SOLUTIONS TO THE
PEC 2013 FINAL EXAMINATION

Problem 1.

(a) Suppose that $G$ does not contain $K_{2\ell}$. Then between every two vertices there are at most $\ell - 1$ common neighbours. Consider the sum of common neighbours over all pairs of vertices in $G$. We just observed that this sum is at most $(\ell - 1)\binom{n}{2}$. We express this sum in a different way. Every vertex is accounted for exactly once in this sum by each pair of its neighbours. So this sum is exactly $\sum_{v \in V} \binom{\deg(v)}{2}$.

(b) The inequality is equivalent to

$$\frac{1}{2} \sum_{v \in V} \deg(v)^2 - \frac{1}{2} \sum_{v \in V} \deg(v) \geq \frac{2m^2}{n} - m$$

Since $\sum_{v \in V} \deg(v) = 2m$, it suffices to show that

$$n \sum_{v \in V} \deg(v)^2 \geq 4m^2.$$ 

Assuming $V = [n]$, we set $x_i = 1$ and $y_i = \deg(i)$ for all $i$, and the inequality follows directly from the hint inequality (which is Cauchy-Schwarz) and the equality $\sum_{v \in V} \deg(v) = 2m$.

(c) The solution is to write out the basic arithmetic for checking $m \left( \frac{2m}{n} - 1 \right) > (\ell - 1)\binom{n}{2}$ under the hypothesis and conclude using (a).

(d) Two circles intersect in at most two points. This implies that the auxiliary graph defined by joining two of the $n$ points by an edge iff they are at distance exactly 1 in $\mathbb{R}^2$, it must be $K_{2,3}$-free. Then the result follows directly from (c).
Problem 2.

(a) Since \( K_{3,3} \) is not planar (a fact which you were allowed to use without proof) we have \( \text{cr}(K_{3,3}) \geq 1 \). To prove that \( \text{cr}(K_{3,3}) = 1 \) it thus suffices to give a drawing of \( K_{3,3} \) with one crossing. Here is an example of such a drawing:

(b) It suffices to give a drawing of \( K_{3,3} \) on the torus without crossings. This can for instance be done like so:

(c) If \( |V(G)| \leq 3 \) then the inequality is obvious. Suppose thus \( |V(G)| \geq 4 \). We create a new toroidal graph \( G^* \supseteq G \) by adding edges to \( G \) until this is no longer possible without creating a non-toroidal graph. Note that \( G^* \) must be have the following property: in every crossing-free drawing of \( G^* \) on the torus, every face is a triangle. (because if some face has \( \geq 4 \) vertices on its boundary then we can add an extra edge inside that face). Double counting incident pairs (edge,face) of a drawing of \( G^* \) gives \( 2e = 3f \). Hence we have \( v - e + (2/3)e \geq 0 \), which implies

\[
|E(G)| \leq |E(G^*)| \leq 3|V(G^*)| = 3|V(G)|,
\]

as desired.

(d) The proof is completely identical to the proof of the (ordinary) crossing number inequality. (In the exam you needed to spell out this proof.)
Problem 3.

(a) By repeatedly applying the Erdős-Szekeres recurrence, \(R(3, 5) \leq R(2, 5) + R(3, 4) \leq 5 + R(2, 4) + R(3, 3) \leq 5 + 4 + R(2, 3) + R(3, 2) = 15\).

Let us label the vertices with the elements of \([13]\) in clockwise order. Let us imagine the given graph drawn in the exam as being the red subgraph of \(K_{13}\), and the complement as the blue subgraph. By symmetry considerations, we need only to verify that vertex 1 is not contained in any red triangle. The only red edges from 1 are to the vertex subset \{2, 6, 9, 13\} and none of these are adjacent, so 1 is not contained in any red triangle.

The blue edges from 1 are to the vertex subset \{3, 4, 5, 7, 8, 10, 11, 12\}. Let us assume WLOG that the blue neighbourhood of 1 contains a blue \(K_4\). By symmetry considerations, we may assume either that it contains 7 (and not 8) or it contains neither of 7 and 8. Assuming it contains 7, then it cannot contain 12, but then the remaining vertices \{3, 4, 5, 10, 11\} induce a blue \(C_5\) which is triangle-free. So we can now assume it contains neither 7 nor 8, in which case it must contain exactly one of 5 or 10 (because \{3, 4, 11, 12\} is not a blue \(K_4\) so let us assume 5; but then it does not contain 4 and at most one of 11 or 12 which is too few vertices. Thus 1 is not in a blue \(K_5\), and we conclude that \(R(3, 5) > 13\).

(b) Fix 0 < \(p < 1\) Let us consider the probability space of \{red, blue\}-edge-coloured \(K_n\)s on vertex set \([n]\) formed by independently flipping a \(p\)-biased coin for each of the \(\binom{n}{2}\) pairs of vertices, and we include a red edge with probability \(p\) and a blue edge with probability \((1-p)\). The sum of expected numbers of red \(K_{\ell_1}\)s and blue \(K_{\ell_2}\)s is exactly \(\binom{n}{\ell_1}p^{\binom{\ell_1}{2}} + \binom{n}{\ell_2}(1-p)^{\binom{\ell_2}{2}}\). If this quantity is less than one, then with positive probability we obtain an edge-coloured \(K_n\) with no red \(K_{\ell_1}\)s and no blue \(K_{\ell_2}\)s, which certifies that \(R(\ell_1, \ell_2) > n\).

(c) Fixe 0 < \(p < 1\) and consider the same probability space of edge-coloured \(K_n\)s as in part (b). The expected number of red \(K_{\ell_1}\)s is \(\binom{n}{\ell_1}p^{\binom{\ell_1}{2}}\) and the expected number of blue \(K_{\ell_2}\)s is \(\binom{n}{\ell_2}(1-p)^{\binom{\ell_2}{2}}\). For any edge-coloured \(K_n\) from this space, we perform the following process: we repeatedly remove a vertex from the graph, each time from a red \(K_{\ell_1}\)s, if one remains, up to \(\binom{n}{\ell_1}p^{\binom{\ell_1}{2}}\) times; then we do the same for blue \(K_{\ell_2}\)s, up to \(\binom{n}{\ell_2}(1-p)^{\binom{\ell_2}{2}}\) times. We end up with an edge-coloured complete graph on \(n - \binom{n}{\ell_1}p^{\binom{\ell_1}{2}} - \binom{n}{\ell_2}(1-p)^{\binom{\ell_2}{2}}\) vertices, and with positive probability we removed all red \(K_{\ell_1}\)s and blue \(K_{\ell_2}\)s, which gives a certificate of the claim.

(d) First pick \(p\) so \(\binom{n}{3}p^{\binom{3}{2}} \leq n/2 \iff p^3 \leq \frac{3}{(n-1)(n-2)}\), let us say \(p = n^{-2/3}\). Now, using the inequalities, we estimate

\[
\binom{n}{\ell_1}(1 - n^{-2/3})^{\binom{\ell_1}{2}} \leq \left(\frac{en}{\ell_1}\right)^{\ell_1} e^{-n^{-2/3}(\ell_1)} = \exp\left(\ell_1 \ln n - n^{-2/3}\left(\frac{\ell_1}{2}\right)\right).
\]

Setting \(n = \ell^{5/4}\), this last expression is at most \(\exp(1.25\ell \ln \ell - 0.25\ell^{-5/6+2}) \to 0\) as \(\ell \to \infty\). Combining these estimates and applying (c) we obtain that \(R(3, \ell) > \frac{1}{2}\ell^{5/4} - o(1)\), required.