KAM theory:
the legacy of Kolmogorov's 1954 paper

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Abstract

Kolmogorov-Arnol'd-Moser (or KAM) theory was developed for conservative dynamical systems that are nearly integrable. Integrable systems in their phase space usually contain lots of invariant tori and KAM theory establishes persistence results for such tori, which carry quasi-periodic motions. We sketch this theory which begins with Kolmogorov's pioneering work [40, 41].

1   Introduction

At the International Mathematical Congress of 1954, held at Amsterdam, A.N. Kolmogorov gave a closing lecture with title "The general theory of dynamical systems and classical mechanics" [41], discussing the paper [40]. The event took place in the Amsterdam Concertgebouw and it has played a major role in the developments of what is now called Kolmogorov-Arnold-Moser (or KAM) theory.

In this lecture Kolmogorov discusses the occurrence of multi- or quasi-periodic motions, which in the phase space are confined to invariant tori. He restricts himself to conservative (or Hamiltonian) dynamical systems, as these are generally used for modelling in classical mechanics. Invariant (Lagrangean) tori that carry quasi-periodic motions were well-known to occur in Liouville integrable systems

and Kolmogorov's lecture deals with their persistence under small, non-integrable perturbation of the Hamiltonian.

The general message of KAM theory is that generically, in small perturbations of integrable systems the union of quasi-periodic Lagrangean tori has positive (Liouville) measure both in the phase space and in the energy hypersurfaces.

On the one hand this statement applies to many concrete models of classical mechanics. Examples are certain versions of the n-body problem, of which the general non-integrability already had been established by Poincaré. Historically its consequences for philosophical issues like the stability of the solar system are of interest. Indeed, perpetual stability of the solar system certainly would be granted if its actual motion were quasi-periodic.

On the other hand, and from a more global point of view, the measure-theoretical part of KAM theory implies that for typical Hamiltonian systems in finitely many degrees of freedom, no ergodicity holds, since the energy hypersurfaces can be decomposed in several disjoint invariant sets of positive measure. This is of particular interest for statistical physics, where the ergodic hypothesis roughly claims that the system, when confined to bounded energy hypersurfaces, is ergodic.

This paradox probably is resolved as the number of particles is increasing since the obstruction to ergodicity provided by the KAM tori then seems to decrease rapidly in importance.

The present paper sketches certain developments related to the Kolmogorov 1954 lecture [41]. We start with discussing the linearization problem near a fixed point of an analytic map of the complex plane. This is the first known occasion where a small divisor problem was solved, the kind of problem which is so central in KAM theory. Next we turn to the dynamics of circle maps, which goes back to Arnold, followed by a digression to area preserving twist maps, where the name of Moser comes in. After this we discuss conservative KAM theory in greater generality, also touching the ergodic 'paradox'. For an earlier version of the present paper see [12].

Remark A property is called typical if it occurs on an open set of the dynamical systems at hand. Regarding the topology on 'function space', one may think of a (weak) Whitney topology for differentiable systems, or of the compact-open topology for analytic systems. The property we have in mind is the occurrence in phase space of quasi-periodic invariant tori with positive measure. Compare with, e.g., [18, 23].

2 Complex linearization

Consider a local holomorphic map (or a germ) $F : (\mathbf{C}, 0) \to (\mathbf{C}, 0)$ of the form $F(z) = \lambda z + f(z)$ with $f(0) = f'(0) = 0$. The question is whether there exists a local biholomorphic transformation $\Phi : (\mathbf{C}, 0) \to (\mathbf{C}, 0)$ such that

$$\Phi \circ F = \lambda \cdot \Phi. \tag{1}$$

We say that Φ linearizes F near its fixed point.

Formal solution

First consider the problem at a formal level. The corresponding results go back to Poincaré, for an overview see Arnold [6]. Given a series expansion $f(z) = \sum_{j \geq 2} f_j z^j$ we look for another series $\Phi(z) = z + \sum_{j \geq 2} \phi_j z^j$, such that the conjugation relation (1) holds formally. It turns out that a formal solution exists whenever $\lambda \neq 0$ is not a root of 1. Indeed, the coefficients ϕ_j can be determined recursively by the following equations:

$$\begin{aligned}
\lambda(1 - \lambda)\phi_2 &= f_2, \\
\lambda(1 - \lambda^2)\phi_3 &= f_3 + 2\lambda f_2 \phi_2, \\
\lambda(1 - \lambda^{n-1})\phi_n &= f_n + \text{ already known terms.}
\end{aligned} \tag{2}$$

From this the claim directly follows.

Convergence

In the hyperbolic case where $0 < |\lambda| \neq 1$ the series for Φ has positive radius of convergence. This was proven by Poincaré, not by considering the series but by a direct iteration method, compare [6].

So there remains the elliptic case with $\lambda \in \mathbf{T}^1$ (the complex unit circle). We observe from the equations (2) that, even if λ is not a root of unity, its powers may accumulate on 1. This would give small divisors in the formal series of Φ, which casts doubt on its convergence.

This problem was succesfully solved by C.L. Siegel [66] in 1942. To this purpose, writing $\lambda = \mathrm{e}^{2\pi i\alpha}$, the following Diophantine conditions were introduced. For some $\gamma > 0$ and $\tau > 0$ it is required that

$$|\alpha - \frac{p}{q}| \geq \frac{\gamma}{q^\tau},$$

for all rationals p/q (with $q > 0$). It turns out that this is sufficient for convergence of the formal solution for Φ.

Let us discuss certain aspects of the set of $\lambda \in \mathbf{T}^1$ satisfying the Diophantine conditions, as these are relevant for us. It can be directly shown that for fixed $\gamma > 0$ and $\tau > 2$, with γ sufficiently small, the Diophantine conditions give rise to a Cantor subset $\lambda \in \mathbf{T}^1$ which is of almost full measure. We recall that the characterizing (topological) properties of a Cantor set are compactness, nowhere denseness and perfectness. It can be easily seen that the measure of the complement is of order $O(\gamma)$ as $\gamma \downarrow 0$. Compare with, e.g., [18, 23]. We note that for small values of γ (as well as for large values of τ) only a small radius of convergence for Φ can be guaranteed [66].

Geometry and number theory

That the Diophantine conditions give rise to a nowhere dense subset of $\mathbf{T}^1$, directly follows from the fact that the roots of unity, which are everywhere dense, are in their complement. The following example, due to H. Cremer [25] in 1927, already indicates that there is more to this. For a nice description of this example in a somewhat different context see [11].

Consider the map $F : \mathbf{C} \to \mathbf{C}$ given by

$$F(z) = \lambda z + z^2,$$

where $\lambda \in \mathbf{T}^1$ is not a root of unity. We shall see that there is a topologically large subset of λ for which the map has periodic points in any neighbourhood of 0. For such λ this provides a geometrical proof that the formal conjugation diverges, as follows. Indeed, since the linear map $z \mapsto \lambda z$ is an irrational rotation all of whose orbits that are everywhere dense in $\mathbf{T}^1$, the existence of periodic orbits in every neighbourhood of 0 implies that the formal linearization must have zero radius of convergence.

To examine periodic points of period q, we consider the equation

$$F^q(z) = z. \tag{3}$$

Using that

$$F^q(z) = \lambda^q z + \cdots + z^{2^q}$$

it follows that

$$F^q(z) - z = z(\lambda^q - 1 + \cdots + z^{2^q - 1}).$$

Abbreviating $N = 2^q - 1$, let $z_1 \cdot z_2, \cdots, z_N$ be the nontrivial solutions of (3). It follows that their product satifies the equation

$$z_1 \cdot z_2 \cdot \cdots \cdot z_N = \lambda^q - 1.$$

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From this we see that there exists a solution within radius

$$|\lambda^q - 1|^{1/N}$$

of $z = 0$. Now consider the set of $\lambda \in \mathbf{T^1}$ satifying

$$\liminf_{q \to \infty} |\lambda^q - 1|^{1/N} = 0.$$

This set turns out to be residual.[1] We recall that a residual set contains a countable intersection of dense-open sets; this property means that the set is large in the sense of topology. The present residual set resembles that of the Liouville-numbers. Notice that our residual set has measure zero, since it has all Diophantine numbers in its complement. For a general background reference dealing with subsets of $\mathbf{R}$ that are large in topology and small in measure, or vice versa, see Oxtoby [57].

We conclude that for all λ contained in this residual set, periodic points of F occur in any neighbourhood of $z = 0$, which makes the above divergence argument work.

By the end of the 20th century J.-C. Yoccoz [69, 70] completely solved the elliptic case, proving that the so-called Bruno-condition [21] on the corresponding continued fractions, is necessary and sufficient for convergence of the formal linearization for all possible higher order terms.

3 Circle Maps

We slightly shift context by considering a smooth (for definiteness, say C^∞ or real analytic) family of circle diffeomorphisms

$$P_{\alpha,\varepsilon} : \mathbf{T}^1 \to \mathbf{T}^1; \ x \mapsto x + 2\pi\alpha + \varepsilon a(x, \alpha, \varepsilon),$$

where ε is a small parameter; so we are in the neighbourhood of a family of rigid circle rotations. For convenience we rewrite this as a 'vertical' family of cylinder maps

$$P_\varepsilon : \mathbf{T}^1 \times [0,1] \to \mathbf{T}^1 \times [0,1]; \ (x, \alpha) \mapsto (x + 2\pi\alpha + \varepsilon a(x, \alpha, \varepsilon), \alpha).$$

The problem is to find a diffeomorphism Φ_ε that conjugates P_ε and P_0. Note that the latter map is associated to the family of rigid rotations. To be precise, we require that the following diagram commutes:

$$
\begin{array}{ccc}
\mathbf{T}^1 \times [0,1] & \xrightarrow{P_\varepsilon} & \mathbf{T}^1 \times [0,1] \\
\uparrow \Phi_\varepsilon & & \uparrow \Phi_\varepsilon \\
\mathbf{T}^1 \times [0,1] & \xrightarrow{P_0} & \mathbf{T}^1 \times [0,1],
\end{array}
$$

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[1] In other words a dense G_δ or a set of 2nd Baire category.

meaning that

$$P_\varepsilon \circ \Phi_\varepsilon = \Phi_\varepsilon \circ P_0. \qquad (4)$$

Small divisors again

We proceed by formally solving equation (4). Indeed, assuming that Φ_ε has the form

$$\Phi_\varepsilon(x, \alpha) = (x + \varepsilon U(x, \alpha, \varepsilon), \alpha + \varepsilon \sigma(\alpha, \varepsilon)) \qquad (5)$$

it follows that (4) can be rewritten as

$$U(x + 2\pi\alpha, \alpha, \varepsilon) - U(x, \alpha, \varepsilon) = 2\pi\sigma(\alpha, \varepsilon) + a\left(x + \varepsilon U(x, \alpha, \varepsilon), \alpha + \varepsilon\sigma(\alpha, \varepsilon), \varepsilon\right)$$

Expanding in powers of ε and comparing lowest order coefficients, equation (4) leads to the linear equation

$$U_0(x + 2\pi\alpha, \alpha) - U_0(x, \alpha) = 2\pi\sigma_0(\alpha) + a_0(x, \alpha).$$

This linear equation can be directly solved by Fourier-series. Indeed, writing

$$\begin{aligned}
a_0(x, \alpha) &= \sum_{k \in \mathbf{Z}} a_{0k}(\alpha) \mathrm{e}^{\mathrm{i}kx}, \\
U_0(x, \alpha) &= \sum_{k \in \mathbf{Z}} U_{0k}(\alpha) \mathrm{e}^{\mathrm{i}kx},
\end{aligned}$$

we find that $\sigma_0 = -\frac{1}{2\pi} a_{00}$, which amounts to a parameter shift, and that

$$U_{0k}(\alpha) = \frac{a_{0k}(\alpha)}{\mathrm{e}^{2\pi \mathrm{i}k\alpha} - 1}.$$

Similar to what we had in the previous case, we observe that there exists a formal solution as soon as α is irrational. As before, the powers of $\mathrm{e}^{2\pi \mathrm{i}\alpha}$ may still accumulate on 1. This gives small divisors in the Fourier series, which puts its convergence in doubt.

A KAM **theorem for circle maps**

To overcome the problem of small divisors, we introduce the same Diophantine conditions as before, requiring that for given numbers $\tau > 2$ and $\gamma > 0$, and for all rationals p/q one has

$$|\alpha - \frac{p}{q}| \geq \frac{\gamma}{q^\tau}.$$

In correspondence to what was said earlier, this gives a Cantor set $[0,1]_c \subseteq [0,1]$, of almost full measure as $\gamma > 0$ is small. As our first example of an actual KAM theorem we now formulate

Theorem 1 *In the above circumstances assume that $\gamma > 0$ is sufficiently small. Then, for $|\varepsilon|$ sufficiently small, there exists a smooth (C^∞) transformation of the cylinder $\Phi_\varepsilon : \mathbf{T}^1 \times [0,1] \to \mathbf{T}^1 \times [0,1]$, conjugating the restriction $P_0|_{\mathbf{T}^1 \times [0,1]_c}$ to a subsystem of P_ε. Moreover, Φ depends smoothly on ε.*

This result goes back to V.I. Arnold [5], also compare with [6]. The present formulation closely fits with [19, 18]. Concerning the smoothness of the map Φ_ε, compare with Pöschel [58] and with [67, 71, 72]. The actual proof of theorem 1 does not deal with the power series in ε. Instead it uses a Newton method and an approximation property by analytic maps, of Whitney smooth maps defined on closed sets.

We observe that the KAM theorem 1 has a strongly perturbative character: indeed it only applies to small perturbations of the rigid rotation family. In contrast to this M.R. Herman [35] and J.-C. Yoccoz [68, 69] have proven a non-perturbative version of this theorem, i.e., for large values of ε. Also see [49].

Discussion

A circle diffeomorphism smoothly conjugated to a rigid rotation $x \mapsto x + 2\pi\alpha$ with α irrational, is called *quasi-periodic*. It is not hard to show that each orbit of such a quasi-periodic map fills the circle densely [26]. Notice that for $\alpha \in [0,1]_c$ the map rigid rotation $P_{\alpha,0}$ certainly is quasi-periodic.

A first consequence of theorem 1 then is that the circle maps $P_{\alpha,\varepsilon}$ that are conjugated to one of the Diophantine – and hence quasi-periodic – rotations $P_{\alpha,0}$, are still quasi-periodic. So we conclude that quasi-periodicity typically occurs with positive measure in the parameter space. Moreover, the fact that a Cantor set is perfect,[2] implies that in this setting quasi-periodicity never occurs as an isolated phenomenon.

As a concrete example consider the Arnold family

$$P_{\alpha,\varepsilon}(x) = x + 2\pi\alpha + \varepsilon \sin x$$

of circle maps, looking at the (α, ε)-plane of parameters. We restrict to $|\varepsilon| < 1$, to ensure that P_ε is a diffeomorphism. It is known [5, 6, 26] that from the points $(\alpha, \varepsilon) = (p/q, 0)$ resonance tongues emanate into the open half-planes $\varepsilon \neq 0$, in such a way that for small $|\varepsilon|$ an open and dense subset is covered. In the tongue

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[2]Which means that it contains no isolated points.

emanating from $(\alpha, \varepsilon) = (p/q, 0)$ the dynamics is periodic, with rotation number p/q (for a definition see, e.g., [26]).

Theorem 1 implies that in the complement of this union of tongues there exists a union of smooth curves that fill out positive measure. For parameter values on these curves, the dynamics is quasi-periodic.

Theorem 1 has a number of applications. Indeed, it applies whenever a system of ordinary differential equations has a 2-torus attractor, of which P_ε is a Poincaré map close to a rigid rotation family. In that case the quasi-periodic subsystem of P_ε, for $|\varepsilon| \ll 1$ often is referred to as a family of quasi-periodic attractors [62]. Also in the general case resonance tongues are (generically) present and we find the same intricate interplay of periodicity and quasi-periodicity as in the Arnold family.

Higher dimensional analogues of the present situation exist, where next to periodicity and quasi-periodicity, also chaotic dynamics coexists [19, 18]. This scenario and the transitions or bifurcations between the various kinds of dynamics, has been associated to the onset of turbulence in fluid dynamics, see Ruelle and Takens et al. [63, 56]. The quasi-periodic state then is seen as intermediate between very orderly and chaotic. Also see [36, 44, 45].

4 The twist theorem

Returning to the conservative setting we consider an annulus $A \subseteq \mathbf{R}^2$ with 'polar' coordinates $(\varphi, I) \in \mathbf{T}^1 \times \mathbf{K}$, where $\mathbf{K}$ is compact. We endow A with the area form $\sigma = d\varphi \wedge dI$. Consider a smooth (say C^∞) map $P_\varepsilon : A \to A$ of the form

$$P_\varepsilon(\varphi, I) = (\varphi + 2\pi\alpha(I), I) + O(\varepsilon), \qquad (6)$$

that preserves the area σ. Note that for $\epsilon = 0$ the map leaves the family of circles $I = cst$ invariant and again the problem is the persistence of this family for $|\varepsilon|$ small. We say that the map P_0 is a (pure) twist map if the map $I \mapsto \alpha(I)$ is a diffeomorphism.

We assume Diophantine conditions as before: For given constants $\tau > 2$ and $\gamma > 0$ let

$$\left| \alpha - \frac{p}{q} \right| \geq \frac{\gamma}{q^\tau},$$

for all rationals p/q. Via the map $\alpha : I \mapsto \alpha(I)$ this set is pulled back to a subset $A_c \subseteq A$, which is the product of a circle and a Cantor set of large measure.

Theorem 2 *In the above circumstances assume that $\gamma > 0$ is sufficiently small. Then for $|\varepsilon|$ sufficiently small, there exists a smooth (C^∞) transformation of the*

annulus $\Phi_\varepsilon : A \to A$, conjugating the restriction $P_0|_{A_c}$ to a subsystem of P_ε. Moreover, Φ depends smoothly on ε.

This theorem is known as the 'twist' theorem and it was first proven by J.K. Moser [50]. The present formulation is close to theorem 1 and, concerning the smoothness of Φ_ε, the same comments apply again here.

Discussion

Regarding the similarity between the theorems 1 and 2 one can say that in theorem 2 the rôle of the parameter α has been taken by the action variable I. In the same spirit as before, we conclude that for the area preserving case, typically quasi-periodicity occurs with positive measure in phase space.

Remark A difference in the settings of theorem 1 and 2 is the following. In the case of KAM theorem 1 for circle maps, the conjugation Φ_ε between the family of pure rotations P_0 and the perturbation P_ε preserves the projection to the parameter space $[0, 1] = \{\alpha\}$, compare with the equation (5). In the present case of theorem 2 the corresponding projection to the action space $K = \{I\}$ generally is not preserved.

As an application consider the mathematical (planar) pendulum with (weak) periodic forcing. As possible equations of motion one may take

$$\ddot{y} + \omega^2 \sin y = \varepsilon \cos t \quad \text{or}$$
$$\ddot{y} + (\omega^2 + \varepsilon \cos t) \sin y = 0,$$

which give rise to volume-preserving 3-dimensional vector fields. For simplicity we only write down the first example

$$\begin{aligned}
\dot{y} &= z \\
\dot{z} &= -\omega^2 \sin y + \varepsilon \cos t \\
\dot{t} &= 1.
\end{aligned}$$

As is common in mechanics, e.g., see [7], we introduce action-angle variables (I, φ) for $\varepsilon = 0$, i.e., for the autonomous planar pendulum. In fact, denoting the energy by $H_0(y, z) = \frac{1}{2}z^2 - \omega^2 \cos y$, we restrict ourselves to the oscillatory region where $H_0(y, z) < \omega^2$. Next consider any level set $H_0(y, z) = h$, with $|h| < \omega^2$. The action variable I then is defined by

$$I(h) = \frac{1}{2\pi} \oint_{H_0(y,z)=h} z \, dy,$$

which is proportional to the area enclosed by the level set. The angle variable φ is obtained by taking the time parametrization of the periodic motion inside this level set scaled to period 2π. Thus one obtains the following canonical equations

$$\dot{I} = 0, \; \dot{\varphi} = \alpha(I)$$

for the oscillatory motions of the planar pendulum.

This implies that the Poincaré (or stroboscopic) map P_ε has the above form (6). A direct computation[3] shows that P_0 is a pure twist map. Therefore the conclusion of quasi-periodicity occurring with positive measure in phase space, applies here.

A related application deals with coupled oscillators

$$\begin{aligned} \ddot{y}_1 &= -\omega_1^2 \sin y_1 - \varepsilon \frac{\partial U}{\partial y_1}(y_1, y_2) \\ \ddot{y}_2 &= -\omega_2^2 \sin y_2 - \varepsilon \frac{\partial U}{\partial y_2}(y_1, y_2), \end{aligned}$$

leading to a 4-dimensional Hamiltonian vector field

$$\begin{aligned} \dot{y}_j &= z_j \\ \dot{z}_j &= -\omega_j^2 \sin y_1 - \varepsilon \frac{\partial U}{\partial y_j}(y_1, y_2), \end{aligned}$$

$j = 1, 2$, with Hamilton function $H_\varepsilon(y_1, z_1, y_2, z_2) = \frac{1}{2}z_1^2 + \frac{1}{2}z_2^2 - \omega_1 \cos y_1 - \omega_2 \cos y_2 + \varepsilon U(y_1, y_2)$. In this case it is the iso-energetic Poincaré map which obtains the form P_ε. This leads to the conclusion of quasi-periodicity occurring with positive measure in energy hypersurfaces of H_ε.

5 Conservative KAM theory

The above discussion has set the stage for a general formulation of the KAM theorem, as a suitable variation on Kolmogorov [40, 41] and Arnold [1].

We begin by introducing the notion of a quasi-periodic invariant torus for an autonomous system of ordinary differential equations. Indeed, we understand this to be an invariant torus, the dynamics on which is smoothly conjugated to that of a constant vector field

$$\begin{aligned} \dot{x}_1 &= \omega_1 \\ \dot{x}_2 &= \omega_2 \\ \cdots \quad & \cdots \quad \cdots \\ \dot{x}_n &= \omega_n, \end{aligned}$$

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[3]Involving an elliptic integral . . .

on the standard n-torus $\mathbf{T}^n = \mathbf{R}^n/(2\pi\mathbf{Z}^n) = \{x_1, x_2, \ldots, x_n\}$, where the angles x_j are counted modulo 2π, $1 \leq j \leq n$. Moreover, the frequencies $\omega_1, \omega_2, \ldots, \omega_n$ are required to be independent over the rationals, which implies that the torus is densely filled by each of its trajectories. In that case we speak of n quasi-periodicity, both for the restriction of the system to the n-torus as for the n-torus itself.

For integrable Hamiltonian systems, the conjugation with a constant vector field is provided by the action angle variables provided by the Liouville-Arnold theorem [7].

The present formulation of the classical KAM theorem now reads

Theorem 3 *In Hamiltonian systems with n degrees of freedom typically n quasi-periodic Lagrangean tori occur with positive Liouville measure, both*

1. *in phase space;*

2. *and in each energy hypersurface.*

As said earlier 'typical' means that there are classes of examples that are 'open in function space'. These examples are close to Liouville integrable systems [7].
The first conclusion of theorem 3 is the standard KAM theorem, while the second is referred to as the iso-energetic KAM theorem. The nondegeneracy condition on the integrable approximation $H = H(I)$ is somewhat different for both cases.
Indeed, for the standard KAM theorem the Kolmogorov nondegeneracy condition is required which states that the frequency vector

$$\omega = \frac{\partial H}{\partial I}$$

as a function of the action variables I should have maximal rank; this implies that the frequency map $I \mapsto \omega(I)$ locally is a diffeomorphism.
The condition for iso-energetic nondegeneracy similarly requires that the corresponding frequency ratio map $I \mapsto (\omega_1(I) : \omega_2(I) : \ldots : \omega_n(I))$ to the $(n-1)$-dimensional projective space should have maximal rank. Compare with [7, 19, 16]. We remark that the conclusion drawn in the example of the two coupled oscillators of the previous section also would follow as an application of the iso-energetic KAM theorem.
In typical cases, however, both nondegeneracy conditions are satisfied, implying that the union of quasi-periodic Lagrangean invariant tori has postive measure in phase space, in such a way that the conditional measure in the energy hypersurfaces also is positive.

In addition to nondegeneracy one again needs Diophantine conditions on the frequencies, in this case defined as follows. For constants $\tau > n - 1$ and $\gamma > 0$, it is required that for all $k \in \mathbf{Z}^n \setminus \{0\}$ one has

$$|\langle \omega, k \rangle| \geq \gamma |k|^{-\tau}.$$

Here $\langle \omega, k \rangle = \sum_{j=1}^n \omega_j k_j$, while $|k| = \sum_{j=1}^n |k_j|$. Similar to the cases met before, the corresponding subset of $\mathbf{R}^n$ is nowhere dense of positive measure. In fact this Diophantine subset is a 'bundle' of closed half-lines, the intersection of which with the unit sphere $\mathbf{S}^{n-1}$, for small γ, contains a Cantor set of almost full measure. Compare with, e.g., [23, 18].

In the case of the general Kolmogorov nondegeneracy, this set is pulled back along the frequency map in a locally diffeomorphic way. Iso-energetic nondegeneracy means that the energy hypersurfaces are transversal to the Diophantine line bundle, compare with [19, 16].

Some applications of the KAM theorem

In the introduction we already mentioned possible applications of theorem 3. Indeed, in many models of classical mechanics one finds Lagrangean KAM tori with quasi-periodic dynamics.

As a 'philosophical' application consider the solar system, first taken as a perturbation of the integrable system obtained when neglecting the interaction between the planets. If the theorem would apply it would follow that the actual evolution has positive probability to be quasi-periodic, when assuming that the initial condition has been chosen at random. In that case one might be inclined to call the solar system stable.

Much has been said about this example [2, 53, 54, 64, 27] and here we just restrict to a few remarks. Firstly the solar system contains quite strong resonances, which necessitate a more suitable integrable approximation than described here [2]. Secondly the interaction between the planets probably is far too strong for an actual application of theorem 3 as a perturbation result. The third remark refers to recent numerical work of Laskar [46], which seems to show that the solar system is entirely chaotic, mankind just does not exist long enough to have noticed . . . .

A large class of examples is provided by considering the neighbourhood of any generic elliptic equilibrium point of a Hamiltonian system. Under a few nonresonance conditions on the eigenvalues of the linear part one can normalize a number of steps according to Birkhoff, thereby making the lower order terms n-torus symmetric. In fact this allows writing the system locally as a nearly integrable system, where the size of the perturbation is controlled by the distance to the equilibrium, compare [53, 54].

As a consequence, in a neighbourhood of the equilibrium there are many KAM tori. Their union is of positive Liouville measure, where the equilibrium point is even a density point of quasi-periodicity [58, 18].

Other applications of KAM-techiques occur in quantum mechanics, in particular in the study of electron – Anderson – localization. If one considers Schrödinger equations with spatially ergodic potentials, a localized (non-conducting) state is an eigenfunction of the Schrödinger equation at some energy eigenvalue. Such a function will decay exponentially in space. As in the localized regime the Schrödinger operator typically has a dense point spectrum, developing a perturbation theory of the corresponding resolvent operator, which diverges at this dense set of eigenvalues, runs into problems very similar to problems encountered in the traditional KAM set-up, where the dense set of resonances gives rise to divergent perurbation expansions. KAM-inspired proofs and analyses of localization have been developed for typical realizations of random potentials in arbitrary dimensions for large interaction strengths or high energies in arbitrary dimensions, see e.g. [30], as well as for quasi-periodic potentials in one dimension, describing electrons in quasicrystals, see e.g. [28, 20, 61].
Yet another field of physics which is notorious for divergent perturbation theory problems and where KAM-like ideas are starting to play a role is Quantum Field Theory [33, 10].

Discussion

There are a great many related issues spread over a vast literature, compare with [7, 8, 53, 54, 64, 27], and we will be restrictive here.
One subject to mention is the difference between the cases $n = 2$ and $n \geq 3$. Indeed, for $n = 2$ the level set of the energy function is 3 dimensional and the Lagrangean tori have dimension 2. In the nearly integrable case these KAM tori generally foliate over nowhere dense sets. However, for open sets of initial conditions the evolution curves are forever trapped in between them, which implies perpetual adiabatic stability.
In contrast, for $n \geq 3$ the Lagrangean tori have codimension larger than 1 and evolution curves may escape. This actually occurs in case of Arnold diffusion [4, 8, 24].
Another subject concerns the general motion in the neighbourhood of the KAM tori. Indeed by refined averaging it follows that nearby evolution curves stay nearby over an exponentially long time, see Nekhoroshev [55]. Colloquially one says that the KAM tori are 'sticky'.

From the first developments of KAM theory on, it was known that the theory extends outside the Hamiltonian frame work of Lagrangean tori. So it turns out

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that analogues of theorem 3 exist, e.g., for lower dimensional (isotropic) invariant tori in Hamiltonian systems. Here, apart from the internal frequencies of the quasi-periodic tori, also normal frequencies play a role and these moreover enter the Diophantine conditions. For a discussion on the dynamical role of the corresponding normal-internal resonances see [13].
There has been quite a lot of interest in the Hamiltonian KAM theory of normally elliptic tori [59, 38]. This also includes cases in which the normal frequencies have strong resonances [37]. Moreover, cases have been studied of normally parabolic tori. The latter two cases involve quasi-periodic torus bifurcations, in certain cases also branching off of higher dimension tori [18, 34, 14, 15].

Similar results go when considering reversible systems instead of Hamiltonian ones [51, 54, 65, 17, 22], or systems that are equivariant with respect to certain Lie groups. Moreover, as section 3 already indicates, there also exist KAM theorems for the general class of (dissipative) dynamical systems, compare with section 3. In fact a unifying Lie algebra approach allows to reach all these results at once. In general one needs external parameters to have a good persistence theory for quasi-periodic invariant tori. Compare with [51, 52, 19, 18].

6 Ergodic hypothesis: a paradox?

As a direct corollary of theorem 3 we have

Theorem 4 *In typical Hamiltonian systems with $n < \infty$ degrees of freedom, energy hypersurfaces are not ergodic under the Hamiltonian flow.*

Indeed, by theorem 3 there exist open classes of Hamiltonian systems with invariant tori, such that the union of these tori has positive (Liouville) measure in the energy hypersurfaces. This leads to a decomposition of these hypersurfaces in several disjoint invariant sets, which contradicts ergodicity. For background on ergodic theory, e.g., see [8].

So the KAM tori form an obstruction against ergodicity and a question is how bad this obstruction is as $n \to \infty$. This question is of importance in statistical physics, where large systems are studied.
A primitive formulation of the ergodic hypothesis asserts that typically, energy hypersurfaces of many-particle systems are ergodic, which clearly would not be compatible with theorem 4. To fix thoughts we give an example.

A lattice system

Consider the 1-dimensional lattice $\mathbf{Z} \subset \mathbf{R}$, at the vertices of which identical nonlinear oscillators are situated. For simplicity think of the lattice being situated on

a horizontal line, where at all vertices identical pendula are suspended, subject to constant vertical gravity. Also we connect nearest neighbour oscillators by weak springs. Here the spring constants can be either constant or decay at infinity.

Let $\Lambda_N \subseteq \mathbf{Z}$ be the box with vertices $(-N, N)$. Then, for $M \leq N$ consider two of these boxes $\Lambda_M \subseteq \Lambda_N$. We 'truncate' the infinite system by ignoring all pendula outside the larger box Λ_N.

First consider the integrable system associated to Λ_N, where all interactions are neglected. Suppose that the oscillators situated at vertices in Λ_M are in motion, while the others are at rest. In the phase space this corresponds to a $2M + 1$-dimensional invariant torus, which is normally elliptic. Moreover, the normal frequencies are in $1 : 1 : \ldots : 1$ resonance.

Then we 'turn on' the activity of the interacting springs. A suitable adaptation [37] of theorem 3 and of [59] for this case, yields the persistence of these elliptic tori for small values of the spring constants. The corresponding kind of motion is a quasi-periodic breather; for a similar kind of motion compare with [48]. The union of elliptic tori has positive $2(2M + 1)$-dimensional Hausdorff measure.

The question now is what is the asymptotics of the 'density' of this measure as N (and M) $\rightarrow \infty$. A partial answer to this question [37], says that this density decays exponentially fast in N, while there is a further polynomial decay in M.

Remark It is worth mentioning that the normal $1 : 1 : \ldots : 1$ resonance of this system gives rise to interesting quasi-periodic branchings [37].

Discussion

The conclusion from this example is that the obstruction to ergodicity given by KAM theory is not too bad as the size of the system tends to infinity. This is in the same line as an earlier result by Arnold [3].

Another problem is what happens if the limit $N \rightarrow \infty$ is really attained. The KAM theory for infinite systems is fully in development [42, 39], but infinite lattice systems still have many secrets.

7 Conclusion

The lasting influence of Kolmogorov, Arnold and Moser on the present state of the art in mathematics, physics and other sciences is enormous and this paper only sketches part of this legacy. Nevertheless we believe that it is an important part, which is still fully in development. The role of the ergodic hypothesis in statistical mechanics has turned out to be much more subtle than was expected, see e.g., [9,

31, 32]. Regarding KAM theory, for further reading we mention the introductory texts [23, 49, 47, 60]. Also reading of [18], [43] and [29] is recommended.

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