeed, the interconnection matrix J only couples position with velocity. To overcome this problem, we propose to enforce a coupling between the flux-linkage φ and the momentum p. For, we modify interconnection structure to

$$J_d = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & -\alpha \\ 0 & \alpha & 0 \end{bmatrix},$$

where α is a constant to be defined, so that (15.20) becomes

$$\begin{aligned}\frac{\partial H_a}{\partial p} &= 0,\\ -\frac{\partial H_a}{\partial q} - \alpha\frac{\partial H_a}{\partial \varphi} &= -\alpha\frac{(1-q)}{k}\varphi,\\ \alpha\frac{\partial H_a}{\partial p} - R\frac{\partial H_a}{\partial \varphi} &= \alpha\frac{p}{m} + u(x).\end{aligned}$$

The third equation defines the control, whereas the second one can be solved to obtain

$$H_a(q,\varphi) = \frac{\varphi^3}{6k\alpha} + \frac{1}{2k}(1-q)\varphi^2 + \Phi\left(q + \frac{\varphi}{\alpha}\right),$$

where $\Phi(\cdot)$ should be chosen such that (15.21) and (15.22) are satisfied.

In the latter example, the control is obtained from (15.20), where J_d is selected based on physical intuition and $R_d = 0$. The remaining unknown 'parameter' $H_a(x)$ is then selected such that the closed-loop energy $H_d(x) = H(x) + H_a(x)$ has its minimum in the desired equilibrium point. In general, the control that achieves the closed-loop control objective (15.19) is given by

$$\begin{aligned}u(x) = &\left(g^T(x)g(x)\right)^{-1} g^T(x)\\ &\times\left(\left[J_d(x) - R_d(x)\right]\frac{\partial H_d}{\partial x}(x) - \left[J(x) - R(x)\right]\frac{\partial H}{\partial x}\right),\end{aligned}$$

where the desired closed-loop Hamiltonian $H_d(x)$ and the desired interconnection and damping matrices are obtained by solving the so-called *matching condition*

$$g^\perp(x)\left[J(x) - R(x)\right]\frac{\partial H}{\partial x} = g^\perp(x)\left[J_d(x) - R_d(x)\right]\frac{\partial H_d}{\partial x}, \qquad (15.23)$$

such that $H_d(x)$ has its minimum at the desired equilibrium.

Solving the matching condition (15.23) can be a tedious process. The following observations about the choice of the desired interconnection and dissipation matrices of (15.19) can be of help in the process:

- The desired interconnection matrix $J_d(x)$ and the dissipation matrix $R_d(x)$ can be freely chosen provided they satisfy skew-symmetry and positive semi-definiteness, respectively.

- The left-annihilator matrix $g^{\perp}(x)$ can be considered as an additional degree of freedom. Hence for a particular problem it can be appropriately chosen to reduce the complexity of the matching condition (15.23).

- The desired Hamiltonian $H_d(x)$ can be partially or completely fixed to satisfy the desired equilibrium condition (15.17).

Using combinations of the stated options, there are three major approaches to solve the PDE's of (15.23):

- *Non-parameterized IDA-PBC*: In this general form, $J_d(x)$ and $R_d(x)$ are fixed and the PDE's (15.23) are solved for the energy function $H_d(x)$. Among the admissible solutions, the one satisfying (15.17) is chosen.

- *Algebraic IDA-PBC*: The desired energy function $H_d(x)$ is fixed thus rendering (15.23) an algebraic set of equations in terms of the unknown matrices $J_d(x)$ and $R_d(x)$.

- *Parameterized IDA-PBC*: Here, the structure of the energy function $H_d(x)$ is fixed. This imposes constraints on the unknown matrices $J_d(x)$ and $R_d(x)$, which need to be satisfied by (15.23).

For a detailed discussion about these approaches and some motivating examples, we refer to Ortega et al. (2001b, 2008), and the references therein.

15.8 Power-shaping control

We conclude this chapter by briefly highlighting an appealing alternative to energy-shaping called *power-shaping*. Power-shaping PBC is based on the Brayton-Moser framework presented in Chapter 11. The idea is as follows. Recall from Section 11.4 that if $Q(z) + Q^T(z) \leq 0$,

then $\dot{P}(z) \leq u^T y'$. Hence, integrating the latter from 0 to τ, yields the power balance inequality

$$P(z(\tau)) - P(z(0)) \leq \int_0^\tau u^T(t) y'(t) \mathrm{d}t. \tag{15.24}$$

Usually, the point where the mixed-potential has a minimum is not the operating point of interest, and we would rather stabilize another admissible equilibrium point. Thus, we look for a control law such that the power supplied by the controller, say P_a, can be expressed as a function of the state z. Indeed, from (15.24) we see that the mixed-potential function $P(z)$ is shaped with the control $u(z)$, where

$$g(z)u(z) = \nabla P_a(z).$$

This yields the closed-loop system $Q(z)\dot{z} = \frac{\partial}{\partial z} P_d(z)$, with total Lyapunov function $P_d(z) := P(z) + P_a(z)$. The equilibrium will be stable if it corresponds to the minimum of the total Lyapunov function.

So far the method of power shaping is shown to be applicable to a large class of nonlinear RLC circuits system Ortega et al. (2003), and is extended to a class of nonlinear systems in García-Canseco et al. (2010); Favache and Dochain (2010). The application of power shaping to switched power converters is a topic of future research. Another control application in which the generation of passive input-output maps is of interest is PI control. In Hernandez-Gomez et al. (2010) it is shown that under certain conditions a passive system can be stabilized by a simple PI controller. A general Brayton-Moser based PI control method is discussed in Dirksz and Scherpen (2012).

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Appendices

A

Proofs

A.1 Proof of Proposition 2.1

Let $\mathcal{D}$ satisfy (2.5). Then for every $(f, e) \in \mathcal{D}$

$$0 = \ll (f, e), (f, e) \gg = < e \mid f > + < e \mid f > = 2 < e \mid f >$$

By non-degeneracy of $\ll, \gg$

$$\dim \mathcal{D}^{\perp\!\!\!\perp} = \dim(\mathcal{F} \times \mathcal{E}) - \dim \mathcal{D} = 2 \dim \mathcal{F} - \dim \mathcal{D}$$

and hence property (2.5) implies $\dim \mathcal{D} = \dim \mathcal{F}$. Conversely, let $\mathcal{D}$ be a Dirac structure and thus satisfying Properties 1 and 2 of 2.1. Let $(f^a, e^a), (f^b, e^b)$ be any vectors contained in $\mathcal{D}$. Then by linearity also $(f^a + f^b, e^a + e^b) \in \mathcal{D}$. Hence by Property 1

$$\begin{aligned} 0 &= < e^a + e^b \mid f^a + f^b > \\ &= < e^a \mid f^b > + < e^b \mid f^a > + < e^a \mid f^a > + < e^b \mid f^b > \\ &= < e^a \mid f^b > + < e^b \mid f^a > = \ll (f^a, e^a), (f^b, e^b) \gg, \end{aligned} \tag{A.1}$$

since by another application of Property 1, $< e^a \mid f^a > = < e^b \mid f^b > = 0$. This implies that $\mathcal{D} \subset \mathcal{D}^{\perp\!\!\!\perp}$. Furthermore, by Property 2 and $\dim \mathcal{D}^{\perp\!\!\!\perp} = 2 \dim \mathcal{F} - \dim \mathcal{D}$ it follows that

$$\dim \mathcal{D} = \dim \mathcal{D}^{\perp\!\!\!\perp},$$

yielding $\mathcal{D} = \mathcal{D}^{\perp\!\!\!\perp}$. For the alternative characterization we note that we have actually shown that Property 1 implies $\mathcal{D} \subset \mathcal{D}^{\perp\!\!\!\perp}$. Together with the fact that $\dim \mathcal{D}^{\perp\!\!\!\perp} = 2 \dim \mathcal{F} - \dim \mathcal{D}$ this implies that any subspace $\mathcal{D}$ satisfying Property 1 has the property that $\dim \mathcal{D} \leq \dim \mathcal{F}$. Thus, as claimed before, a Dirac structure is a linear subspace *of maximal dimension* satisfying Property 1.

A.2 Proof of Proposition 2.2

It is immediately seen that any subspace $\mathcal{K} \times \mathcal{K}^\perp$ satisfies (2.7), and is a Dirac structure. Conversely, let the Dirac structure $\mathcal{D}$ satisfy (2.7). Define the following subspaces

$$\mathcal{F}_0 = \{f \in \mathcal{F} \mid (f,0) \in \mathcal{D}\} \qquad \mathcal{F}_1 = \{f \in \mathcal{F} \mid \exists e \in \mathcal{E} \text{ s.t. } (f,e) \in \mathcal{D}\}$$
$$\mathcal{E}_0 = \{e \in \mathcal{E} \mid (0,e) \in \mathcal{D}\} \qquad \mathcal{E}_1 = \{e \in \mathcal{E} \mid \exists f \in \mathcal{F} \text{ s.t. } (f,e) \in \mathcal{D}\}$$

It is readily seen Dalsmo & van der Schaft (1999) that for any Dirac structure $\mathcal{E}_1 = (\mathcal{F}_0)^\perp, \mathcal{E}_0 = (\mathcal{F}_1)^\perp$. We will now show that (2.7) implies that $\mathcal{F}_0 = \mathcal{F}_1 =: \mathcal{K}$ (and hence $\mathcal{E}_0 = \mathcal{E}_1 =: \mathcal{K}^\perp$). Clearly, $\mathcal{F}_0 \subset \mathcal{F}_1$. Let now $(f_a, e_a) \in \mathcal{D}$ and thus $f_a \in \mathcal{F}_1$. Then for all $(f_b, e_b) \in \mathcal{D}$

$$\ll (f_a,0),(f_b,e_b) \gg := < e_b \mid f_a > + < 0 \mid f_b > = < e_b \mid f_a > = 0$$

by (2.7). Hence, also $(f_a, 0) \in \mathcal{D}$ and thus $f_a \in \mathcal{F}_0$. By definition $\mathcal{F}_0 \times \mathcal{E}_0 \subset \mathcal{D}$, and hence $\mathcal{K} \times \mathcal{K}^\perp \subset \mathcal{D}$. Finally, since the dimension of $\mathcal{K} \times \mathcal{K}^\perp$ equals the dimension of $\mathcal{F}$ equality results.

A.3 Extension of Proposition 2.1

Finally, we mention the following adaptation of the first definition of a Dirac structure (Definition 2.1), which also applies to the infinite-dimensional case, and is in the spirit of the definition of a maximal monotone relation as discussed in Chapter 9.

Proposition A.1. $\mathcal{D} \subset \mathcal{F} \times \mathcal{E}$ is a Dirac structure if and only if it satisfies Property 1 in Definition 2.1 and is *maximal* with respect to this property, that is, if some subspace $\mathcal{D}'$ satisfies Property 1 while $\mathcal{D} \subset \mathcal{D}'$, then $\mathcal{D} = \mathcal{D}'$.

B

Physical meaning of efforts and flows

In science and engineering, the ideas and concepts developed in one branch of science and engineering are often transferred to other branches. One approach to transferring ideas and concepts is by the use of analogies. Historically, the first attempt to relate mechanical and electrical systems was due to James Clerk Maxwell and Lord Kelvin in the 19th century by using the similarity between force and voltage, as is also apparent from the early use of the term electromotive force (emf). This force-voltage analogy implies that a mechanical mass is analogous to an electrical inductor. Once the force-voltage analogy had been established, some scientists started to address some of its limitations. These limitations led to the alternative force-current analogy (often referred to as the Firestone or mobility analogy), which implies that a mechanical mass is analogous to an electrical capacitor. For a further discussion, see Jeltsema & Scherpen (2009) and the references therein.

Throughout the present work it is shown that a physical system can be characterized by interconnections between energy storage elements, energy dissipating elements, and the environment. One remarkable feature of the storage elements in each domain is that their

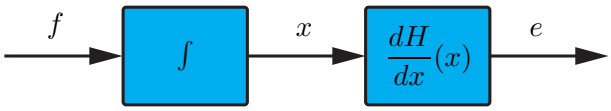

Figure B.1: Structure of a storage element.

structure is identical. Indeed, an inductor and a capacitor, for example, although being dual elements, can both be characterized by a port-Hamiltonian representation of the form

$$\dot{x} = f$$
$$e = \frac{dH}{dx}(x),$$

which corresponds to the structure shown in Fig. B.1.

In contrast to common port-based modeling approaches, such as the standard bondgraph formalism Paynter (1960) or classical energy- and power-based approaches Jeltsema & Scherpen (2009), the port-Hamiltonian framework uses only one type of storage, continuing the work of Breedveld Breedveld (1984) on port-based modeling of thermodynamics. From this perspective, it is natural to discriminate among domains depending on the kind of energy that a certain part of the system can store and split the usual physical domains into two subdomains. Consequently, we do not speak of electrical, mechanical, or hydraulic domains, but of electric *and* magnetic, mechanical kinetic *and* mechanical potential, or hydraulic kinetic *and* hydraulic potential subdomains. The chemical and thermic domains have no dual subdomains, which is related to the irreversible transformation of energy in thermodynamics. This subdomain classification is based on the generalized bond graph (GBG) framework introduced in Breedveld (1984); Duindam et al. (2009), where f is referred to as the *flow* (rate of change of state) and e as the *effort* (equilibrium-determining variable). By considering only one type of storage, the state variables associated to each subdomain are all treated at the same footing. Table B.1 shows a complete overview.

The energy associated to each of these subdomains is then computed as follows. Let $e = \Phi(x)$ represent the constitutive relationship

Table B.1: Domain classification in port-Hamiltonian framework.

physical subdomain	flow $f \in \mathcal{F}$	effort $e \in \mathcal{E}$	storage state $x \in \mathcal{X}$
electric	current	voltage	charge
magnetic	voltage	current	flux linkage
potential translation	velocity	force	displacement
kinetic translation	force	velocity	momentum
potential rotation	angular velocity	torque	angular displacement
kinetic rotation	torque	angular velocity	angular momentum
potential hydraulic	volume flow	pressure	volume
kinetic hydraulic	pressure	volume flow	flow tube momentum
chemical	molar flow	chemical potential	number of moles
thermal	entropy flow	temperature	entropy

of a one-port energy storing element. Then, the associated Hamiltonian is computed as

$$H(x) = \int \Phi(x) dx,$$

which essentially represents the area underneath the curve in the (e, x)–plane. Note that the resistive elements in each subdomain are simply represented by relationships between e and f of the form $\mathcal{R}(e, f) = 0$.

An advantage of considering only one type of storage is that the concept of mechanical force has no unique meaning as it may play the role of an 'input' (rate of change of state or flow) in the kinetic domain or an 'output' (equilibrium-determining variable or effort) in the potential domain. Hence, from a port-Hamiltonian perspective, the discussion which analogy—when it exists—is superior is left a non-issue.

Another (mathematical) advantage is that the state space manifold $\mathcal{X}$ of the port-Hamiltonian system is one single object, with the flow variables all belonging to the tangent space $T_x\mathcal{X}$ at a given state x and the effort variables belonging to the co-tangent space $T_x^*\mathcal{X}$. This is especially important if the state space is not a linear space.

Notice, however, that in the treatment of Chapter 12 the state variables may either belong to the vertex space or its dual, respectively, to the edge space or its dual. For example, the state variable corresponding to a capacitor naturally belongs to the edge space, while the state variable corresponding to an inductor naturally belongs to the dual of the edge space. This is very much in line with the use of *through* and *across* variables MacFarlane (1970).

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