Bartlett Correction for likelihood ratio test

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1 Likelihood

The likelihood ratio statistic is given as:

$$\Lambda(X,Y) = -2\sup\left\{l(\theta|X,Y;\theta\in\Theta_0)\right\} - \sup\left\{l(\theta|X,Y;\theta\in\Theta)\right\},$$

We then maximize this function under two conditions,

$$l_{H_0} = \sup_{\alpha,\delta,\mu_j^x=\mu_j^y}\left\{l(\alpha,\delta,\mu_1,..,\mu_m)\right\} \quad (1)$$

$$l_{H_1} = \sup_{\alpha,\delta,\mu_1,..,\mu_m}\left\{l(\alpha,\delta,\mu_1,..,\mu_m)\right\} \quad (2)$$

The full log-likelihood can be written as

$$l(\alpha,\delta,\mu_j^x,\mu_j^y) = m\left(\ln\Gamma(\alpha+\frac{n}{2}) - \ln\Gamma(\alpha) - \frac{n}{2}\ln(2\pi\delta)\right) - (\alpha+\frac{n}{2})\sum_{j=1}^{m}\ln\left(\frac{\zeta_j}{2}+\delta\right), \quad (3)$$

where for notational simplicity, we introduce the quantities $\zeta_j = \sum_{i=1}^{n_x}(x_{ij}-\mu_j^x)^2 + \sum_{i=1}^{n_y}(y_{ij}-\mu_j^y)^2$. and $n = n_x + n_y$. Applying an asymptotic expansion for the gamma function to Equation (3), the log-likelihood at the maximum is given by,

$$\lim_{\delta\to\infty} l(\delta) = \frac{nm}{2}\left(\ln\left(\frac{nm}{2\pi\sum_{j=1}^{m}\zeta_j}\right) - 1\right). \quad (4)$$

2 Bartlett correction

We define the Bartlett correction as

$$\begin{aligned} BC &= E_{H_0}\left\{\Lambda(x,y)\right\} \\ &= -2E_{H_0}(l_{H_0} - l_{H_1}) \quad (5) \end{aligned}$$

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with l_{H_0} and l_{H_1} defined in (1) and (2). By defining the small sample likelihood ratio statistic as $\Lambda_{\mathrm{BC}}(x,y) = m\frac{\Lambda(x,y)}{BC}$, we achieve that precisely the Bartlett corrected likelihood ratio statistic has $E_{H_0}\{LR_{\mathrm{BC}}\} = m$ as would be expected for a χ^2_m distribution.

Calculation of these two expectations in general is very involved. If we use the characterization for $l(\delta)$ defined in (4), we can get an explicit approximate expression for the Bartlett correction,

$$\begin{aligned} BC &\approx -2E_{H_0}(\lim_{\delta\to\infty} l(\delta|H_0) - \lim_{\delta\to\infty} l(\delta|H_1)) && (6)\\ &= mnE\ln\left(\frac{\sum_{j=1}^{m}\zeta_j^{H0}}{\sum_{j=1}^{m}\zeta_j^{H1}}\right) && (7)\\ &= nmE\ln\left(1+\frac{\sum_{j=1}^{m} S_{\Delta,j}}{\sum_{j=1}^{m}(S_{x,j}+S_{y,j})}\right) && (8)\end{aligned}$$

whereby

$$\begin{aligned} \zeta_j^{H0} &= \sum_{i=1}^{n_x}(x_{ij}-\widehat{\mu}_j)^2 + \sum_{i=1}^{n_y}(y_{ij}-\widehat{\mu}_j)^2 \\ \zeta_j^{H1} &= \underbrace{\sum_{i=1}^{n_x}(x_{ij}-\overline{x}_{\cdot j})^2}_{S_{x,j}} + \underbrace{\sum_{i=1}^{n_y}(y_{ij}-\overline{y}_{\cdot j})^2}_{S_{y,j}} \\ S_{\Delta,j} &= \frac{n_x n_y(\overline{x}_{\cdot j}-\overline{y}_{\cdot j})^2}{n}, \end{aligned}$$

with $\widehat{\mu}_j = \frac{n_x\overline{x}_{\cdot j}+n_y\overline{y}_{\cdot j}}{n_x+n_y}$ and $\overline{x}_{\cdot j} = \frac{1}{n_x}\sum_{i=1}^{n_x} x_{ij}$. Since for small values of m, the supremum of the likelihood in (1) and (2) are actually found at such degenerate δ, the approximation in (6) can be exact in certain cases. The log expressions are normally distributed, so that under the assumption of no differential expression and in the degenerate case $\sigma_j^2 = \sigma^2$, we have that $\overline{x}_{\cdot j} - \overline{y}_{\cdot j} \sim N(0, \frac{\sigma^2(n_x+n_y)}{n_x n_y})$ and therefore, $\sum_{j=1}^{m}\frac{S_{\Delta,j}}{\sigma^2} \sim \chi^2_m$ and $\sum_{j=1}^{m}\frac{S_{x,j}+S_{y,j}}{\sigma^2} \sim \chi^2_{m(n-2)}$. The ratio of χ^2 distributions yields an F distribution, thus

$$BC \approx nmE\ln\left(1+\frac{F_{m,m(n-2)}}{n-2}\right). \qquad (9)$$

The density of $F_{m,m(n-2)}$ [Hogg et al., 2005, p.185] is

$$f(x) = \frac{(n-2)^{m/2}x^{m/2-1}}{B(m,m(n-2))(1+x/(n-2))^{m(n-1)/2}}, \qquad x > 0 \qquad (10)$$

and $B(x,y)$ is the Beta function. The expected value in (9) can be found by applying a transformation theorem [Hogg et al., 2005, p.55], [Gradshteyn and Ryzhik, 2000, p.555]

$$BC \approx \frac{mn}{B(m,m(n-2))} \times \int_0^\infty \frac{ln(1+x/(n-2))\nu^{m/2}x^{m/2-1}}{(1+x/(n-2))^{m(n-1)/2}}\,dx \qquad (11)$$

$$
\begin{aligned}
&= mn\left[\psi\left(\frac{m(n-1)}{2}\right)-\psi\left(\frac{m(n-2)}{2}\right)\right] \\
&= mn\left[\ln\left(1+\frac{m/2}{m(n-2)/2}\right)+O(n^{-2})\right] && (12)\\
&\approx mn\ln\left(\frac{n-1}{n-2}\right) && (13)\\
&\approx m\frac{n}{n-2} && (14)
\end{aligned}
$$

From this it follows that the Bartlett corrected likelihood ratio statistic is found as

$$
\begin{aligned}
\Lambda_{BC}(x,y) &\approx \frac{\Lambda(x,y)}{n\left[\psi\left(\frac{m(n-1)}{2}\right)-\psi\left(\frac{m(n-2)}{2}\right)\right]} && (15)\\
&\approx \frac{\Lambda(x,y)}{n\ln\left(\frac{n-1}{n-2}\right)} && (16)\\
&\approx \frac{n-2}{n}\Lambda(x,y) && (17)
\end{aligned}
$$

References

I. S. Gradshteyn and I. M. Ryzhik. *Table of Integrals, Series, and Products.* Academic Press, 2000.

R. V. Hogg, J. W. McKean, and A. T. Craig. *Introduction to Mathematical Statistics.* Pearson Prentice Hall, 2005.