Problem 1, 30 points.
Given graphs $G$ and $H$, the Ramsey number $R(G, H)$ is defined as the least $n$ such that in any colouring of the edges of $K_n$ with red or blue there is guaranteed to be a subgraph isomorphic to $G$ all of whose edges are red or a subgraph isomorphic to $H$ all of whose edges are blue.

(a) Prove that $R(K_{\ell_1}, K_{\ell_2}) \leq \binom{\ell_1 + \ell_2 - 2}{\ell_1 - 1}$ for any positive integers $\ell_1, \ell_2$.

(b) Show that $R(G, H)$ exists for any $G, H$.

(c) Prove that $R(K_{\ell}, K_{\ell}) \geq n$ if $\binom{n}{\ell} 2^{\binom{\ell}{2}} < 1$.

(d) Prove that $6 \leq R(C_4, C_4) \leq 20$ where $C_4$ is a cycle on four vertices.

Problem 2, 30 points.
A family $\mathcal{F}$ of sets is intersecting if every pair of members of $\mathcal{F}$ have an element in common.

(a) Show that an intersecting family $\mathcal{F} \subseteq \binom{[2^n]}{k}$ has at most $\binom{2^n - 1}{k - 1}$ members.

A family $\mathcal{F}$ of subsets of $[n]$ is regular if every element of $[n]$ is contained in a constant number of members of $\mathcal{F}$. For the next part, you may assume the following result: if $k$ is not a power of 2, then $\binom{2^n - 1}{k - 1}$ is even and there exists a regular intersecting subfamily of $\binom{[2^n]}{k}$ with $\binom{2^n - 1}{k - 1}$ members.

(b) Assuming $k$ is not a power of 2, construct a regular intersecting family of subsets of $[k]$ with $2^{k-1}$ members.

(c) If $k \in \{2, 4\}$, show that there is no such family.
Problem 3, 30 points.
Let $G = (V, E)$ be a triangle-free graph with $|V| = n$. Recall for $A, B \subseteq V$ the density of $(A, B)$ is

$$d(A, B) = \frac{e(A, B)}{|A||B|}$$

and for $\varepsilon > 0$ the pair $(A, B)$ is $\varepsilon$-regular if, for any $A' \subseteq A, B' \subseteq B$, with $|A'| \geq \varepsilon|A|, |B'| \geq \varepsilon|B|$, we have

$$|d(A', B') - d(A, B)| \leq \varepsilon.$$

(a) State the Szemerédi regularity lemma. Let $\varepsilon > 0$ and apply the lemma to obtain an $\varepsilon$-regular partition $V = V_0 \cup V_1 \cup \cdots \cup V_k$ of $G$.

Let $R = ([k], E(R))$ be the auxiliary graph formed by letting $ij \in E(R)$ if and only if $(V_i, V_j)$ is an $\varepsilon$-regular pair with density greater than $\varepsilon$.

(b) Letting $G_R = (V \setminus V_0, E(G_R))$ be the subgraph of $G$ which includes only the edges between parts $V_i$ and $V_j$ for which $ij \in E(R)$, show by choosing $\varepsilon$ small enough and a parameter in the regularity lemma large enough that $(1 - \varepsilon)n \leq |V \setminus V_0| \leq n$ and $0 \leq e(G) - e(G_R) < 2\varepsilon n^2$.

(c) Show that $R$ is triangle-free.